Quiver Yangian from crystal melting

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DIAS, July 22, 2020

Reference

Mainly based on

- **Quiver Yangian from crystal melting**

  [arxiv:2003.08909] with *Masahito Yamazaki*

Related

- Gluing two affine Yangian of $\mathfrak{gl}_1$

  [arxiv:1905.03076] with *Pietro Longhi*
- Gluing two affine Yangians with bi-fundamentals

  [arxiv:1910.10129]

Also see

- Twin plane partitions and $\mathcal{N} = 2$ affine yangian

  [arXiv:1807.11304] with *Matthias Gaberdiel and Cheng Peng*
- Higher Spins and Yangian Symmetries

  [arXiv:1702.05100]

  with *Matthias Gaberdiel, Rajesh Gopakumar, and Cheng Peng*

Main question

What is the algebraic structure underlying the BPS sector of a 4D $\mathcal{N} = 2$ gauge theory?

$$\boxed{\text{multiplication:} \qquad \mathcal{H}_{\mathrm{BPS}} \otimes \mathcal{H}_{\mathrm{BPS}} \to \mathcal{H}_{\mathrm{BPS}}}$$

- Analogue of chiral algebra of 2D $\mathcal{N} = 2$ SCFT
- Robust information invariant under duality transformation
- Control many aspects of theory
  (threshold correction, BPS counting, wall crossing, etc)

History

- Single-particle BPS sector of 4D $\mathcal{N}=2$ gauge theory *Harvey-Moore '96*
  - Fusion of two single-particle BPS states

$$\boxed{\text{multiplication:} \qquad \mathcal{H}_{\mathrm{BPS}}^{\mathrm{single}} \otimes \mathcal{H}_{\mathrm{BPS}}^{\mathrm{single}} \to \mathcal{H}_{\mathrm{BPS}}^{\mathrm{single}}}$$

  - for some cases: Lie algebra

- Include multi-particle BPS sectors *Kontsevich-Soibelman '10*

$$\boxed{\text{multiplication:} \qquad \mathcal{H}_{\mathrm{BPS}} \otimes \mathcal{H}_{\mathrm{BPS}} \to \mathcal{H}_{\mathrm{BPS}}}$$

  - Cohomological Hall Algebra (CoHA)

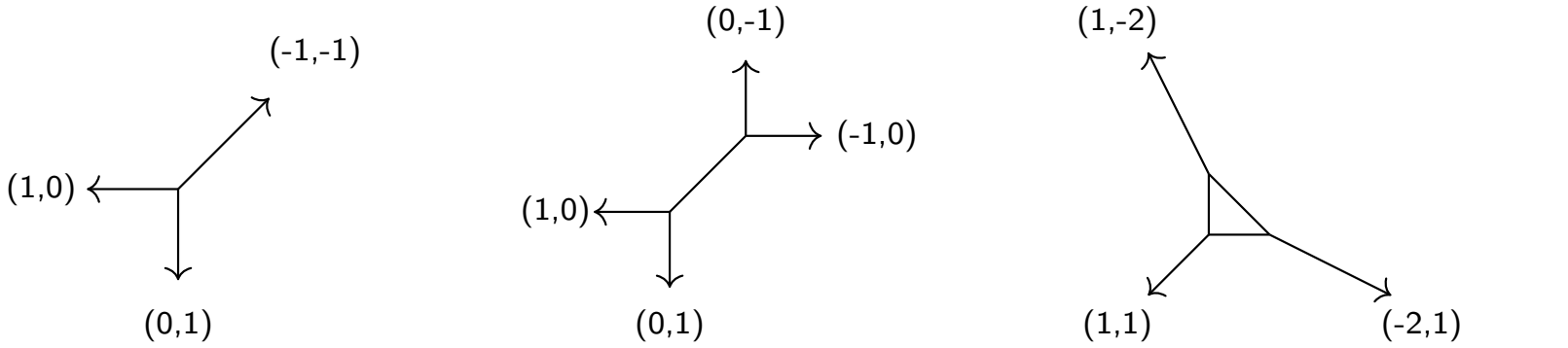
- Include decaying process *Davison '13*

$$\boxed{\text{co-multiplication:} \qquad \mathcal{H}_{\mathrm{BPS}} \to \mathcal{H}_{\mathrm{BPS}} \otimes \mathcal{H}_{\mathrm{BPS}}}$$

  - Drinfeld double of CoHA $\longrightarrow$ affine Yangian algebra ($\simeq$ UEA($\mathcal{W}$ algebra))

Setup

- Type IIA string in non-compact toric Calabi-Yau threefolds



$\mathbb{C}^3$ conifold $\mathbb{C}^3/\mathbb{Z}_3$

- 4D $\mathcal{N} = 2$ gauge theory
- Effective description of BPS sector

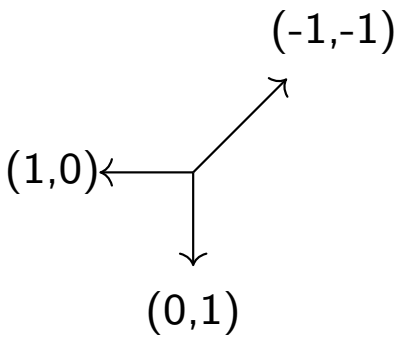
$$\boxed{\text{supersymmetric quiver quantum mechanics } (Q, W)}$$

- BPS algebra

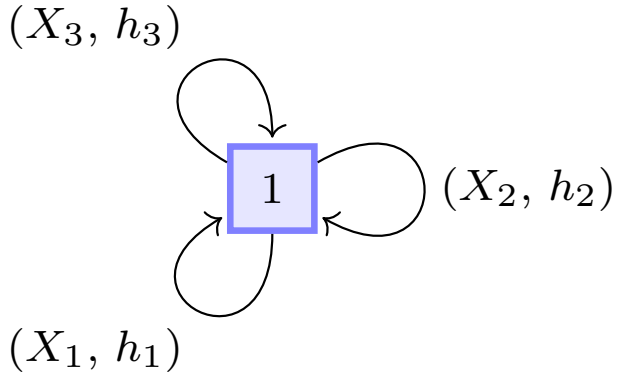
$$\boxed{\text{BPS algebra } (\Gamma_{\text{toric}}) = \text{BPS algebra } (Q, W)}$$

BPS algebra for type IIA string in $\mathbb{C}^3$

Γ_{toric}:



$\Longleftrightarrow$ (Q, W):



$$W = \mathrm{Tr}[-X_1\, X_2\, X_3 + X_1\, X_3\, X_2]$$

1. Quiver quantum mechanics (Q, W)

$\Downarrow$ define

$\{$ BPS states $\} = \{$ Plane partitions $\}$

act $\Uparrow$

2. BPS algebra = affine Yangian of $\mathfrak{gl}_1 \simeq \mathrm{UEA}(\mathcal{W}_{1+\infty})$

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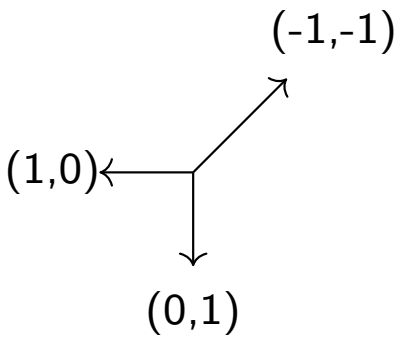

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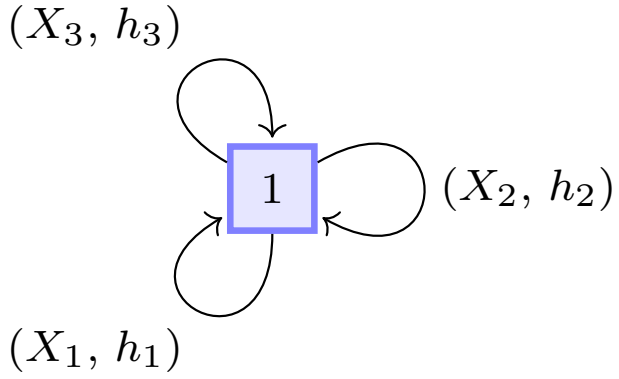

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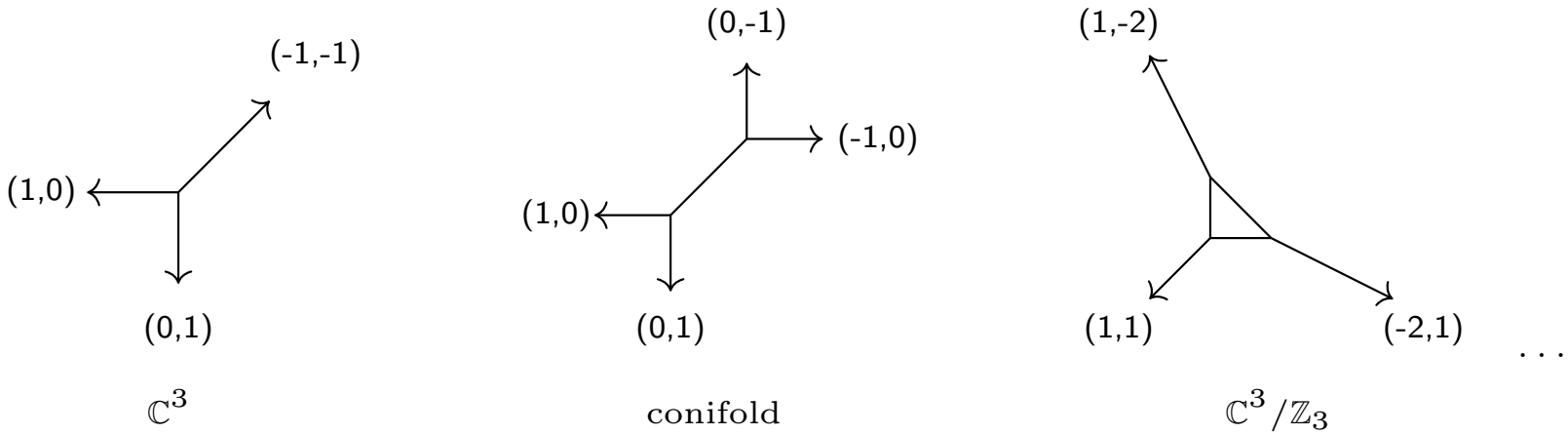
Gluing construction of BPS algebra (Γ_{toric})

- BPS sector of type IIA string in non-compact toric Calabi-Yau threefolds



- BPS algebra of $\mathbb{C}^3$: $\hat{\mathcal{Y}}(\mathfrak{gl}_1) \equiv$ affine Yangian of $\mathfrak{gl}_1$ ($\simeq$ UEA($\mathcal{W}_{1+\infty}$))
- Gluing construction
  - vertex: affine Yangian of $\mathfrak{gl}_1$
  - internal leg: additional operator (bi-module w.r.t. two $\hat{\mathcal{Y}}(\mathfrak{gl}_1)$)
  - face: ?

Gluing construction of BPS algebra (Γ_{toric})

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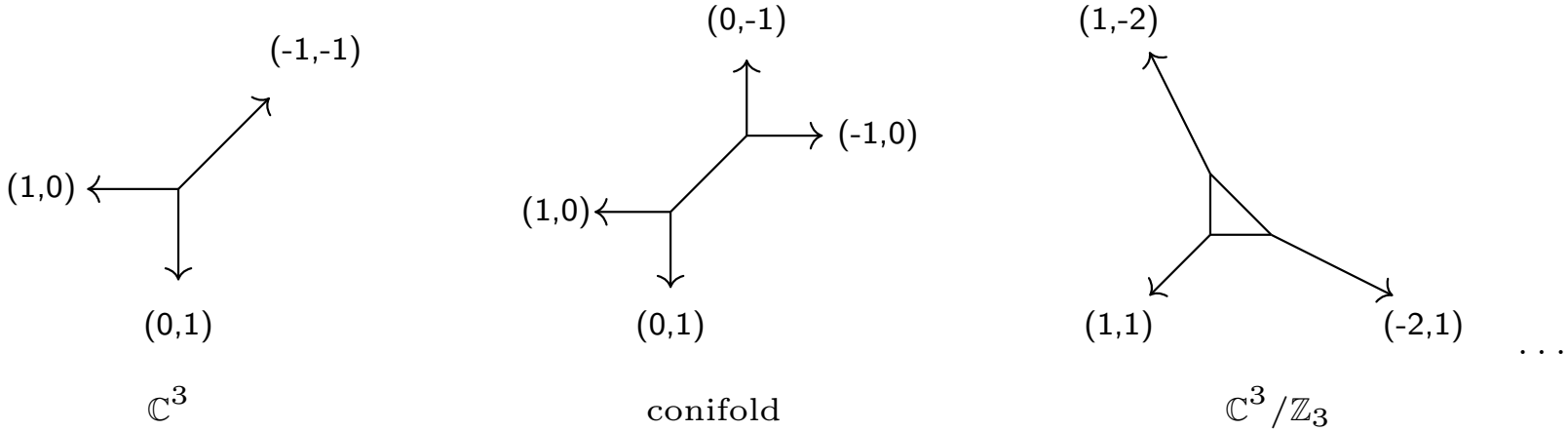
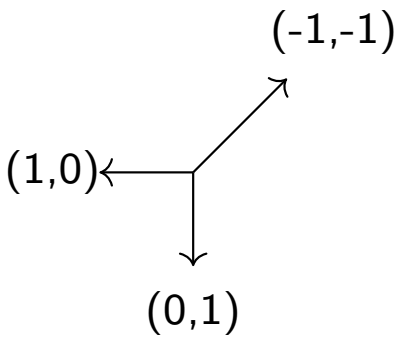


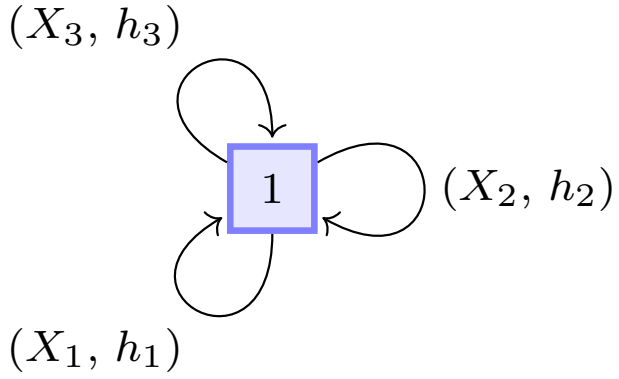
- BPS algebra of $\mathbb{C}^3$: $\hat{\mathcal{Y}}(\mathfrak{gl}_1) \equiv$ affine Yangian of $\mathfrak{gl}_1$ ($\simeq$ UEA($\mathcal{W}_{1+\infty}$))
- Gluing construction
  - resolved conifold and $\mathbb{C}^2/Z_2 \times \mathbb{C}$: affine Yangian of $\mathfrak{gl}_{1|1}$ and $\mathfrak{gl}_2$ (bootstrap from glued plane partitions)
    Gaberdiel-L-Peng '18, L-Longhi '19, L '19
  - toric Calabi-Yau without compact 4-cycles: affine Yangian of $\mathfrak{gl}_{n|m}$ (?)
  - generic toric Calabi-Yau: ?

