Topological Complexity for Quantum Information

Zhengwei Liu
Tsinghua University
Joint with Arthur Jaffe, Xun Gao, Yunxiang Ren and Shengtao Wang

Seminar at Dublin IAS, Oct 14, 2020

Topological Complexity: The complexity of computing matrix products or contractions of tensors may grows exponentially using the state sum over the basis. Using topological ideas, such as knot-theoretical isotopy, one could compute the tensors in a more efficient way.

Fractionalization: We open the "black box" in tensor network and explore "internal pictorial relations" using the 3D quon language, a fractionalization of tensor network. (Euler's formula, Yang-Baxter equation/relation, star-triangle equation, Kramer-Wannier duality, Jordan-Wigner transformation etc.)

Applications: We show that two well-known efficiently classically simulable families, Clifford gates and matchgates, correspond to two kinds of topological complexities.
Besides the two simulable families, we introduce a new method to design efficiently classically simulable tensor networks and new families of exactly solvable models.

Qubits and Gates

A qubit is a vector state in $\mathbb{C}^2$.

An n-qubit is a vector state $|\phi\rangle$ in $(\mathbb{C}^2)^n$.

A n-qubit gate is a unitary on $(\mathbb{C}^2)^n$.

Quantum Simulation

Feynman and Manin both proposed Quantum Simulation in 1980.

Simulate a quantum process by local interactions on qubits.

Present an n-qubit gate as a composition of 1-qubit gates and adjacent 2-qubits gates.

For example, Shor's algorithm of factorization, quantum Fourier transform as $O(n^3)$ adjacent 2-qubits gates.

Quantum supremacy by Google 2019, efficient quantum simulation of a distribution in the lab, no efficient classical simulation.

In this talk, Efficient $\leftrightarrow$ Polynomial Time

Some Usual Gates

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$H = 2^{-1/2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix},$$

$$CZ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Kitaev: Any 2-qubit gate is approximately a composition of $O(\varepsilon^{-3})$ gates above. (Topological Quantum Computation)
Euler's formula: any 1-qubit gate

$$U = e^{i\alpha_1 X} e^{i\alpha_2 Z} e^{i\alpha_3 X}.$$

Classical Simulation: Clifford gates

The n-qubit Pauli group is projectively generated by $\{X, Y, Z\}$ (on n-qubits). The Clifford group is the stabilizer group of the Pauli group. It is generated by $\{X, Y, Z, H, S, CZ\}$.

Gottesman 95: The n-qubit Clifford group can be classically simulated by the symplectic group $Sp(2n)$ over $\mathbb{F}(2)$, therefore the complexity of computing compositions of Clifford gates reduced from $O(2^{3n})$ to $O(n^3)$.

Classical Simulation: Matchgates

Valiant 02: n-qubit matchgates are efficiently classically simulable.

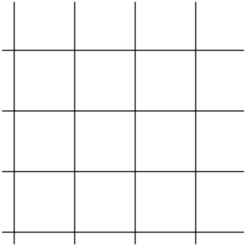
Generating match gates: $e^{i\alpha Z}$, $e^{i\theta(X\otimes X)}$,
(or equivalently $e^{i\alpha X}$, $e^{i\theta(Z\otimes Z)}$,)

$$\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \; and \; \begin{bmatrix} * & 0 & 0 & * \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ * & 0 & 0 & * \end{bmatrix}.$$

2D Ising model

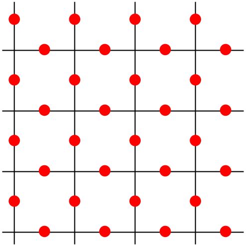
2D (periodic) Ising model on $m \times m$ square lattice (no magnetic field): each vertex i is assigned a spin σ_i (or a qubit) with nearest interaction J at each edge.

$$H(\sigma) = \sum_{(i,j)} J_{i,j}\sigma_i\sigma_j;$$
$$Z = \sum_{\sigma} e^{-\beta H(\sigma)}.$$



Onsager 1944, computing the partition function for $m \to \infty$.
Computing the partition function efficiently:
(1) Majorana fermions;
(2) Pfaffian;
(3) Yang-Baxter equation and transfer matrices.