BPS algebra for type IIA string in $\mathbb{C}^3$

Γ_{toric}:



$\Longleftrightarrow$ (Q, W):



$$W = \mathrm{Tr}[-X_1\, X_2\, X_3 + X_1\, X_3\, X_2]$$

1. Quiver quantum mechanics (Q, W)

$\Downarrow$ define

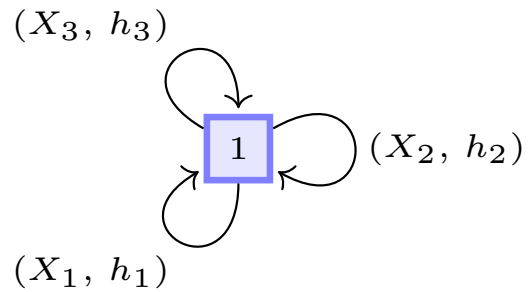
$\{$ BPS states $\} = \{$ Plane partitions $\}$

act $\Uparrow \Downarrow$ bootstrap

2. BPS algebra = affine Yangian of $\mathfrak{gl}_1 \simeq \mathrm{UEA}(\mathcal{W}_{1+\infty})$

BPS algebra (Q, W) from BPS crystal

- BPS sector of type IIA string in non-compact toric Calabi-Yau threefolds



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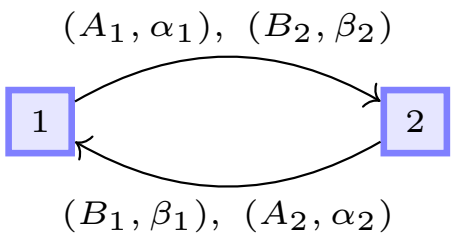
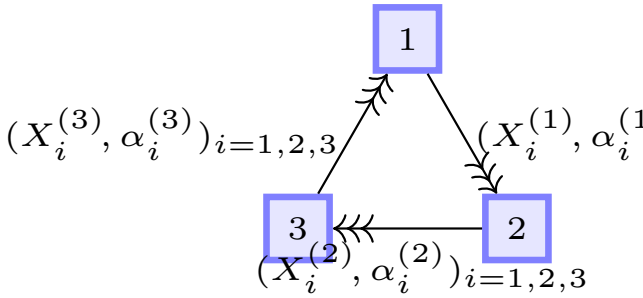


$W = \mathrm{Tr}[-A_1 B_1 B_2 A_2 + A_1 A_2 B_2 B_1]$



$W = \sum_{i,j,k=1}^{3} \varepsilon^{ijk} \mathrm{Tr}(X_i^{(1)} X_j^{(2)} X_k^{(3}$

1. Quiver quantum mechanics (Q, W)

$\Downarrow$ define

$\{$ BPS states $\} = \{$ colored crystals $\}$

act $\Uparrow \Downarrow$ bootstrap

2. BPS algebra = quiver Yangian (Q, W) *L-Yamazaki '20*

Summary of construction

Given a non-compact toric Calabi-Yau threefold X, consider the BPS sector of D-brane system of type IIA string on X

1. Write down (Q, W)

$$\Downarrow \quad \text{define}$$

2. $\{$ BPS states $\} = \{$ colored crystals $\}$

$$\Downarrow \quad \text{bootstrap}$$

3. BPS quiver Yangian $Y(Q, W)$

Summary of construction

Given a non-compact toric Calabi-Yau threefold X, consider the BPS sector of D-brane system of type IIA string on X

1. Write down (Q, W)

2. Write down $Y(Q, W)$ according to the prescription

Advantages

1. Explicit algebraic relations
2. Apply to all non-compact toric Calabi-Yau threefolds

Truncation of quiver Yangian (Q, W)

- give new rational VOAs
- relate to other systems
  1. Vertex VOA and the gluing thereof

     Gaiotto-Rapcak '17, Rapcak-Prochazka'17-'18
  2. Spike instantons and generalized AGT

     Nekrasov '16, Nekrasov-Prabhakar '16

     Rapcak-Soibelman-Yang-Zhao '18

Outline

1. Introduction
2. BPS crystal
3. $\mathbb{C}^3$
4. General toric CY3
5. Summary

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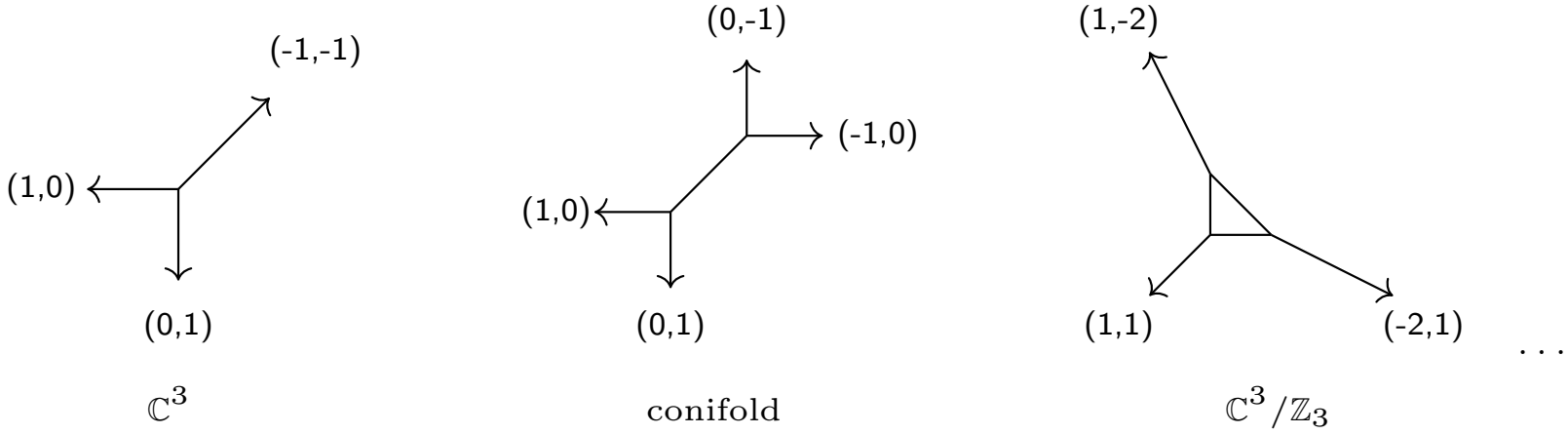
Setup

Type IIA string compactified on a non-compact toric Calabi-Yau threefolds X_3

- $X_3 = \mathbb{T}^2 \times \mathbb{R}$ fibered overe $\mathbb{R}^3$



- D-brane bound states:

  D6/D4/D2/D0 branes wrapping holomorphic 6/4/2/0 cycles of X_3

- $\frac{1}{2}$-BPS states of 4D $\mathcal{N} = 2$ gauge theory with D-brane charge

$$\#(\mathrm{D6},\ \mathrm{D4},\ \mathrm{D2},\ \mathrm{D0}) = (n_6, n_4^i, n_2^j, n_0)$$

How to describe these states?

Setup

Type IIA string compactified on a non-compact toric Calabi-Yau threefolds X_3

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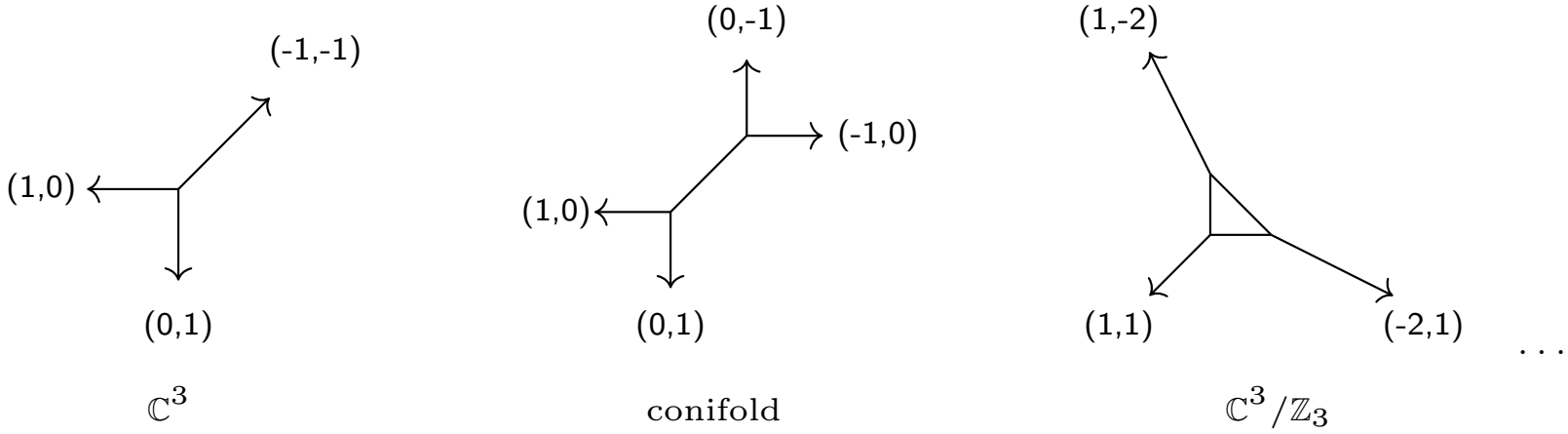

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$$\#(\mathrm{D6},\ \mathrm{D4},\ \mathrm{D2},\ \mathrm{D0}) = (n_6, n_4^i, n_2^j, n_0) \quad \longrightarrow (1, 0, n_2^j, n_0)$$

How to describe these states?

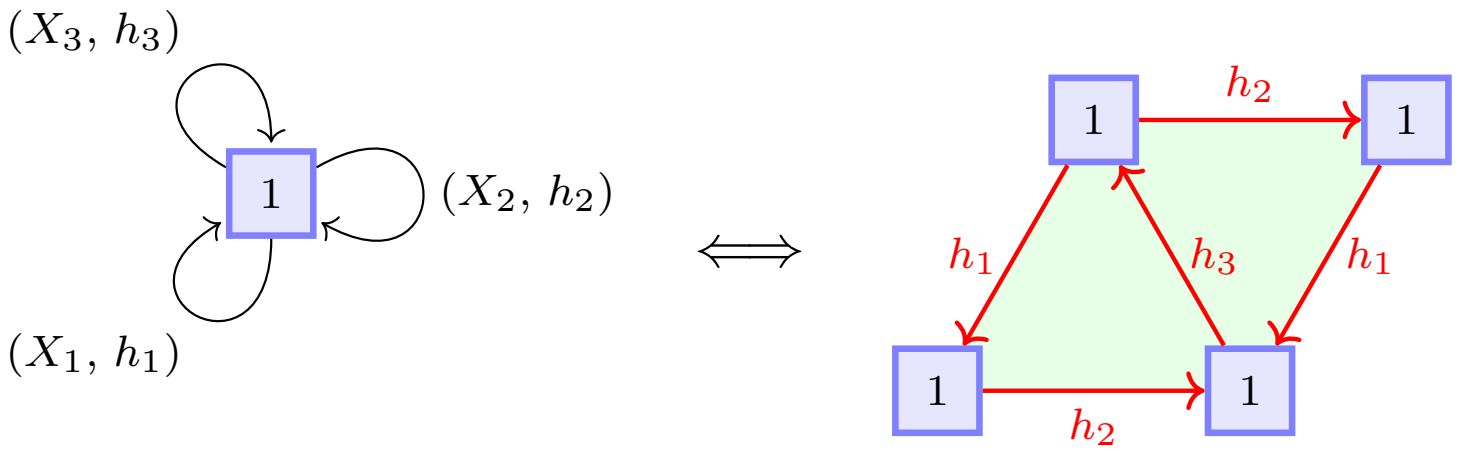
Low energy effective theory on D-brane bound state

Reduce to 1D: $\mathcal{N} = 4$ quiver quantum mechanics (Q, W)

- Quiver $Q = (Q_0, Q_1)$, with $\quad Q_0 = \{a\}$ (gauge group)
  $\quad Q_1 = \{I\}$ (bi-fundamentals)
- superpotential W
- $(Q, W) \iff$ periodic quiver $\tilde{Q} = (Q_0, Q_1, Q_2)$

$$W = \sum_{F \in Q_2} \pm \mathrm{Tr}\left(\prod_{e \in F} \Phi_e\right)$$

$\mathbb{C}^3$:



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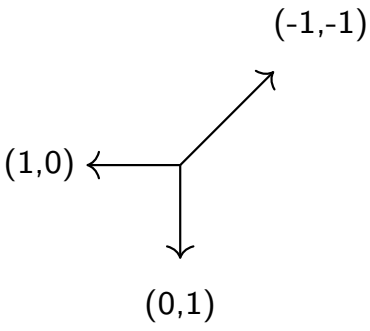
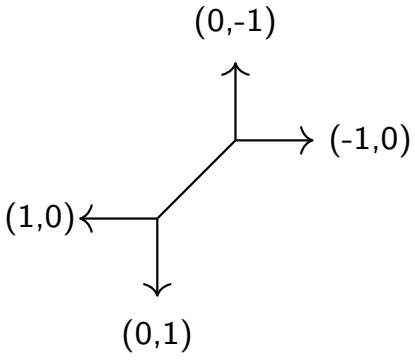
Brane tiling

Γ_{toric} for toric Calabi-Yau X_3 $\iff$ (Q, W) = periodic quiver

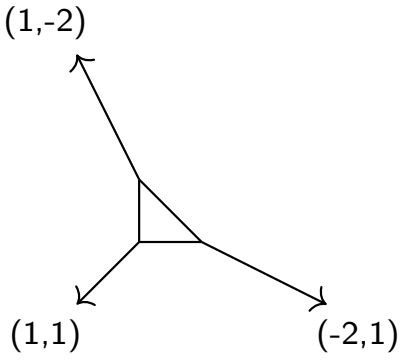
Hanany et al



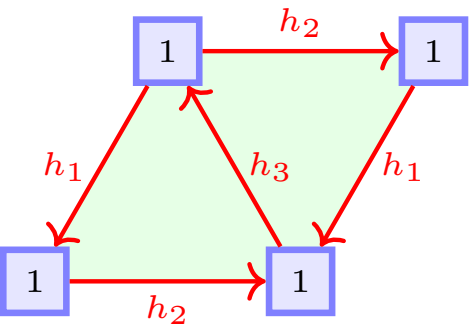
$\mathbb{C}^3$

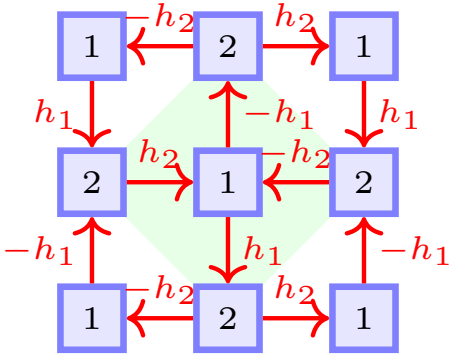


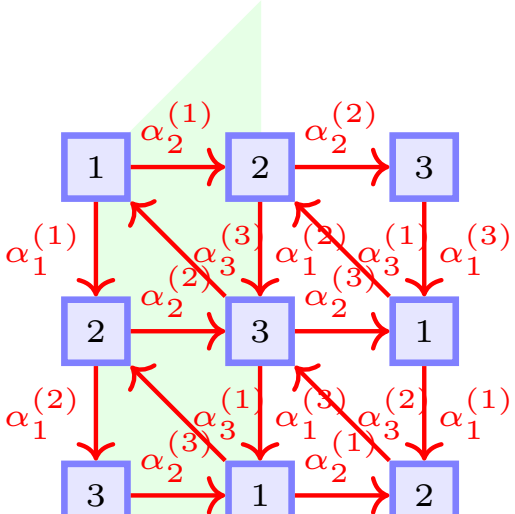
conifold



$\mathbb{C}^3/\mathbb{Z}_3$ ...







From quiver quantum mechanics to BPS crystal

1. set of $\mathbb{T}^2$-invariant states of quiver quantum mechanics (Q, W):

$$\text{path algebra of } Q \,/\, \text{ideal from F-term constraint}$$

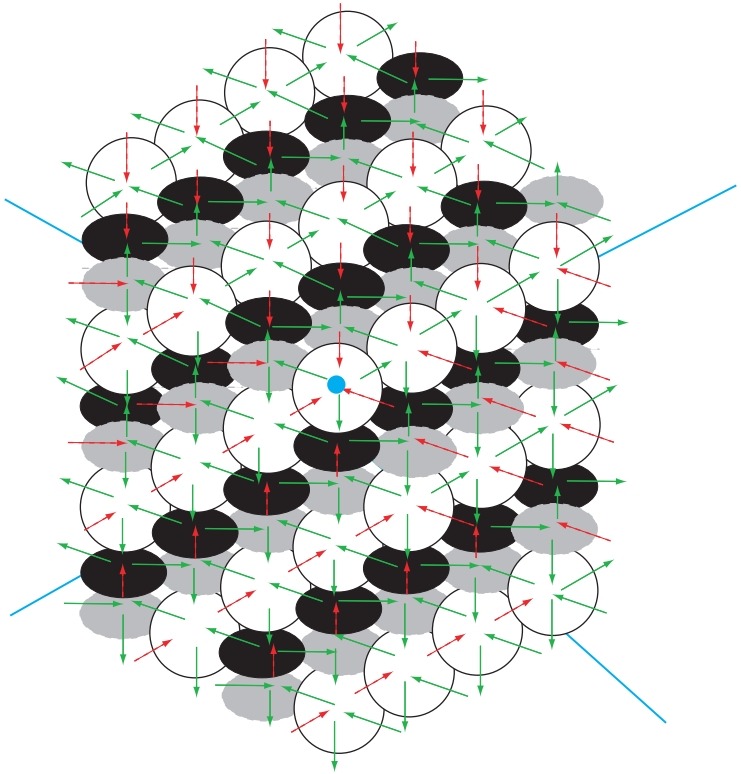
2. Can be described given 3D colored crystals

3. 1-to-1 correspondence

$$\text{a crystal config.} \quad \Longleftrightarrow \quad \text{a D-brane bound state w/}$$
$$\#(\text{D6},\ \text{D4},\ \text{D2},\ \text{D0}) = \ (1, n_4^i, n_2^j, n_0)$$

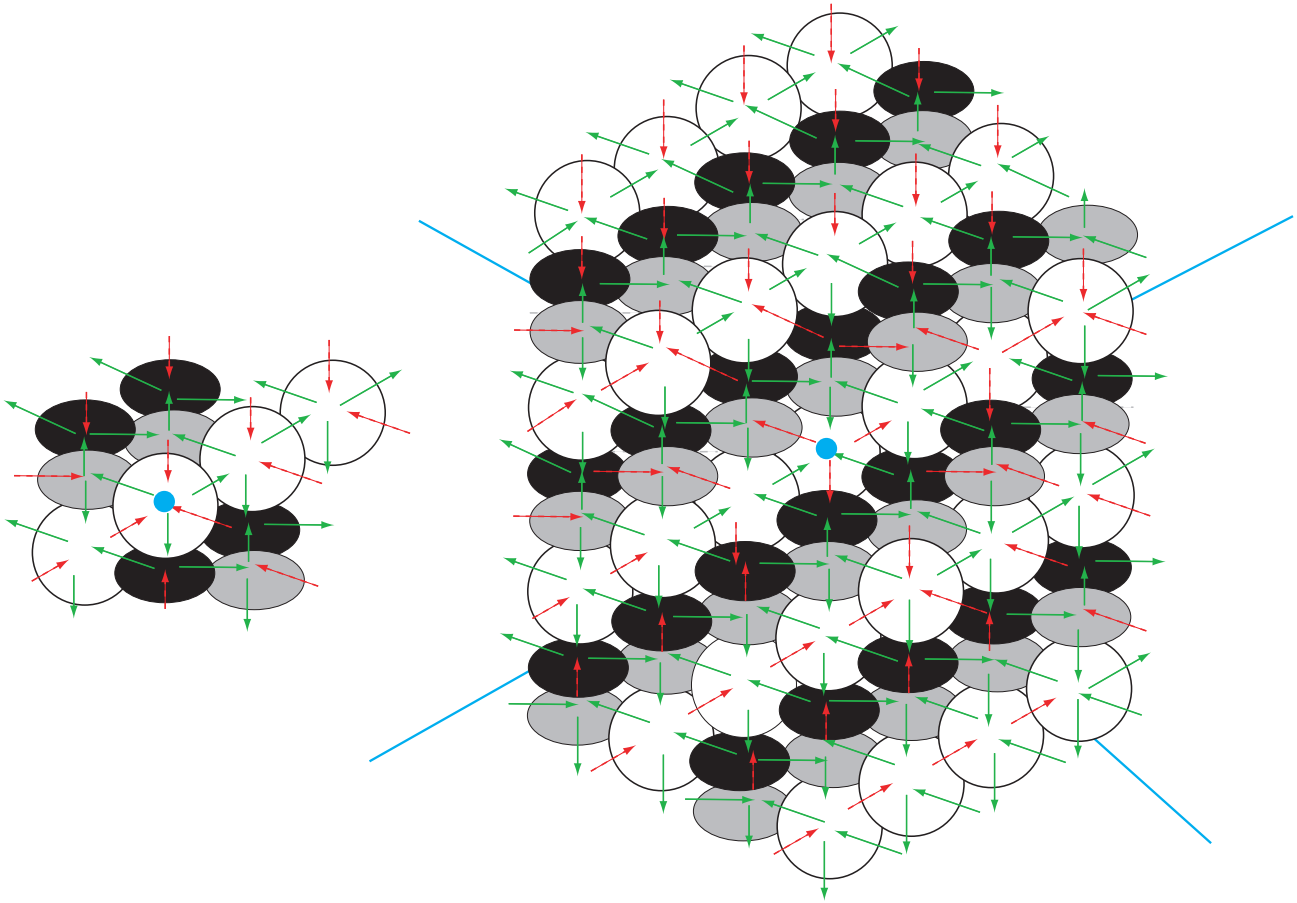
BPS crystal

BPS crystal for the Suspended Pinched Point singularity



three colors: labeled by $q_{1,2,3}$, each atom carry certain D2-D0 charge.

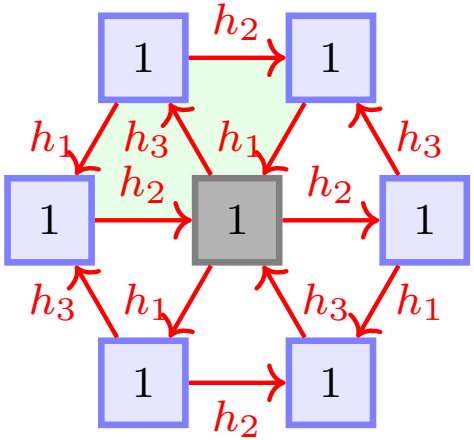
Molten crystal

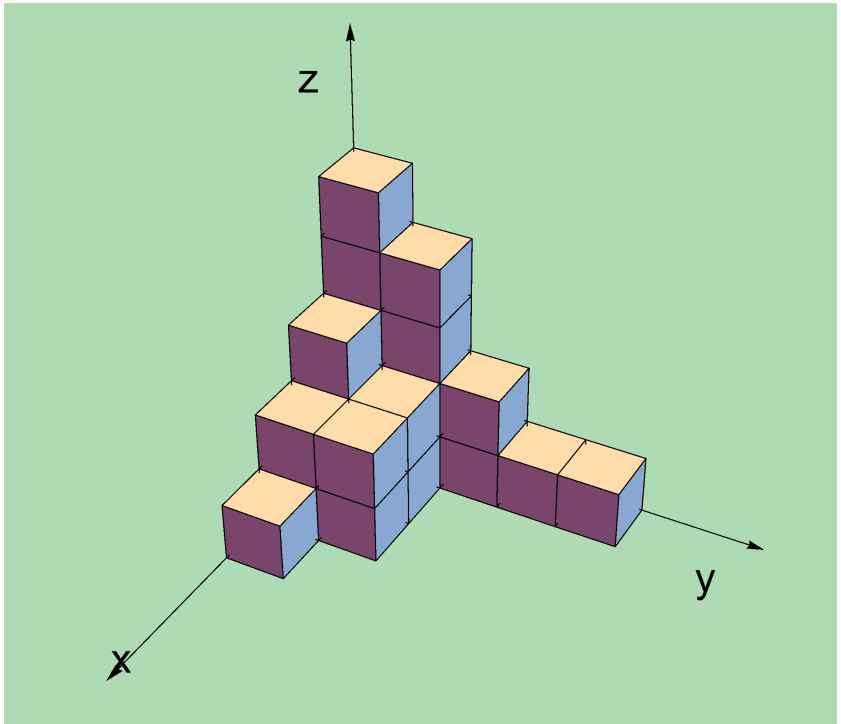


Melted piece: contributes a term $q_1^4 q_2^3 q_3^4$ to the BPS partition function.

From periodic quiver to BPS crystal

- choose origin $\mathfrak{o}$
- atom $\leftrightarrow$ path from $\mathfrak{o}$
- Rule: if $\boxed{\text{a}} \notin \mathrm{K}$, then $I \cdot \boxed{\text{a}} \notin \mathrm{K}$
- equivalence of paths
- for $p_{\mathfrak{o},\mathfrak{a}} \prod_F (\omega_F)^{n_F}$, depth$=\sum_F n_F$





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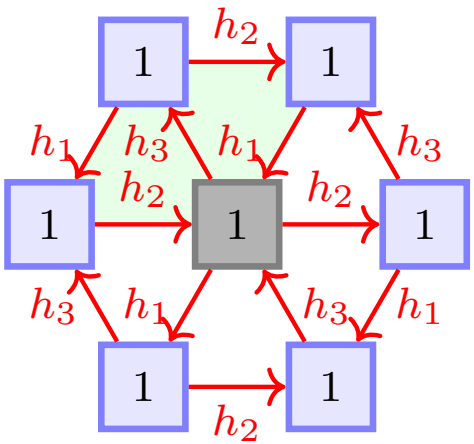
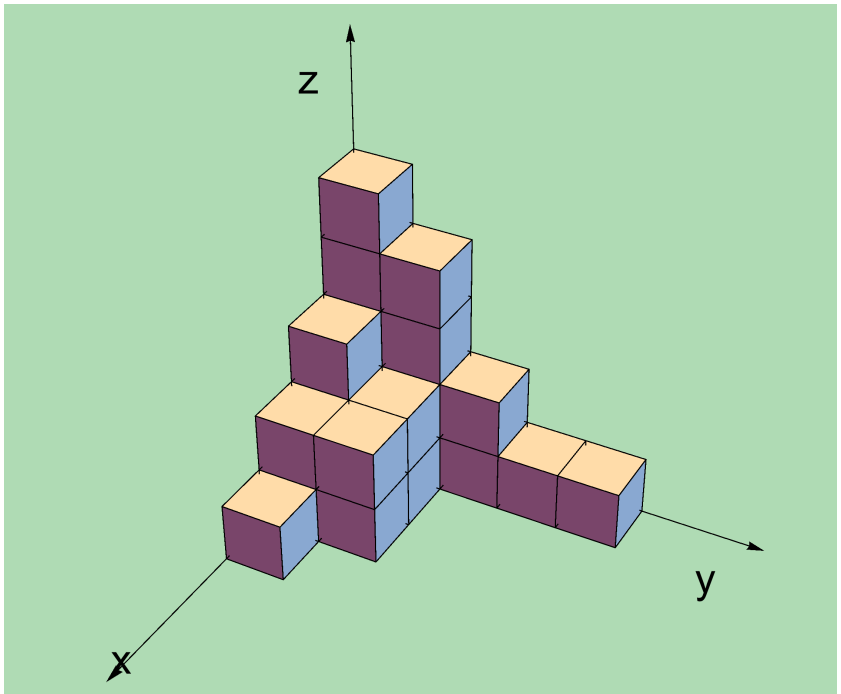
- coordinate of atom
  $h(\boxed{\text{a}}) = \sum_{I \in \mathrm{path}[\mathfrak{o} \to \boxed{\text{a}}]} h_I$
- Loop constraint (global symmetry)
  $\sum_{I \in L} h_I = 0$
- Vertex constraints (gauge symmetry)
  $\sum_{I \in a} \mathrm{sign}_a(I)\, h_I = 0$





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3. $\mathbb{C}^3$
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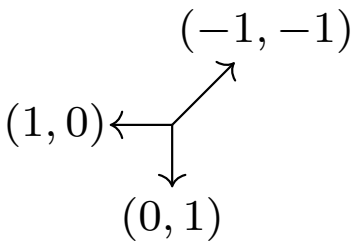
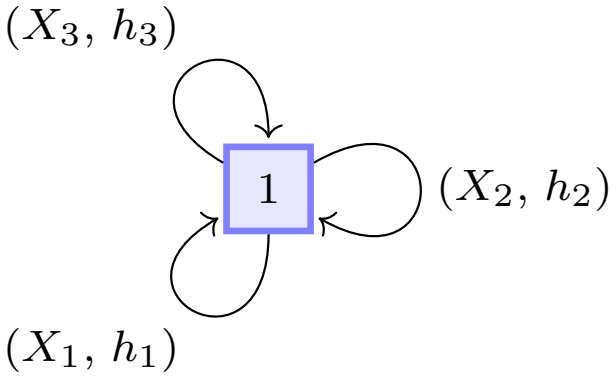
$\mathbb{C}^3$

- From toric data to quiver data



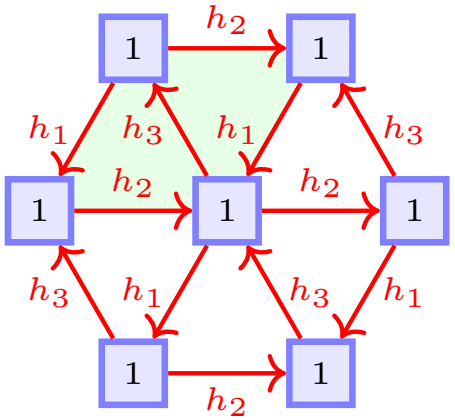
quiver and superpotential



$$W = \mathrm{Tr}[-X_1\, X_2\, X_3 + X_1\, X_3\, X_2]$$

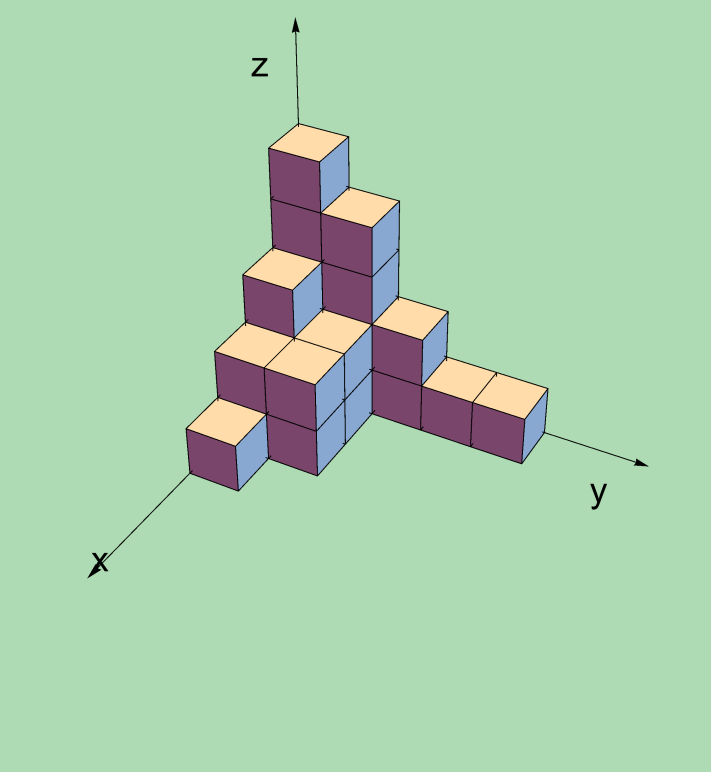
$\Longleftrightarrow$

periodic quiver



- loop constraint = vertex constraint: $h_1 + h_2 + h_3 = 0$

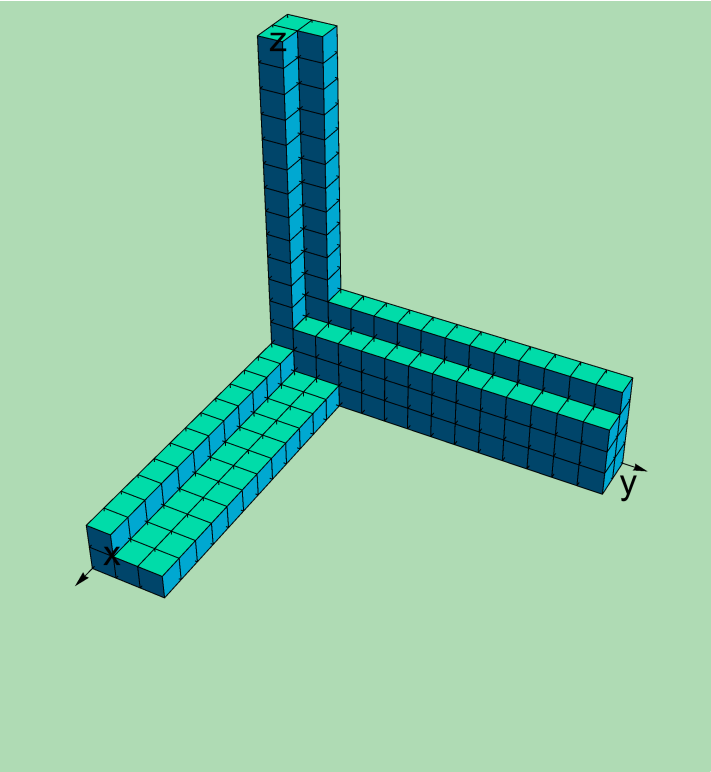
BPS crystal for $\mathbb{C}^3$: plane partitions



$$\sum_{n=0} M(n) q^n = \prod_{k=1}^{\infty} \frac{1}{(1-q^k)^k} = 1 + q + 3\,q^2 + 6\,q^3 + 13\,q^4 + 24\,q^5 + 48\,q^6 + \cdots$$

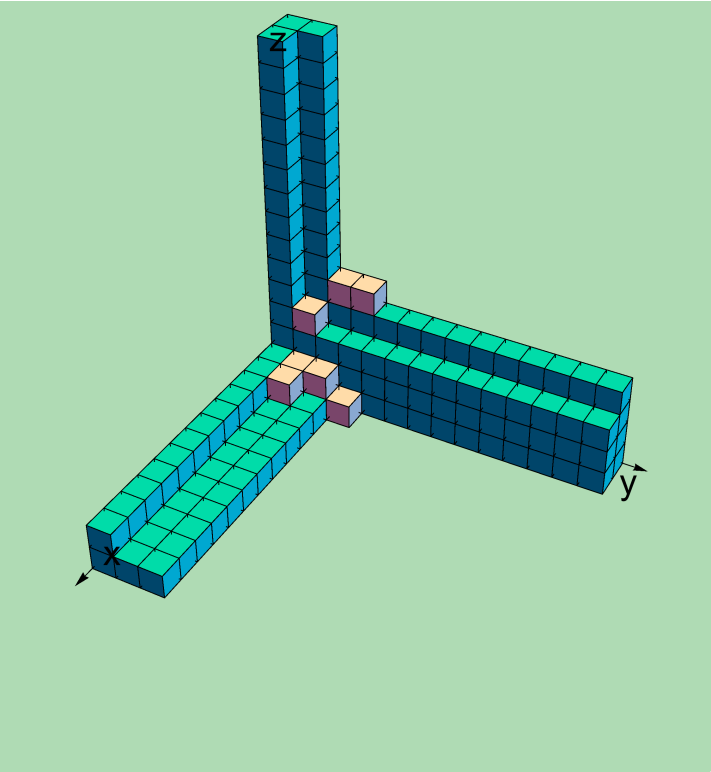
Plane partition with non-trivial asymptotics

Ground state of $(\Lambda_x, \Lambda_y, \Lambda_z)$



Plane partition with non-trivial asymptotics

a level-7 excited states of $(\Lambda_x, \Lambda_y, \Lambda_z)$



Affine Yangian of $\mathfrak{gl}_1$

Associative algebra with generators e_j, f_j and ψ_j with $j \in \mathbb{Z}_0$

- Defining relations

$$
\begin{aligned}
0 =& [\psi_j, \psi_k] \\
\psi_{j+k} =& [e_j, f_k] \\
\sigma_3\{\psi_j, e_k\} =& ([\psi_{j+3}, e_k] - 3[\psi_{j+2}, e_{k+1}] + 3[\psi_{j+1}, e_{k+2}] - [\psi_j, e_{k+3}]) \\
&+ \sigma_2 ([\psi_{j+1}, e_k] - [\psi_j, e_{k+1}]) \\
-\sigma_3\{\psi_j, f_k\} =& ([\psi_{j+3}, f_k] - 3[\psi_{j+2}, f_{k+1}] + 3[\psi_{j+1}, f_{k+2}] - [\psi_j, f_{k+3}]) \\
&+ \sigma_2 ([\psi_{j+1}, f_k] - [\psi_j, f_{k+1}]) \\
\sigma_3\{e_j, e_k\} =& ([e_{j+3}, e_k] - 3[e_{j+2}, e_{k+1}] + 3[e_{j+1}, e_{k+2}] - [e_j, e_{k+3}]) \\
&+ \sigma_2 ([e_{j+1}, e_k] - [e_j, e_{k+1}]) \\
-\sigma_3\{f_j, f_k\} =& ([f_{j+3}, f_k] - 3[f_{j+2}, f_{k+1}] + 3[f_{j+1}, f_{k+2}] - [f_j, f_{k+3}]) \\
&+ \sigma_2 ([f_{j+1}, f_k] - [f_j, f_{k+1}])
\end{aligned}
$$

- Initial conditions

$$
\begin{aligned}
&[\psi_0, e_j] = 0, \quad [\psi_1, e_j] = 0, \quad [\psi_2, e_j] = 2e_j \\
&[\psi_0, f_j] = 0, \quad [\psi_1, f_j] = 0, \quad [\psi_2, f_j] = -2f_j
\end{aligned}
$$

- Serre relation

$$
\mathrm{Sym}_{(j_1,j_2,j_3)} [e_{j_1}, [e_{j_2}, e_{j_3+1}]] = \mathrm{Sym}_{(j_1,j_2,j_3)} [f_{j_1}, [f_{j_2}, f_{j_3+1}]] = 0
$$

$\mathcal{Y}[\widehat{\mathfrak{gl}_1}] \cong \mathrm{UEA}[\mathcal{W}_{1+\infty}[\lambda]]$

Procházka '15; Gaberdiel Gopakumar WL Peng '17

Virasoro algebra $\quad T(z)T(w) = \frac{c/2}{(z-w)^4} + [\frac{2}{(z-w)^2} + \frac{\partial}{(z-w)}]T(w)$

conserved current: stress energy tensor T $(h = 2 = s)$

$\mathcal{W}$ algebra $\quad$ conserved current: $\quad T,\ W^{(3)},\ \ldots$

$$T(z)W^{(s)}(w) = [\frac{s}{(z-w)^2} + \frac{\partial}{(z-w)}]W^{(s)}(w)$$

$\mathcal{W}_{1+\infty}$ algebra: one field per spin $s = 1, 2, 3, \cdots, \infty$, only two parameters

central charge $\quad c$

OPE coefficient $\quad W^{(3)}(z)W^{(3)}(w) \sim c_{33}^4 \frac{W^{(4)}(w)}{(z-w)^2} + \ldots$

Modes of $\mathcal{W}_{1+\infty}$

$$W^{(s)}(z) = \sum_{n\in\mathbb{Z}} \frac{W_n^{(s)}}{z^{n+s}} \qquad s = 1, 2, 3, \ldots$$

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
spin-5	$\ldots$	X_{-4}	X_{-3}	X_{-2}	X_{-1}	X_0	X_1	X_2	X_3	X_4	$\ldots$
spin-4	$\ldots$	U_{-4}	U_{-3}	U_{-2}	U_{-1}	U_0	U_1	U_2	U_3	U_4	$\ldots$
spin-3	$\ldots$	W_{-4}	W_{-3}	W_{-2}	W_{-1}	W_0	W_1	W_2	W_3	W_4	$\ldots$
spin-2	$\ldots$	L_{-4}	L_{-3}	L_{-2}	L_{-1}	L_0	L_1	L_2	L_3	L_4	$\ldots$
spin-1	$\ldots$	J_{-4}	J_{-3}	J_{-2}	J_{-1}	J_0	J_1	J_2	J_3	J_4	$\ldots$

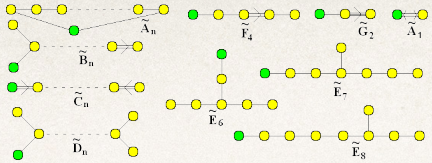
Regrouping the modes

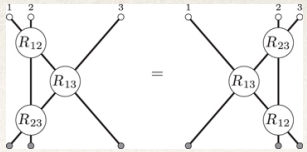
$$W^{(s)}(z) = \sum_{n\in\mathbb{Z}} \frac{W_n^{(s)}}{z^{n+s}} \qquad s = 1, 2, 3, \ldots$$

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
spin-5	$\ldots$	X_{-3}	X_{-2}	$X_{-1} \sim e_4$	$X_0 \sim \psi_5$	$X_1 \sim f_4$	X_2	X_3	X_4
spin-4	$\ldots$	U_{-3}	U_{-2}	$U_{-1} \sim e_3$	$U_0 \sim \psi_4$	$U_1 \sim f_3$	U_2	U_3	U_4
spin-3	$\ldots$	W_{-3}	W_{-2}	$W_{-1} \sim e_2$	$W_0 \sim \psi_3$	$W_1 \sim f_2$	W_2	W_3	W_4
spin-2	$\ldots$	L_{-3}	L_{-2}	$L_{-1} \sim e_1$	$L_0 \sim \psi_2$	$L_1 \sim f_1$	L_2	L_3	L_4
spin-1	$\ldots$	J_{-3}	J_{-2}	$J_{-1} \sim e_0$	$J_0 \sim \psi_1$	$J_1 \sim f_0$	J_2	J_3	J_4

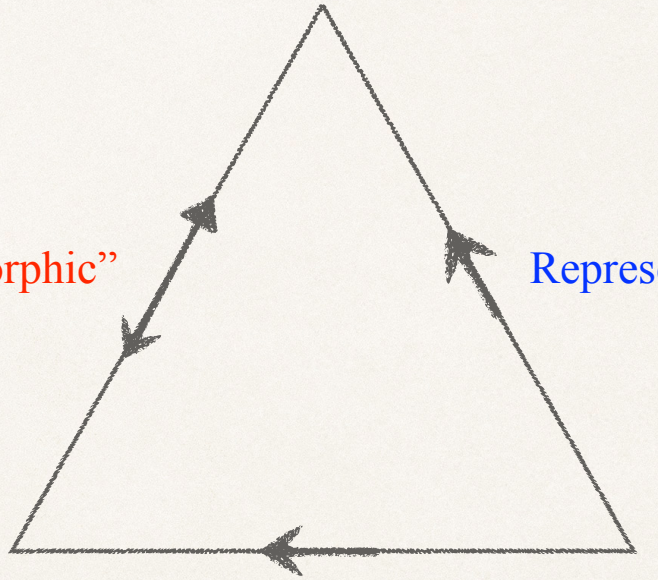
map between Yangian parameter (σ_2, σ_3)

and $\mathcal{W}_{1+\infty}$ parameter (c, λ)



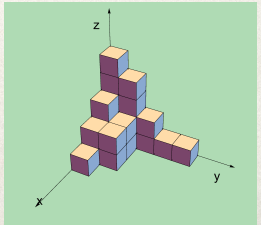


Affine Yangian of gl(1)



W symmetry

Plane partitions



Affine Yangian of $\mathfrak{gl}_1$

Associative algebra with generators e_j, f_j and ψ_j with $j \in \mathbb{Z}_0$

- Defining relations

$$
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& + \sigma_2 ([\psi_{j+1}, e_k] - [\psi_j, e_{k+1}]) \\
-\sigma_3\{\psi_j, f_k\} =& ([\psi_{j+3}, f_k] - 3[\psi_{j+2}, f_{k+1}] + 3[\psi_{j+1}, f_{k+2}] - [\psi_j, f_{k+3}]) \\
& + \sigma_2 ([\psi_{j+1}, f_k] - [\psi_j, f_{k+1}]) \\
\sigma_3\{e_j, e_k\} =& ([e_{j+3}, e_k] - 3[e_{j+2}, e_{k+1}] + 3[e_{j+1}, e_{k+2}] - [e_j, e_{k+3}]) \\
& + \sigma_2 ([e_{j+1}, e_k] - [e_j, e_{k+1}]) \\
-\sigma_3\{f_j, f_k\} =& ([f_{j+3}, f_k] - 3[f_{j+2}, f_{k+1}] + 3[f_{j+1}, f_{k+2}] - [f_j, f_{k+3}]) \\
& + \sigma_2 ([f_{j+1}, f_k] - [f_j, f_{k+1}])
\end{aligned}
$$

- Initial conditions

$$
\begin{aligned}
&[\psi_0, e_j] = 0, \quad [\psi_1, e_j] = 0, \quad [\psi_2, e_j] = 2e_j \\
&[\psi_0, f_j] = 0, \quad [\psi_1, f_j] = 0, \quad [\psi_2, f_j] = -2f_j
\end{aligned}
$$

- Serre relation

$$
\mathrm{Sym}_{(j_1,j_2,j_3)} [e_{j_1}, [e_{j_2}, e_{j_3+1}]] = \mathrm{Sym}_{(j_1,j_2,j_3)} [f_{j_1}, [f_{j_2}, f_{j_3+1}]] = 0
$$

Affine Yangian of $\mathfrak{gl}_1$

Associative algebra with generators e_j, f_j and ψ_j with $j \in \mathbb{Z}_0$

- Defining relations

$$\begin{aligned} 0 =& [\psi_j\,,\psi_k] \\ \psi_{j+k} =& [e_j\,,f_k] \\ \sigma_3\{\psi_j\,,e_k\} =& \left([\psi_{j+3}\,,e_k] - 3[\psi_{j+2}\,,e_{k+1}] + 3[\psi_{j+1}\,,e_{k+2}] - [\psi_j\,,e_{k+3}]\right) \\ &+ \sigma_2\left([\psi_{j+1}\,,e_k] - [\psi_j\,,e_{k+1}]\right) \end{aligned}$$

1. Group the generators into three families

$$\psi(z) \equiv 1 + \sigma_3 \sum_{j=0}^{\infty} \frac{\psi_j}{z^{j+1}} \qquad e(z) \equiv \sum_{j=0}^{\infty} \frac{e_j}{z^{j+1}} \qquad f(z) \equiv \sum_{j=0}^{\infty} \frac{f_j}{z^{j+1}}$$

2. Reparametrize $(\sigma_2, \sigma_3) \longrightarrow (h_1, h_2, h_3)$ with

$$h_1 + h_2 + h_3 = 0 \qquad \sigma_2 \equiv h_1h_2 + h_2h_3 + h_1h_3 \qquad \sigma_3 \equiv h_1h_2h_3$$

3. Translate algebra relations in terms of $(\psi(z), e(z), f(z))$

$$[\psi(z)\,,\psi(w)] \sim 0 \qquad [e(z)\,,f(w)] \sim -\frac{1}{\sigma_3}\,\frac{\psi(z)-\psi(w)}{z-w}$$

$$\psi(z)\,e(w) \sim \varphi_3(z-w)\,e(w)\,\psi(z) \qquad \varphi_3(z) \equiv \frac{(z+h_1)(z+h_2)(z+h_3)}{(z-h_1)(z-h_2)(z-h_3)}$$

Affine Yangian of $\mathfrak{gl}_1$

Associative algebra with generators e_j, f_j and ψ_j with $j \in \mathbb{Z}_0$

- Generators

$$\psi(z) = 1 + \sigma_3 \sum_{j=0}^{\infty} \frac{\psi_j}{z^{j+1}} \qquad e(z) = \sum_{j=0}^{\infty} \frac{e_j}{z^{j+1}} \qquad f(z) = \sum_{j=0}^{\infty} \frac{f_j}{z^{j+1}}$$

- Parameters (h_1, h_2, h_3) with

$$h_1 + h_2 + h_3 = 0 \qquad \sigma_2 \equiv h_1 h_2 + h_2 h_3 + h_1 h_3 \qquad \sigma_3 \equiv h_1 h_2 h_3$$

- One $\mathcal{S}_3$ invariant function

$$\varphi_3(z) \equiv \frac{(z+h_1)(z+h_2)(z+h_3)}{(z-h_1)(z-h_2)(z-h_3)}$$

- Defining relations

$$[\psi(z)\,,\psi(w)] \sim 0 \qquad [e(z)\,,f(w)] \sim -\frac{1}{\sigma_3}\,\frac{\psi(z)-\psi(w)}{z-w}$$

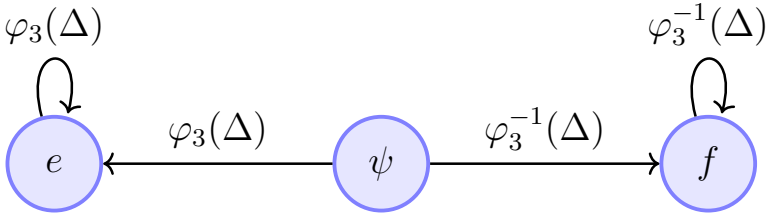
$$\psi(z)\,e(w) \sim \varphi_3(z-w)\,e(w)\,\psi(z) \qquad \psi(z)\,f(w) \sim \varphi_3^{-1}(z-w)\,f(w)\,\psi(z)$$

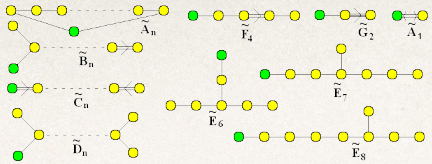
$$e(z)\,e(w) \sim \varphi_3(z-w)\,e(w)\,e(z) \qquad f(z)\,f(w) \sim \varphi_3^{-1}(z-w)\,f(w)\,f(z)$$

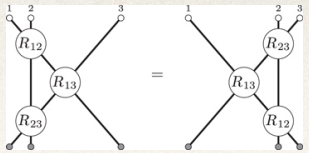
- Serre relations

$$\mathrm{Sym}_{(z_1,z_2,z_3)}(z_2-z_3)\,e(z_1)\,e(z_2)\,e(z_3) \sim \mathrm{Sym}_{(z_1,z_2,z_3)}(z_2-z_3)\,f(z_1)\,f(z_2)\,f(z_3) \sim 0$$

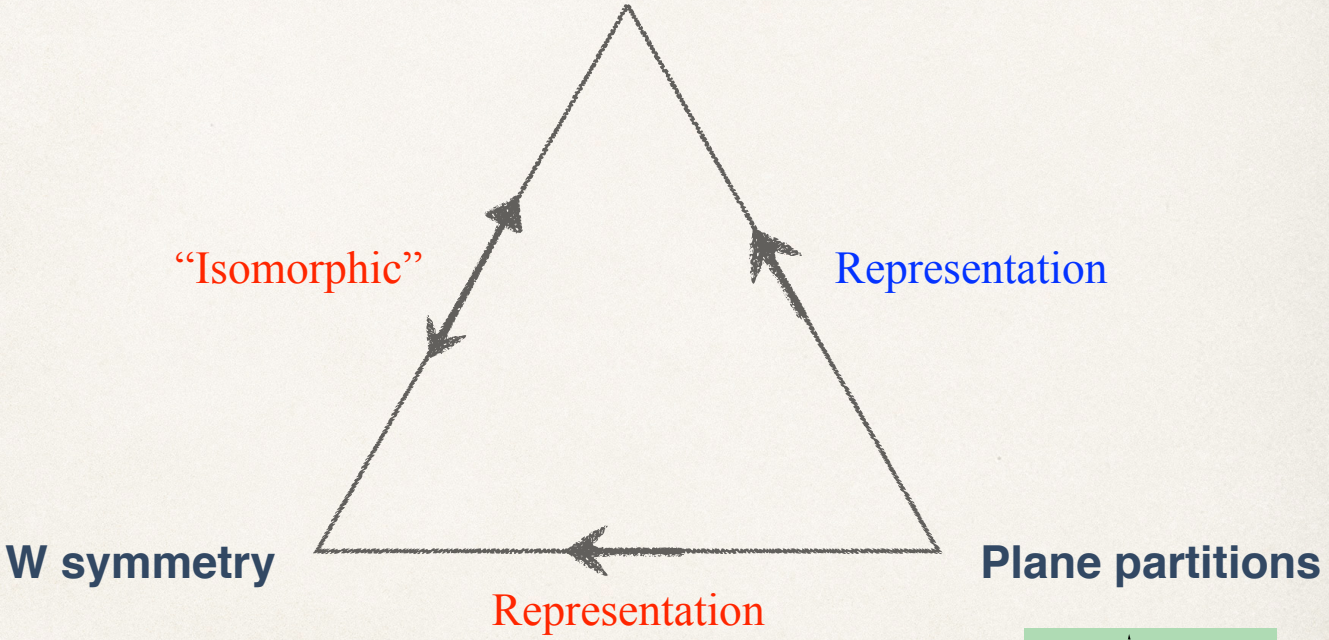
Affine Yangian of $\mathfrak{gl}_1$

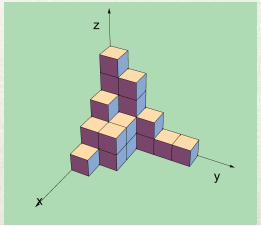






Affine Yangian of gl(1)





Action of $\hat{\mathcal{Y}}(\mathfrak{gl}_1)$ on a plane partition

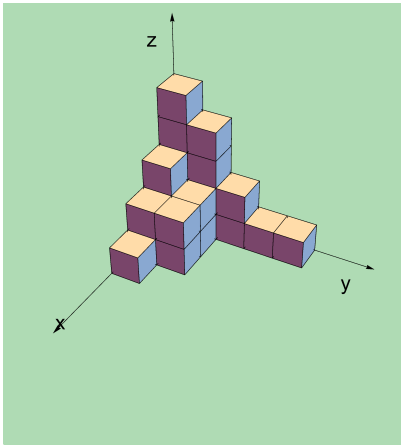
Tsymbaliuk '14, Prochazka '15

$$\begin{cases} \psi(z)|\Lambda\rangle = \Psi_\Lambda(z)|\Lambda\rangle\,, \qquad \psi_\Lambda(z) \equiv \left(1+\dfrac{\psi_0\sigma_3}{z}\right)\prod\limits_{\square\in(\Lambda)}\varphi_3(z-h(\square)) \\ \qquad\qquad\qquad\qquad h(\square) \equiv h_1\,x_1(\square)+h_2\,x_2(\square)+h_3\,x_3(\square) \\ e(z)|\Lambda\rangle = \displaystyle\sum_{\square\in\mathrm{Add}(\Lambda)} \frac{\left[-\frac{1}{\sigma_3}\mathrm{Res}_{w=h(\square)}\Psi_\Lambda(w)\right]^{\frac{1}{2}}}{z-h(\square)}|\Lambda+\square\rangle \\ f(z)|\Lambda\rangle = \displaystyle\sum_{\square\in\mathrm{Rem}(\Lambda)} \frac{\left[+\frac{1}{\sigma_3}\mathrm{Res}_{w=h(\square)}\Psi_\Lambda(w)\right]^{\frac{1}{2}}}{z-h(\square)}|\Lambda-\square\rangle \end{cases}$$

1. Every Λ is eigenstate of $\psi(z)$
   - Each $\square$ in Λ contributes a factor of $\varphi_3(z-h(\square))$
   - Position of $\square$ is encoded by $h(\square)$
2. Actions of $e(z)$ and $f(z)$ are controlled by the eigenvalue $\Psi_\Lambda(z)$
   - Poles of $\Psi_\Lambda(z)$ encode the positions of $\square\in\mathrm{Add}(\Lambda)$ and $\square\in\mathrm{Rem}(\Lambda)$
   - Need loop constraint $h_1+h_2+h_3=0$

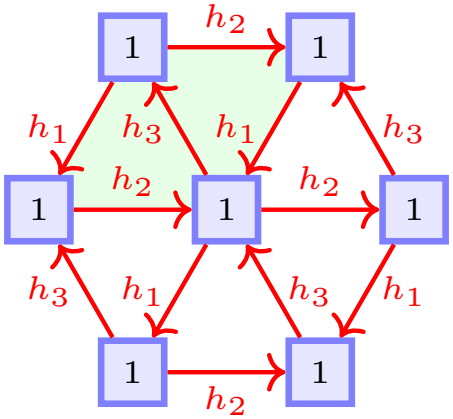
Poles of $\Psi_\Lambda(z)$ encode the positions of $\square \in \mathrm{Add}(\Lambda)$ and $\square \in \mathrm{Rem}(\Lambda)$

For $e(z)$ to create the entire crystal automatically
$f(z)$ kill



- Any pole of $\Psi_\Lambda(z)$ must correspond to a new atom that can be added or an atom that can be removed
- $\Longrightarrow$ $\Psi_\Lambda(z)$ can only have poles near the surface of the crystal

$\mathbb{C}^3$ crystal: vacuum $\Longrightarrow$ level-1

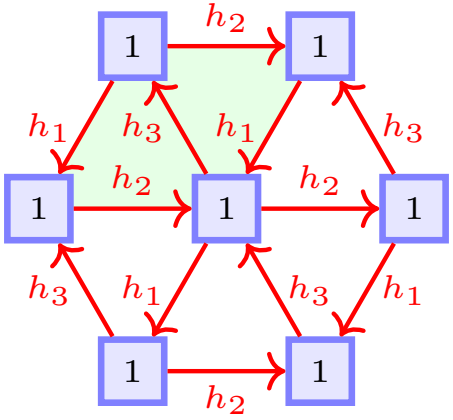


- vacuum $|\emptyset\rangle$:

$$\psi(z)|\emptyset\rangle = \psi_0(z)|\emptyset\rangle = \frac{z+C}{z}|\emptyset\rangle$$

- Pole: $z = 0$

$\mathbb{C}^3$ crystal: vacuum $\Longrightarrow$ level-1

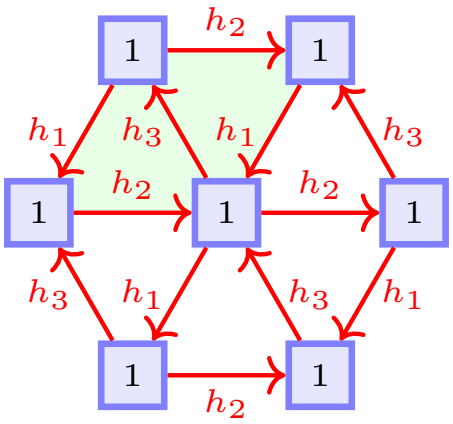


- vacuum $|\emptyset\rangle$:

$$\psi(z)|\emptyset\rangle = \psi_0(z)|\emptyset\rangle = \frac{z+C}{z}|\emptyset\rangle$$

- Pole: $z = 0 \Longrightarrow h(\square) = 0$
- $e(z)|\emptyset\rangle = \frac{\#}{z}|\square\rangle$

$\mathbb{C}^3$ crystal: level-1 $\Longrightarrow$ level-2

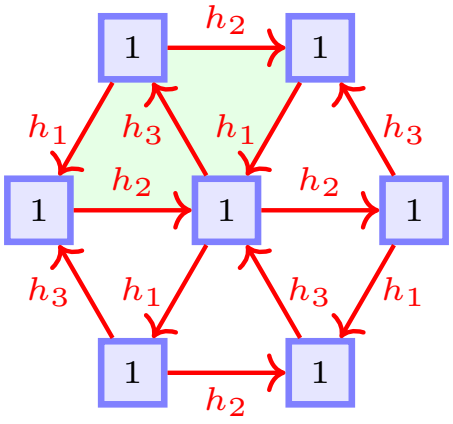


- $h(\square) = 0$:

$$\Psi_\Lambda(z) = \psi_0(z) \cdot \varphi_3(z - h(\square))$$
$$= \frac{(z+\psi_0)}{z} \cdot \frac{(z+h_1)(z+h_2)(z+h_3)}{(z-h_1)(z-h_2)(z-h_3)}$$

- Poles: $z = 0$ and $z = h_{1,2,3}$

$\mathbb{C}^3$ crystal: level-1 $\Longrightarrow$ level-2



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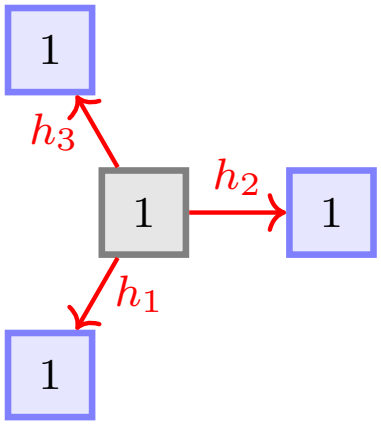
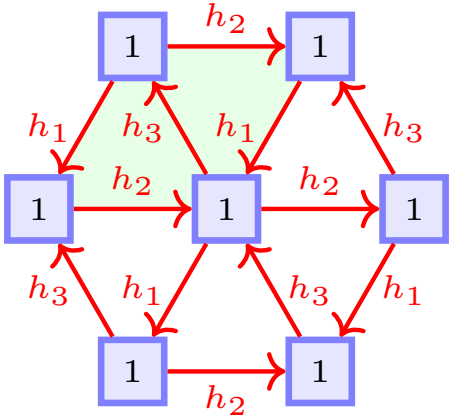
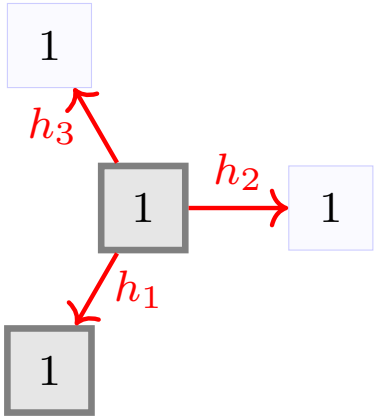
$$= \frac{(z+\psi_0)}{z} \cdot \frac{(z+h_1)(z+h_2)(z+h_3)}{(z-h_1)(z-h_2)(z-h_3)}$$



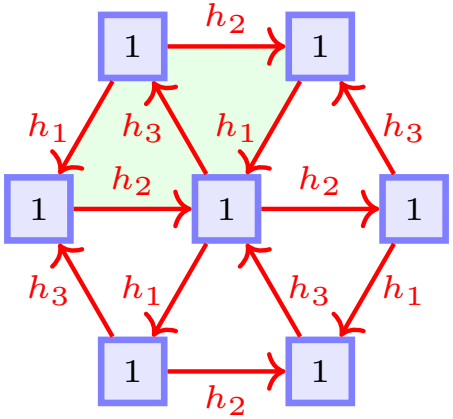
- Poles: $z = 0$ and $z = h_{1,2,3}$
- $e(z)|\square\rangle = \sum_{i=1,2,3} \frac{\#}{z-h_i} |\square\square_i\rangle$
- $f(z)|\square\rangle = \frac{\#}{z}|\emptyset\rangle$

$\mathbb{C}^3$ crystal: level-2 $\Longrightarrow$ level-3





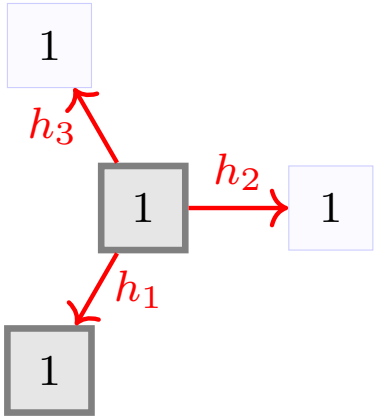
$\mathbb{C}^3$ crystal: level-2 $\Longrightarrow$ level-3



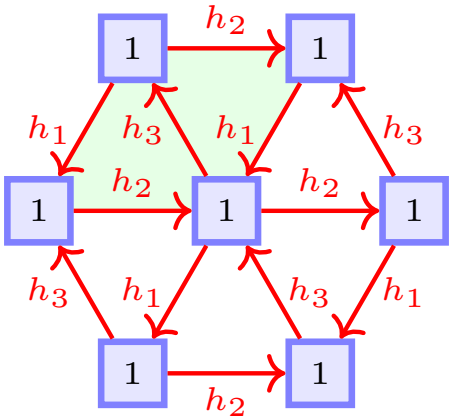
- $h(\square_0) = 0$ and $h(\square_0) = h_1$

$$\Psi_\Lambda(z) = \psi_0(z) \cdot \varphi_3(z - h(\square_0)) \cdot \varphi_3(z - h(\square_1))$$
$$= \frac{(z+\psi_0)}{\cancel{z}} \cdot \frac{(z+h_1)\cancel{(z+h_2)}\cancel{(z+h_3)}}{(z-h_1)(z-h_2)(z-h_3)} \cdot \frac{\cancel{(z)}(z+h_2-h_1)(z+h_3-h_1)}{(z-2h_1)\cancel{(z+h_3)}\cancel{(z+h_2)}}$$

- Poles: $z = h_1$ and $z = h_2, h_3, 2h_1$



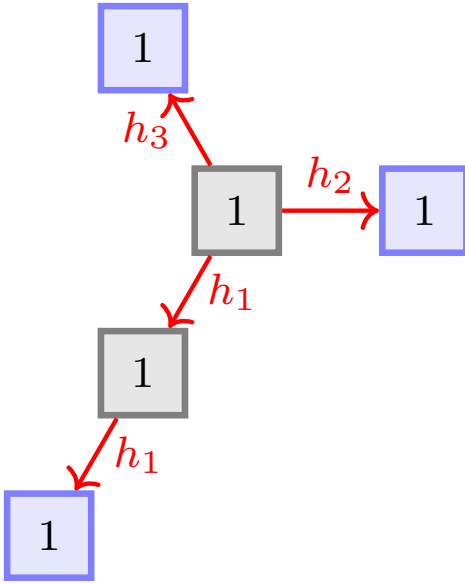
$\mathbb{C}^3$ crystal: level-2 $\Longrightarrow$ level-3



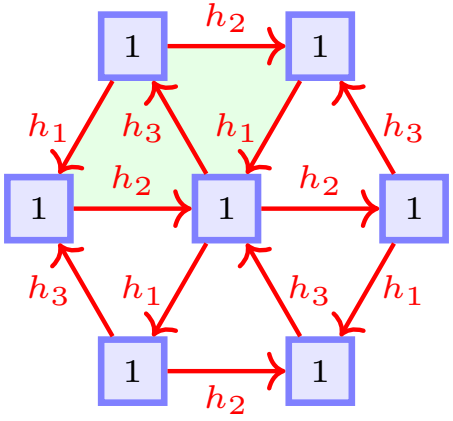
- $h(\square_0) = 0$ and $h(\square_0) = h_1$

$$\Psi_\Lambda(z) = \psi_0(z) \cdot \varphi_3(z - h(\square_0)) \cdot \varphi_3(z - h(\square_1))$$
$$= \frac{(z+\psi_0)}{\cancel{z}} \cdot \frac{(z+h_1)\cancel{(z+h_2)}\cancel{(z+h_3)}}{(z-h_1)(z-h_2)(z-h_3)} \cdot \frac{\cancel{(z)}(z+h_2-h_1)(z+h_3-h_1)}{(z-2h_1)\cancel{(z+h_3)}\cancel{(z+h_2)}}$$

- Poles: $z = h_1$ and $z = h_2, h_3, 2h_1$



Poles of $\Psi_\Lambda(z)$ encode the positions of $\square \in \mathrm{Add}(\Lambda)$ and $\square \in \mathrm{Rem}(\Lambda)$



1. Each $\square$ in Λ contributes a factor of $\varphi_3(z - h(\square))$
2. Position of $\square$ is encoded by $h(\square) \equiv h_1\, x_1(\square) + h_2\, x_2(\square) + h_3\, x_3(\square)$
3. $\varphi_3(z) \equiv \frac{(z+h_1)(z+h_2)(z+h_3)}{(z-h_1)(z-h_2)(z-h_3)}$
4. Need loop constraint $h_1 + h_2 + h_3 = 0$

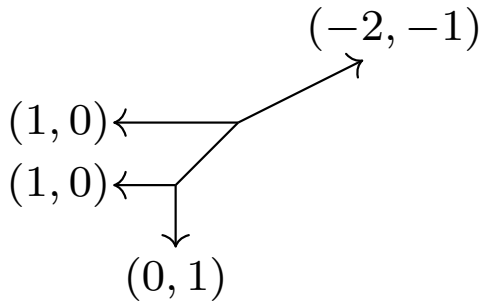
Poles are always pushed to the surface of plane partition

Outline

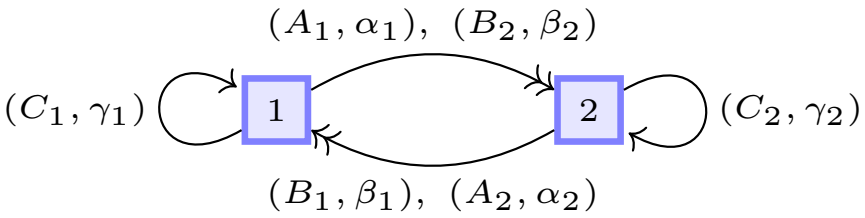
1. Introduction
2. BPS crystal
3. $\mathbb{C}^3$
4. General toric CY3
5. Summary

$(\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{C}$

- From toric data to quiver data



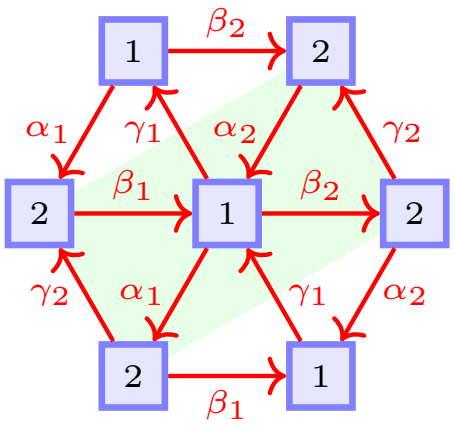
quiver and superpotential



$$W = \mathrm{Tr}[-C_1\,A_1\,B_1 + C_1\,B_2\,A_2 - C_2\,A_2\,B_2 + C_2\,B_1\,A_1]$$

$\Longleftrightarrow$

periodic quiver

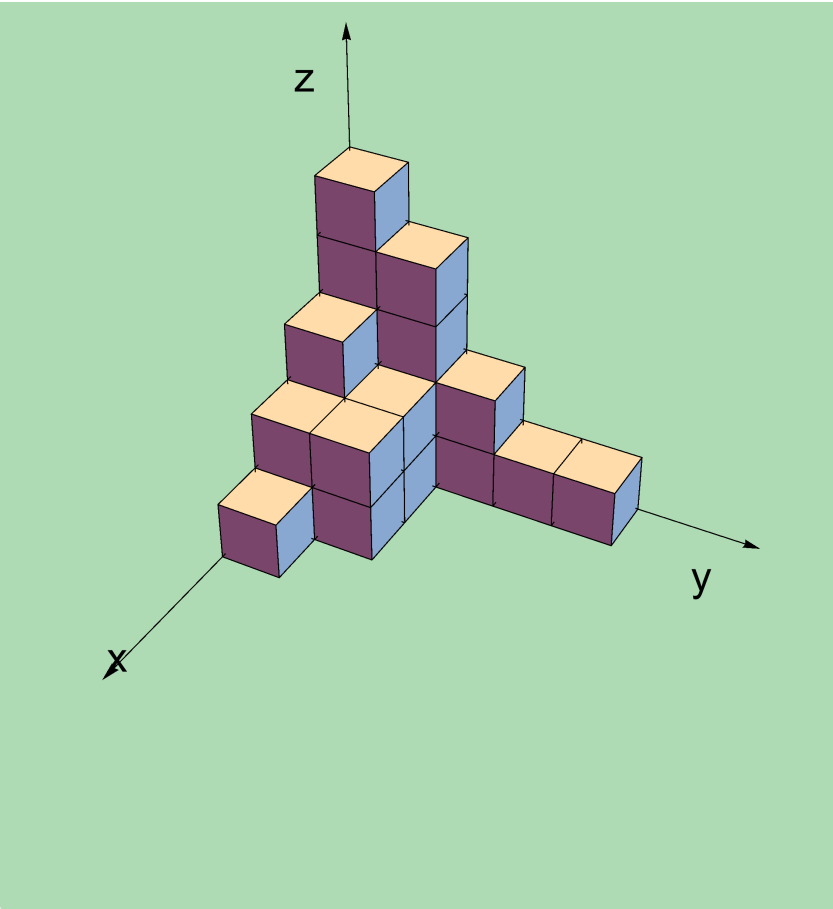


- After loop and vertex constraints

$$\alpha_1 = \alpha_2 \equiv h_1\ , \qquad \beta_1 = \beta_2 \equiv h_2\ , \qquad \gamma_1 = \gamma_2 = \gamma \equiv h_3\ ,$$
$$\text{and} \qquad h_1 + h_2 + h_3 = 0\ .$$

$(\mathbb{C}^2/\mathbb{Z}_n) \times \mathbb{C}$ crystal

n-colored plane partitions

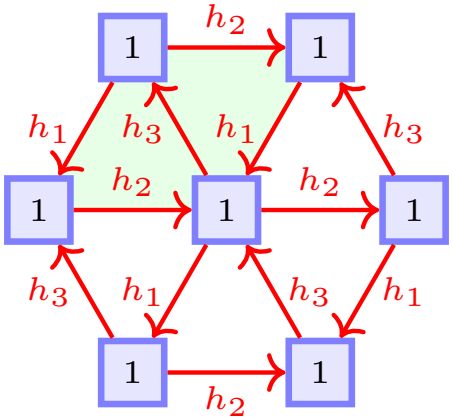


Action of $\hat{\mathcal{Y}}(\mathfrak{gl}_1)$ on a plane partition

$$\begin{cases} \psi(z)|\Lambda\rangle = \Psi_\Lambda(z)|\Lambda\rangle\,, \qquad \psi_\Lambda(z) \equiv \left(1+\frac{\psi_0\sigma_3}{z}\right)\prod_{\square\in(\Lambda)}\varphi_3(z-h(\square)) \\ \qquad\qquad\qquad\qquad h(\square) \equiv h_1\,x_1(\square)+h_2\,x_2(\square)+h_3\,x_3(\square) \\ e(z)|\Lambda\rangle = \sum_{\square\in\mathrm{Add}(\Lambda)} \frac{\left[-\frac{1}{\sigma_3}\mathrm{Res}_{w=h(\square)}\Psi_\Lambda(w)\right]^{\frac{1}{2}}}{z-h(\square)}|\Lambda+\square\rangle \\ f(z)|\Lambda\rangle = \sum_{\square\in\mathrm{Rem}(\Lambda)} \frac{\left[\frac{1}{\sigma_3}\mathrm{Res}_{w=h(\square)}\Psi_\Lambda(w)\right]^{\frac{1}{2}}}{z-h(\square)}|\Lambda-\square\rangle \end{cases}$$

1. Every Λ is eigenstate of $\psi(z)$
   - Each $\square$ in Λ contributes a factor of $\varphi_3(z-h(\square))$
   - Position of $\square$ is encoded by $h(\square)$
2. Actions of $e(z)$ and $f(z)$ are controlled by the eigenvalue $\Psi_\Lambda(z)$
   - Poles of $\Psi_\Lambda(z)$ encode the positions of $\square\in\mathrm{Add}(\Lambda)$ and $\square\in\mathrm{Rem}(\Lambda)$
   - Need loop constraint $h_1+h_2+h_3=0$

Poles of $\Psi_\Lambda(z)$ encode the positions of $\square \in \mathrm{Add}(\Lambda)$ and $\square \in \mathrm{Rem}(\Lambda)$



1. Each $\square$ in Λ contributes a factor of $\varphi_3(z - h(\square))$
2. $h(\square) \equiv h_1\, x_1(\square) + h_2\, x_2(\square) + h_3\, x_3(\square)$
3. $\varphi_3(z) \equiv \frac{(z+h_1)(z+h_2)(z+h_3)}{(z-h_1)(z-h_2)(z-h_3)}$
4. loop constraint $h_1 + h_2 + h_3 = 0$

Poles are always pushed to the surface of plane partition

Action of quiver Yangian on colored crystal

$$\mathcal{Y}(\mathfrak{gl}_1):\quad \begin{cases} \psi(z)|\Lambda\rangle = \Psi_\Lambda(z)|\Lambda\rangle\,, \qquad \psi_\Lambda(z) \equiv \left(1+\dfrac{\psi_0\sigma_3}{z}\right)\displaystyle\prod_{\square\in(\Lambda)}\varphi_3(z-h(\square)) \\[2ex] e(z)|\Lambda\rangle = \displaystyle\sum_{\square\in\mathrm{Add}(\Lambda)} \dfrac{\left[-\frac{1}{\sigma_3}\mathrm{Res}_{w=h(\square)}\Psi_\Lambda(w)\right]^{1/2}}{z-h(\square)}|\Lambda+\square\rangle \\[2ex] f(z)|\Lambda\rangle = \displaystyle\sum_{\square\in\mathrm{Rem}(\Lambda)} \dfrac{\left[\frac{1}{\sigma_3}\mathrm{Res}_{w=h(\square)}\Psi_\Lambda(w)\right]^{1/2}}{z-h(\square)}|\Lambda-\square\rangle \end{cases}$$

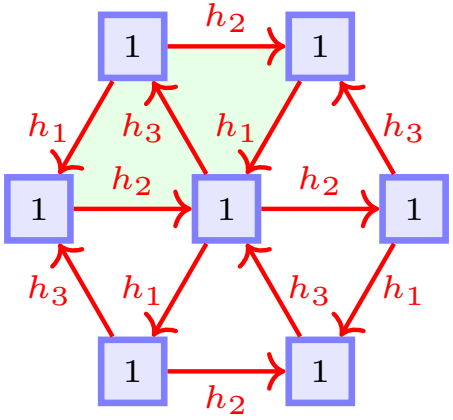
$$\mathcal{Y}(Q,W):\quad \begin{cases} \psi^{(a)}(z)|\mathrm{K}\rangle = \Psi^{(a)}_{\mathrm{K}}(z)|\mathrm{K}\rangle\,, \qquad \Psi^{(a)}_{\mathrm{K}}(u) \equiv \left(1+\dfrac{C}{z}\right)^{\delta_{a,1}}\displaystyle\prod_{b\in Q_0}\prod_{\boxed{b}\in\mathrm{K}}\varphi^{b\Rightarrow a}(u-h(\boxed{b})) \\[2ex] e^{(a)}(z)|\mathrm{K}\rangle = \displaystyle\sum_{\boxed{a}\in\mathrm{Add}(\mathrm{K})} \dfrac{\left[p^{(a)}\mathrm{Res}_{u=h(\boxed{a})}\Psi^{(a)}_{\mathrm{K}}(u)\right]^{1/2}}{z-h(\boxed{a})}|\mathrm{K}+\boxed{a}\rangle\,, \\[2ex] f^{(a)}(z)|\mathrm{K}\rangle = \displaystyle\sum_{\boxed{a}\in\mathrm{Rem}(\mathrm{K})} \dfrac{\left[q^{(a)}\mathrm{Res}_{u=h(\boxed{a})}\Psi^{(a)}_{\mathrm{K}}(u)\right]^{1/2}}{z-h(\boxed{a})}|\mathrm{K}-\boxed{a}\rangle\,, \end{cases}$$

Rules for action of quiver Yangian on color crystals

1. Every K is eigenstate of $\psi^{(a)}(z)$ for all a.
   - Position of $\boxed{b}$ is encoded by $h(\boxed{b}) \equiv \sum_{I \in \mathrm{path}[\mathfrak{o} \to \boxed{b}]} h_I$
   - Each $\boxed{b}$ in K contributes a factor of $\varphi^{b \Rightarrow a}(z - h(\boxed{b}))$
2. Actions of $e^{(a)}(z)$ and $f^{(a)}(z)$ are controlled by $\Psi_{\mathrm{K}}^{(a)}(z)$
   - Poles of $\Psi_{\mathrm{K}}^{(a)}(z)$ encode positions of $\boxed{a} \in \mathrm{Add}(\mathrm{K})$ and $\mathrm{Rem}(\mathrm{K})$
   - Applying $e^{(a)}(z)$ repeatedly on vacuum creates all colored crystals
   - Need loop constraint $\sum_{I \in \mathrm{loop}\ L} h_I = 0$

Poles of $\Psi_{\mathrm{K}}^{(a)}(z)$ encode the positions of $\boxed{a} \in \mathrm{Add(K)}$ and $\boxed{a} \in \mathrm{Rem(K)}$

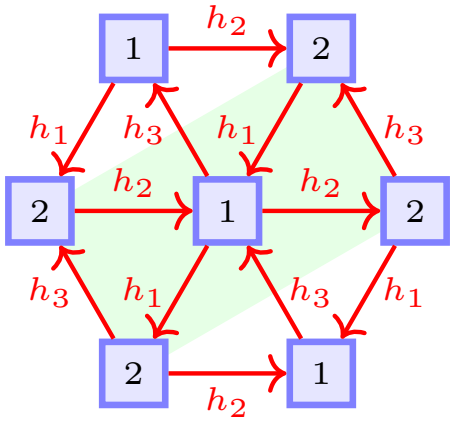
Poles are always pushed to the surface of crystal

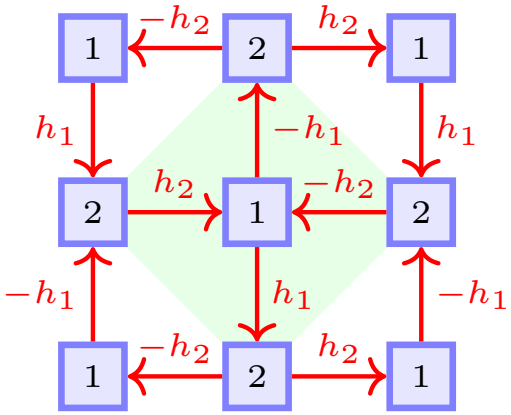


1. Each $\square$ in Λ contributes a factor of $\varphi_3(z - h(\square))$
2. $h(\square) \equiv h_1\, x_1(\square) + h_2\, x_2(\square) + h_3\, x_3(\square)$
3. $\boxed{\varphi_3(z) \equiv \dfrac{(z+h_1)(z+h_2)(z+h_3)}{(z-h_1)(z-h_2)(z-h_3)}}$
4. loop constraint $h_1 + h_2 + h_3 = 0$

Poles of $\Psi_{\mathrm{K}}^{(a)}(z)$ encode the positions of $\boxed{a} \in \mathrm{Add}(\mathrm{K})$ and $\boxed{a} \in \mathrm{Rem}(\mathrm{K})$

Poles are always pushed to the surface of crystal

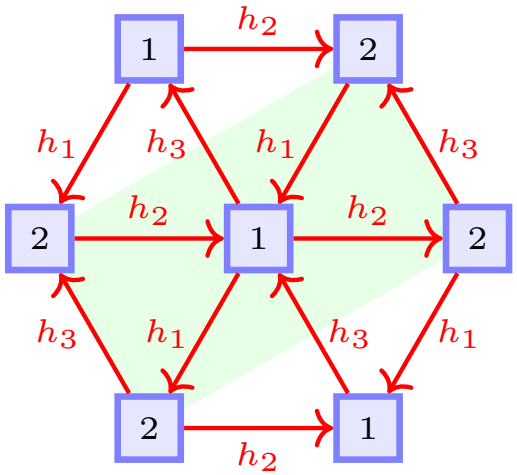




1. Each $\boxed{b}$ in K contributes a factor of $\varphi^{b\Rightarrow a}(z - h(\boxed{b}))$ to $\Psi_{\mathrm{K}}^{(a)}(z)$
2. $h(\boxed{b}) \equiv \sum_{I \in \mathrm{path}[\mathfrak{o}\to\boxed{b}]} h_I$
3. $$\boxed{\varphi^{b\Rightarrow a}(u) \equiv \frac{\prod_{I\in\{a\to b\}}(u+h_I)}{\prod_{I\in\{b\to a\}}(u-h_I)}}$$
4. loop constraint $\sum_{I\in\text{loop } L} h_I = 0$

$(\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{C}$

- periodic quiver

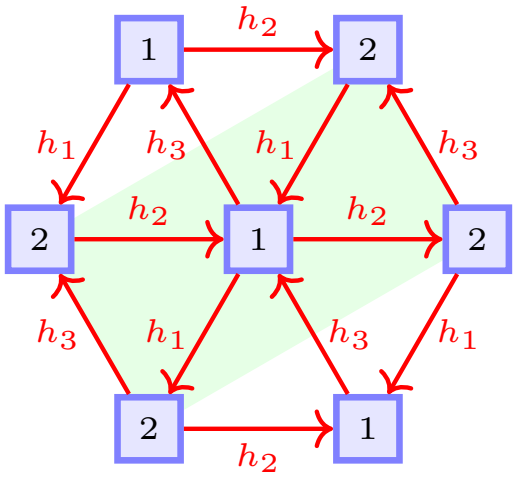


- building blocks of $\Psi_{\mathrm{K}}^{(a)}(u)$

$$\varphi^{a\Rightarrow a}(u) = \frac{u+h_3}{u-h_3} \qquad \text{and} \qquad \varphi^{a+1\Rightarrow a}(u) = \frac{(u+h_1)(u+h_2)}{(u-h_1)(u-h_2)}$$

- Loop constraint $h_1 + h_2 + h_3 = 0$

$(\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{C}$ crystal: vacuum $\Longrightarrow$ level-1



- vacuum $|\emptyset\rangle$:

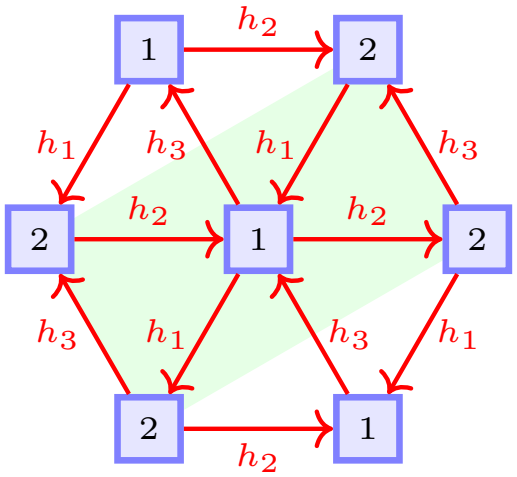
$$\psi^{(1)}(z)|\emptyset\rangle = \psi_0(z)|\emptyset\rangle = \frac{z+C}{z}|\emptyset\rangle$$

$$\psi^{(2)}(z)|\emptyset\rangle = |\emptyset\rangle$$

- Pole for $\boxed{1}$: $z = 0$

  Pole for $\boxed{2}$: none

$(\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{C}$ crystal: vacuum $\Longrightarrow$ level-1



- vacuum $|\emptyset\rangle$:

$$\psi^{(1)}(z)|\emptyset\rangle = \psi_0(z)|\emptyset\rangle = \frac{z+C}{z}|\emptyset\rangle$$

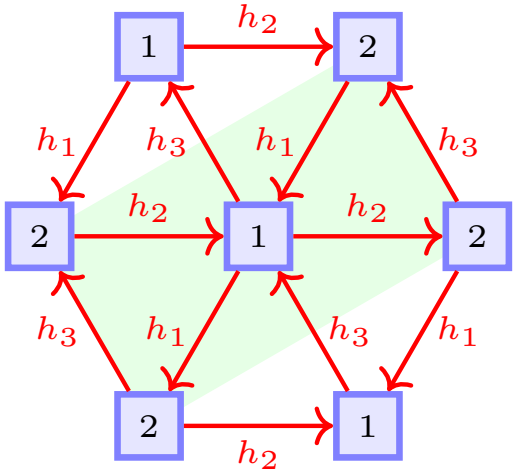
$$\psi^{(2)}(z)|\emptyset\rangle = |\emptyset\rangle$$

$\boxed{1}$

- Pole for $\boxed{1}$: $z=0 \quad \Longrightarrow \quad e^{(1)}(z)|\emptyset\rangle = \frac{\#}{z}|\boxed{1}\rangle$

  Pole for $\boxed{2}$: none $\quad \Longrightarrow \quad e^{(2)}(z)|\emptyset\rangle = 0$

$(\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{C}$ crystal: level-1 $\Longrightarrow$ level-2



- $h(\boxed{1}) = 0$:

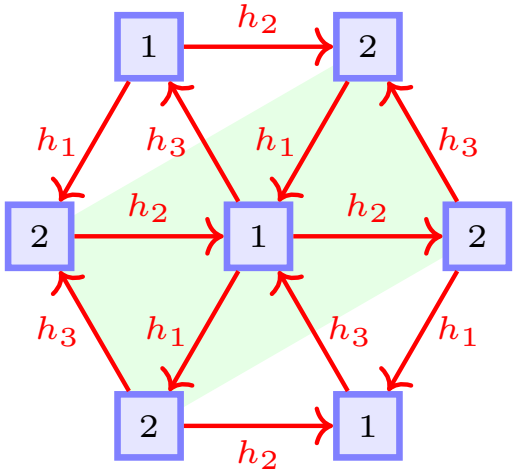
$$\psi_{\mathrm{K}}^{(1)}(z) = \psi_0(z) \cdot \varphi^{1\Rightarrow 1}(z - h(\boxed{1})) = \frac{z + C}{z} \cdot \frac{z + h_3}{z - h_3}$$

$$\psi_{\mathrm{K}}^{(2)}(z) = \varphi^{1\Rightarrow 2}(z - h(\boxed{1})) = \frac{(z + h_1)(z + h_2)}{(z - h_1)(z - h_2)}$$

- Pole for $\boxed{1}$: $z = 0$ and $z = h_3$

  Pole for $\boxed{2}$: $z = h_1$ and $z = h_2$

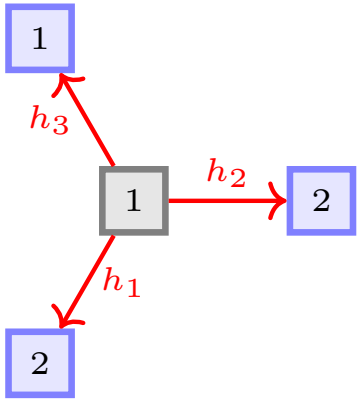
$(\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{C}$ crystal: level-1 $\Longrightarrow$ level-2



- $h(\boxed{1}) = 0$:

$$\psi_{\mathrm{K}}^{(1)}(z) = \psi_0(z) \cdot \varphi^{1\Rightarrow 1}(z - h(\boxed{1})) = \frac{z+C}{z} \cdot \frac{z+h_3}{z-h_3}$$

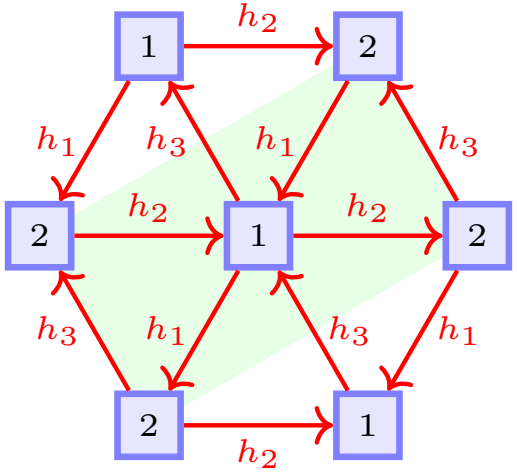
$$\psi_{\mathrm{K}}^{(2)}(z) = \varphi^{1\Rightarrow 2}(z - h(\boxed{1})) = \frac{(z+h_1)(z+h_2)}{(z-h_1)(z-h_2)}$$

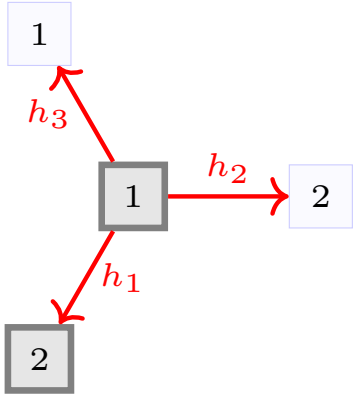


- Pole for $\boxed{1}$: $z = 0$ and $z = h_3$

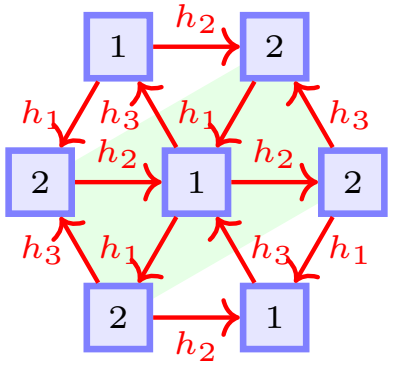
  Pole for $\boxed{2}$: $z = h_1$ and $z = h_2$

$(\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{C}$ crystal: level-1 $\Longrightarrow$ level-2



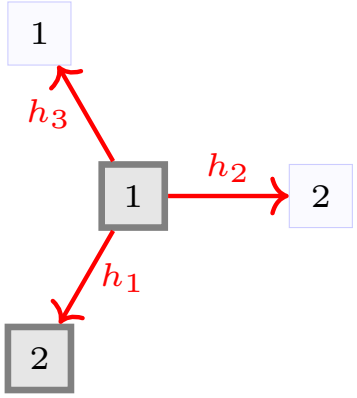


$(\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{C}$ crystal: level-2 $\Longrightarrow$ level-3



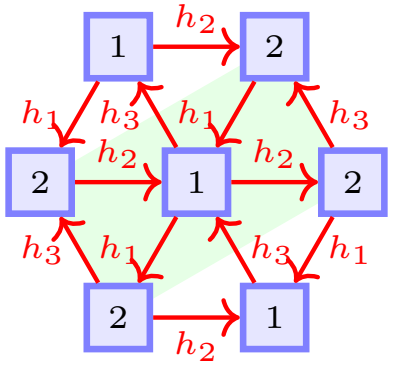
- $h(\boxed{1}) = 0$ and $h(\boxed{2}) = h_1$

$$\Psi_{\mathrm{K}}^{(1)}(z) = \psi_0(z) \cdot \varphi^{1\Rightarrow 1}(z - h(\boxed{1})) \cdot \varphi^{2\Rightarrow 1}(z - h(\boxed{1}))$$
$$= \frac{(z+C)}{\cancel{z}} \cdot \frac{(z+h_3)}{(z-h_3)} \cdot \frac{\cancel{(z)}(z+h_2-h_1)}{(z-2h_1)\cancel{(z+h_3)}}$$
$$\Psi_{\mathrm{K}}^{(2)}(z) = \varphi^{1\Rightarrow 2}(z - h(\boxed{1})) \cdot \varphi^{2\Rightarrow 2}(z - h(\boxed{1}))$$
$$= \frac{(z+h_1)\cancel{(z+h_2)}}{(z-h_1)(z-h_2)} \cdot \frac{(z+h_3-h_1)}{\cancel{(z+h_2)}}$$



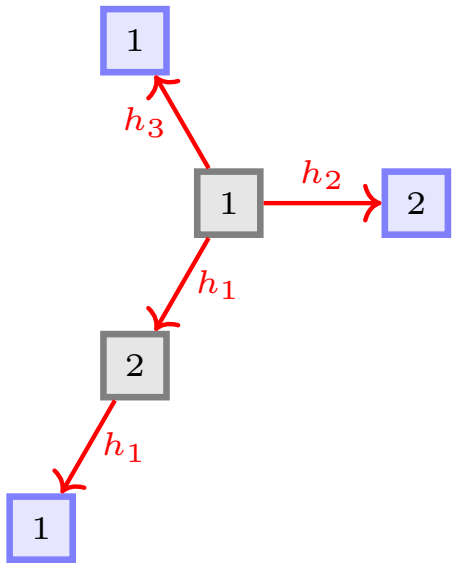
- Pole for $\boxed{1}$: $z = h_3$ and $z = 2h_1$; pole for $\boxed{2}$: $z = h_{1,2}$.

$(\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{C}$ crystal: level-2 $\Longrightarrow$ level-3



- Using $h(\boxed{1}) = 0$ and $h(\boxed{2}) = h_1$

$$\Psi_{\mathrm{K}}^{(1)}(z) = \psi_0(z) \cdot \varphi^{1\Rightarrow 1}(z - h(\boxed{1})) \cdot \varphi^{2\Rightarrow 1}(z - h(\boxed{1}))$$
$$= \frac{(z + C)}{\cancel{z}} \cdot \frac{(z + h_3)}{(z - h_3)} \cdot \frac{\cancel{(z)}(z + h_2 - h_1)}{(z - 2h_1)\cancel{(z + h_3)}}$$
$$\Psi_{\mathrm{K}}^{(2)}(z) = \varphi^{1\Rightarrow 2}(z - h(\boxed{1})) \cdot \varphi^{2\Rightarrow 2}(z - h(\boxed{1}))$$
$$= \frac{(z + h_1)\cancel{(z + h_2)}}{(z - h_1)(z - h_2)} \cdot \frac{(z + h_3 - h_1)}{\cancel{(z + h_2)}}$$



- Pole for $\boxed{1}$: $z = h_3$ and $z = 2h_1$; pole for $\boxed{2}$: $z = h_{1,2}$.

$$\text{action}: \quad \begin{cases} \psi^{(a)}(z)|\mathrm{K}\rangle = \Psi_{\mathrm{K}}^{(a)}(z)|\mathrm{K}\rangle \ , \\ e^{(a)}(z)|\mathrm{K}\rangle = \displaystyle\sum_{\boxed{a}\in \mathrm{Add}(\mathrm{K})} \frac{\sqrt{p^{(a)}\mathrm{Res}_{u=h(\boxed{a})}\Psi_{\mathrm{K}}^{(a)}(u)}}{z-h(\boxed{a})}|\mathrm{K}+\boxed{a}\rangle \ , \\ f^{(a)}(z)|\mathrm{K}\rangle = \displaystyle\sum_{\boxed{a}\in \mathrm{Rem}(\mathrm{K})} \frac{\sqrt{q^{(a)}\mathrm{Res}_{u=h(\boxed{a})}\Psi_{\mathrm{K}}^{(a)}(u)}}{z-h(\boxed{a})}|\mathrm{K}-\boxed{a}\rangle \ , \end{cases}$$

$$\text{with}: \quad \Psi_{\mathrm{K}}^{(a)}(u) \equiv \psi_0(u)^{\delta_{a,1}} \prod_{b\in Q_0}\prod_{\boxed{b}\in\mathrm{K}} \varphi^{b\Rightarrow a}(u-h(\boxed{b})) \ ,$$

$$\psi_0(z) \equiv 1+\frac{C}{z} \ , \qquad \text{and} \qquad \varphi^{b\Rightarrow a}(u) \equiv \frac{\prod_{I\in\{a\to b\}}(u+h_I)}{\prod_{I\in\{b\to a\}}(u-h_I)} \ ,$$

$$h(\boxed{a}) \equiv \sum_{I\in \mathrm{path}[\mathfrak{o}\to\boxed{a}]} h_I \ ,$$

$$\text{subject to}: \quad \sum_{I\in \text{loop } L} h_I = 0 \ ,$$

From action on plane partitions to algebra: $\hat{\mathcal{Y}}(\mathfrak{gl}_1)$

$$\begin{cases} \psi(z)|\Lambda\rangle = \Psi_\Lambda(z)|\Lambda\rangle\,, \qquad \psi_\Lambda(z) \equiv \left(1+\dfrac{\psi_0\sigma_3}{z}\right)\displaystyle\prod_{\square\in(\Lambda)}\varphi_3(z-h(\square)) \\ e(z)|\Lambda\rangle = \displaystyle\sum_{\square\in\mathrm{Add}(\Lambda)} \dfrac{\left[-\frac{1}{\sigma_3}\mathrm{Res}_{w=h(\square)}\Psi_\Lambda(w)\right]^{\frac{1}{2}}}{z-h(\square)}|\Lambda+\square\rangle \\ f(z)|\Lambda\rangle = \displaystyle\sum_{\square\in\mathrm{Rem}(\Lambda)} \dfrac{\left[\frac{1}{\sigma_3}\mathrm{Res}_{w=h(\square)}\Psi_\Lambda(w)\right]^{\frac{1}{2}}}{z-h(\square)}|\Lambda-\square\rangle \end{cases}$$

with $\varphi_3(z) = \dfrac{(z+h_1)(z+h_2)(z+h_3)}{(z-h_1)(z-h_2)(z-h_3)}$ and $h_1+h_2+h_3=0$

Defining relations

$$[\psi(z)\,,\psi(w)] \sim 0 \qquad [e(z)\,,f(w)] \sim -\frac{1}{\sigma_3}\,\frac{\psi(z)-\psi(w)}{z-w}$$

$$\psi(z)\,e(w) \sim \varphi_3(z-w)\,e(w)\,\psi(z) \qquad \psi(z)\,f(w) \sim \varphi_3^{-1}(z-w)\,f(w)\,\psi(z)$$

$$e(z)\,e(w) \sim \varphi_3(z-w)\,e(w)\,e(z) \qquad f(z)\,f(w) \sim \varphi_3^{-1}(z-w)\,f(w)\,f(z)$$

From action on plane partitions to algebra: $\hat{\mathcal{Y}}(\mathfrak{gl}_1)$

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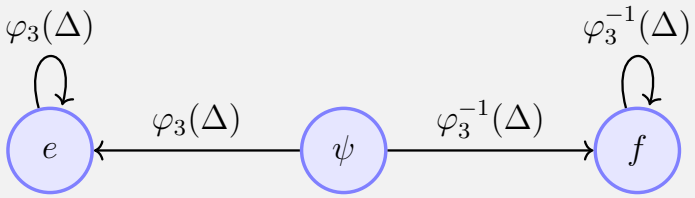
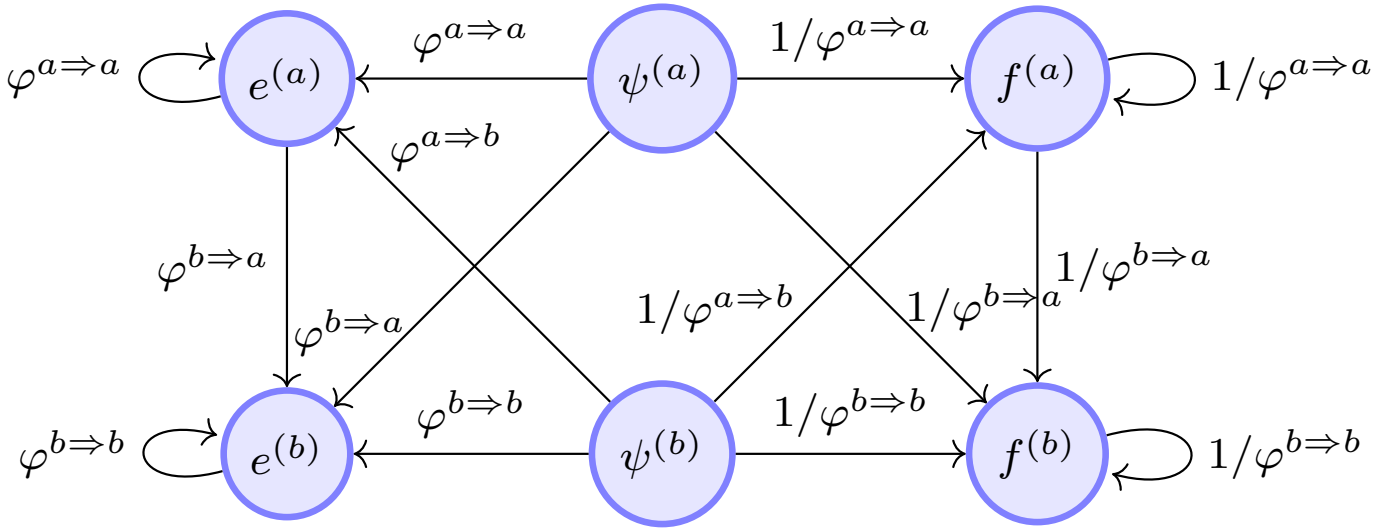


General Quiver Yangian

$$
\begin{aligned}
\psi^{(a)}(z)\,\psi^{(b)}(w) &= \psi^{(b)}(w)\,\psi^{(a)}(z)\ ,\\
\psi^{(a)}(z)\,e^{(b)}(w) &\simeq \varphi^{b\Rightarrow a}(z-w)\,e^{(b)}(w)\,\psi^{(a)}(z)\ ,\\
e^{(a)}(z)\,e^{(b)}(w) &\sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(z-w)\,e^{(b)}(w)\,e^{(a)}(z)\ ,\\
\psi^{(a)}(z)\,f^{(b)}(w) &\simeq \varphi^{b\Rightarrow a}(z-w)^{-1}\,f^{(b)}(w)\,\psi^{(a)}(z)\ ,\\
f^{(a)}(z)\,f^{(b)}(w) &\sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(z-w)^{-1}\,f^{(b)}(w)\,f^{(a)}(z)\ ,\\
\left[e^{(a)}(z), f^{(b)}(w)\right\} &\sim -\delta^{a,b}\,\frac{\psi^{(a)}(z)-\psi^{(b)}(w)}{z-w}\ ,
\end{aligned}
$$



Statistics of $e^{(a)}$ and $f^{(a)}$

$$\begin{aligned} \text{Boson}: &\quad |a| = 0 \\ \text{Fermion}: &\quad |a| = 1 \end{aligned}$$

$$(-1)^{|a|} = -\varphi^{a \Rightarrow a}(0)$$

$$|a| = \begin{cases} 0 & \text{(odd number of self-loops for vertex } a) \\ 1 & \text{(even number of self-loops for vertex } a) \end{cases}$$

Relations in terms of modes

$$\left[\psi_n^{(a)}\,,\,\psi_m^{(b)}\right] = 0\,,$$

$$\sum_{k=0}^{|b\to a|}(-1)^{|b\to a|-k}\,\sigma^{b\to a}_{|b\to a|-k}\,[\psi_n^{(a)}\;e_m^{(b)}]_k = \sum_{k=0}^{|a\to b|}\sigma^{a\to b}_{|a\to b|-k}\,[e_m^{(b)}\;\psi_n^{(a)}]^k\,,$$

$$\sum_{k=0}^{|b\to a|}(-1)^{|b\to a|-k}\,\sigma^{b\to a}_{|b\to a|-k}\,[e_n^{(a)}\;e_m^{(b)}]_k = (-1)^{|a||b|}\sum_{k=0}^{|a\to b|}\sigma^{a\to b}_{|a\to b|-k}\,[e_m^{(b)}\;e_n^{(a)}]^k\,,$$

$$\sum_{k=0}^{|a\to b|}\sigma^{a\to b}_{|a\to b|-k}\,[\psi_n^{(a)}\;f_m^{(b)}]_k = \sum_{k=0}^{|b\to a|}(-1)^{|b\to a|-k}\,\sigma^{b\to a}_{|b\to a|-k}\,[f_m^{(b)}\;\psi_n^{(a)}]^k\,,$$

$$\sum_{k=0}^{|a\to b|}\sigma^{a\to b}_{|a\to b|-k}\,[f_n^{(a)}\;f_m^{(b)}]_k = (-1)^{|a||b|}\sum_{k=0}^{|b\to a|}(-1)^{|b\to a|-k}\,\sigma^{b\to a}_{|b\to a|-k}\,[f_m^{(b)}\;f_n^{(a)}]^k,$$

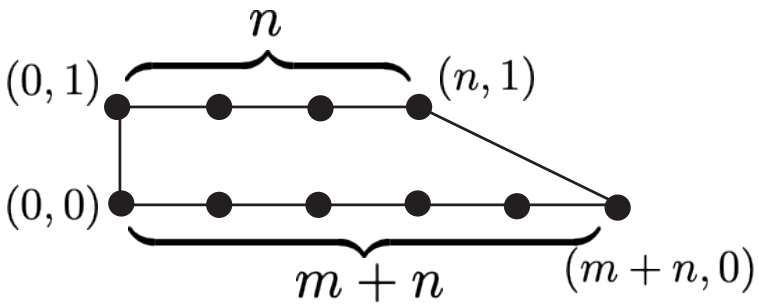
$$\left[e_n^{(a)}\,,\,f_m^{(b)}\right\} = \delta^{a,b}\,\psi^{(a)}_{n+m}\,,$$

$$[A_n\;B_m]_k \equiv \sum_{j=0}^{k}(-1)^j\,\binom{k}{j}\,A_{n+k-j}\;B_{m+j}\,,$$

$$[B_m\;A_n]^k \equiv \sum_{j=0}^{k}(-1)^j\,\binom{k}{j}\,B_{m+j}\;A_{n+k-j}\,.$$

Relation with known algebras

- non-compact Calabi-Yau threefold without compact 4-cycle ($xy = z^m w^n$)

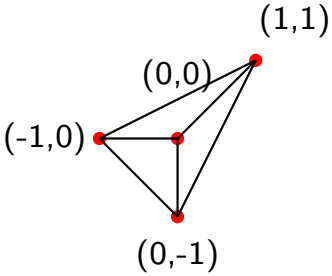


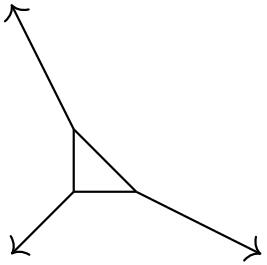
quiver Yangian: affine Yangian of $\mathfrak{gl}_{m|n}$

- non-compact Calabi-Yau threefolds with compact 4-cycle

example:

$\mathbb{C}^3/\mathbb{Z}_3$





quiver Yangian: new

Outline

1. Introduction
2. BPS crystal
3. $\mathbb{C}^3$
4. General toric CY3
5. Summary

Summary: BPS algebra for general toric Calabi-Yau X_3

- Consider the BPS sector of D-brane system of type IIA string in non-compact toric Calabi-Yau threefolds
- Construct BPS algebra
  1. periodic quiver $(Q, W) \Longrightarrow \varphi^{b\Rightarrow a}(z-w)$ and $|a|$
  2. quiver Yangian $Y(Q, W)$

$$
\begin{aligned}
\psi^{(a)}(z)\,\psi^{(b)}(w) &= \psi^{(b)}(w)\,\psi^{(a)}(z)\ , \\
\psi^{(a)}(z)\,e^{(b)}(w) &\simeq \varphi^{b\Rightarrow a}(z-w)\,e^{(b)}(w)\,\psi^{(a)}(z)\ , \\
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\left[e^{(a)}(z), f^{(b)}(w)\right\} &\sim -\delta^{a,b}\,\frac{\psi^{(a)}(z)-\psi^{(b)}(w)}{z-w}\ ,
\end{aligned}
$$

Summary: bootstrap from BPS crystals

- Bootstrap from BPS crystals
  1. periodic Quiver (Q, W)

     $\Downarrow$ define

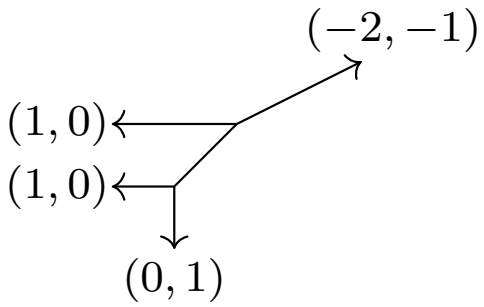
     $\{$ BPS states $\} = \{$ colored crystal $\}$

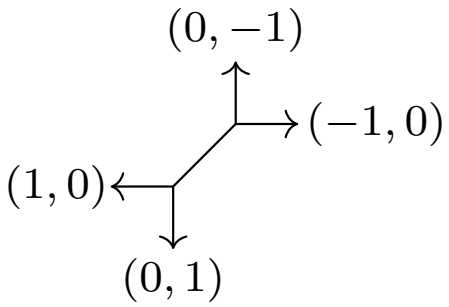
     act $\Uparrow\Downarrow$ bootstrap
  2. Quiver Yangian $Y(Q, W)$
- Advantages
  1. Explicit algebraic relations
  2. Apply to all non-compact toric Calabi-Yau threefolds

Future directions

- Relation to gluing constructions

e.g.





L-Longhi '19, L '19

- Translate to $\mathcal{W}$ algebras basis
- Non-vacuum modules of the algebra
- Truncation of the algebra
  1. Correspond to non-zero D4 charge
  2. Produce VOA of rational CFT
  3. Relations to other systems

Truncation of $\hat{\mathcal{Y}}(\mathfrak{gl}_1)$ and rational VOA

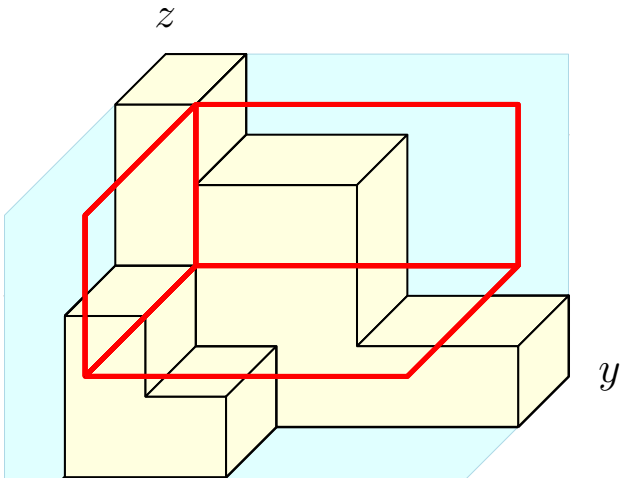
- affine Yangian of $\mathfrak{gl}_1$ = UEA $(\mathcal{W}_{1+\infty})$
- $\mathcal{W}_{1+\infty}$ can truncation to give $\mathcal{W}_{N,k}$ minimal model

How to truncate $\hat{\mathcal{Y}}(\mathfrak{gl}_1)$? *Prochazka '15, Rapcak-Prochazka '17*

1. Use parameter $(\lambda_1, \lambda_2, \lambda_3)$ with $\sum_{i=1}^{3} \frac{1}{\lambda_i} = 0$
2. Truncation labeled by (N_1, N_2, N_3)

$$\boxed{\frac{N_1}{\lambda_1} + \frac{N_2}{\lambda_2} + \frac{N_3}{\lambda_3} = 1}$$

How does plane partition representation truncate ? *Gaiotto-Rapcak '17*



$$(N_1, N_2, N_3) = (1, 1, 2)$$

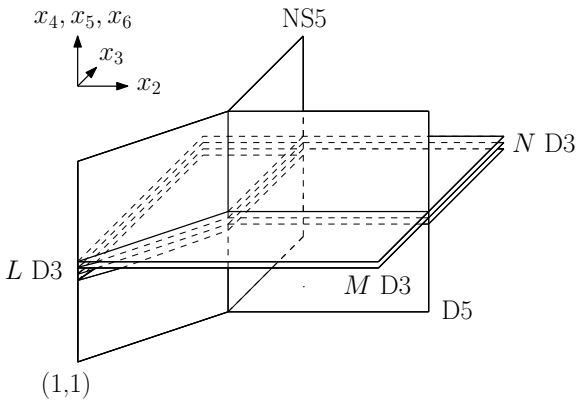
Truncation of quiver Yangian (Q, W)

Goal: classify all possible truncations of quiver Yangian (Q, W)

- correspond to adding D4 charges
- give new rational VOAs
- relate to other systems
  1. Vertex VOA and the gluing thereof
     Gaiotto-Rapcak '17, Rapcak-Prochazka'17-'18
  2. Spike instantons and generalized AGT
     Nekrasov '16, Nekrasov-Prabhakar '16
     Rapcak-Soibelman-Yang-Zhao '18
  3. ...

VOA for (p, q) web of Y-junctions

- The Vertex VOA



$\times \underset{x_0, x_1}{C} \times \underset{x_7, x_8, x_9}{R^3}$

VOA on the 2D junction:

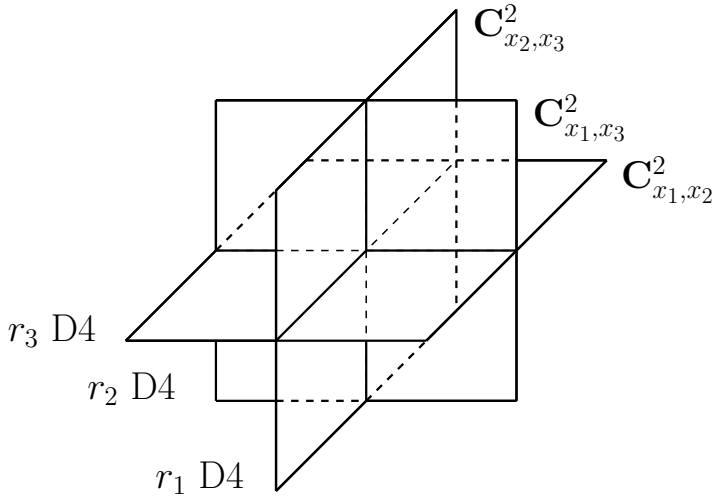
(L, M, N) truncation of $\mathcal{W}_{1+\infty}$

Gaiotto-Rapcak '17

- Gluing of the finite truncations of $\mathcal{W}_{1+\infty}$ should give chiral algebra of Y-junction webs *Rapcak-Prochazka'17, '18*
  1. $\mathcal{W}$ algebra basis
  2. explicit OPEs?
  3. only known for toric Calabi-Yau without compact 4-cycles
- Can (1) use Yangian basis and (2) first glue and then truncate *L-Longhi '19, L '19*
- How to map to truncated quiver Yangian (Q, W) directly?

Truncated quiver Yangian for generalized AGT

- AGT correspondence:
  Nekrasov instanton partition function of 4D $\mathcal{N}=2$ SU(N) gauge theory form $\mathcal{W}_N$ conformal blocks conjecture: *Alday-Gaiotto-Tachikawa '09*
  proof: *Schiffmann-Vasserot '12, Maulik-Okounkov '12*
- Spike instanton and generalized AGT



Spike instanton partition function transform under (r_1, r_2, r_3) truncation of $\hat{\mathcal{Y}}(\mathfrak{gl}_1)$

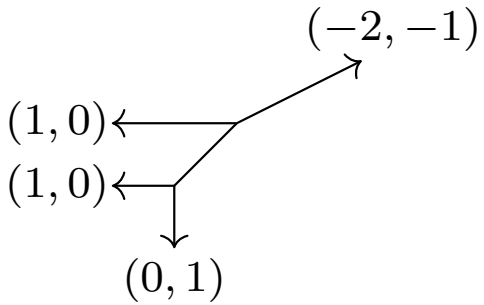
Rapcak-Soibelman-Yang-Zhao '18

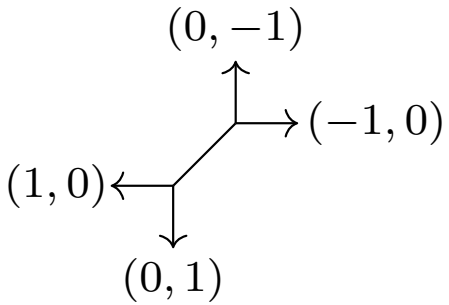
- Gluing of the finite truncations of $\hat{\mathcal{Y}}(\mathfrak{gl}_1)$ should give Yangian algebra for generalized AGT for general toric Calabi-Yau X_3
  - Construction via CoHA
- How to map to truncated quiver Yangian (Q, W) directly?

Future directions

- Relation to gluing constructions

e.g.





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Thank you for your attention !