Tensor Network

The partition function Z of the Ising model can be represented as the value of a tensor network: ,

each 4-valent vertex represents the identity tensor $|0000\rangle + |1111\rangle$;
each 2-valent red bullet represents the tensor
$J = a(|00\rangle + |11\rangle) + b(|01\rangle + |10\rangle)$,

Penrose: gluing end points $\leftrightarrow$ contractions

Fisher-Kasteleyn-Temperley Algorithm

Fisher-Kasteleyn-Temperley (FKT) algorithm:
For any planar graph, whose vertices represent the identity tensor and edges represent the tensor $J = a(|00\rangle + |11\rangle) + b(|01\rangle + |10\rangle)$, ($a, b$ may depend on edges,) the value of this tensor network is the Pfaffian of certain matrix. Therefore, it can be computed efficiently.

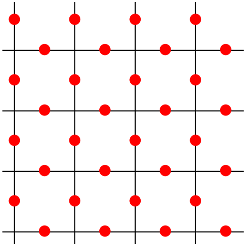
Special cases:
(1) Matchgates;
(2) Partition function of the Ising model;
(3) Perfect matching for planar graphs,

$$a = \cosh(\pi/3), b = \cosh(\pi/3).$$

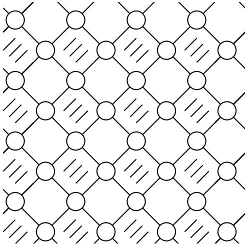
Jones' spin model

Jones' spin model in 1989 is a 2-fold fractionalization of tensor network.
k-valent tensor in tensor network $\leftrightarrow$ $2k$-valent diagram in spin model.
Each shaded region is assigned a spin in spin model.

Tensor Network



Spin Model



Kramer-Wannier duality: switch the alternating shading for regions.

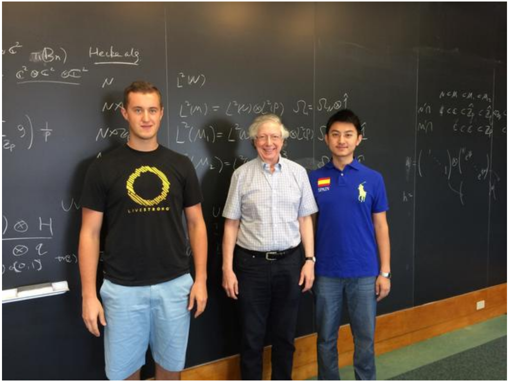
Majorana zero mode

Majorana fermions: $\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{i,j}$.
Kitaev's map: $X = i\gamma_1\gamma_4$, $Y = i\gamma_1\gamma_3$, $Z = i\gamma_1\gamma_2$,
1-qubit space is the eigenspace of $-\gamma_1\gamma_2\gamma_3\gamma_4 = 1$.
Group $4n$ Majorana fermions four by four as n-qubit transformations.

This has been extensively studied as the Majorana zero mode,
see Sarma-Freedman-Nayak 2015 njp QI for a recent survey.

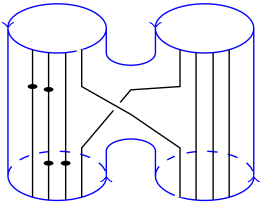
Quon Language

In the joint paper with Arthur Jaffe and Alex Wozniakowski (PNAS 2017), we introduce the **quon language** as a 3D picture language for quantum information.



Quon Language for Qubits

The quon language for qubits is an Ising TQFT with **charges**, the functor F is extended to a **projective** monoidal functor from the category of 1+1 cobordisms with braided strings and pairs of charges $\mathbf{Cob_{BCS}}$ to $\mathbf{Vec}$.

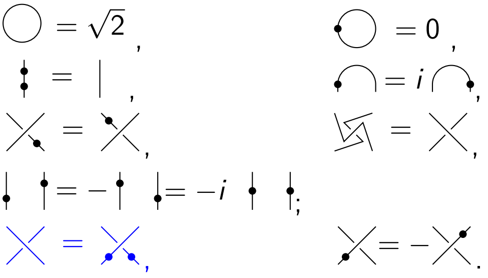


The quon language is projective in the bulk and linear on the boundary. This is good to simulate quantum theory, as the state space is linear in terms of the super position, and the transformations are defined projectively. (The functor F can be further extended to be super.)

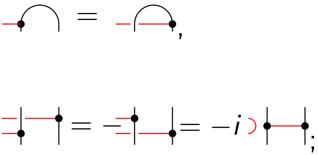
Bulk Relations

The braid satisfies Reidemeister moves of type I, II, III.

The charge behaves like a Majorana fermion:

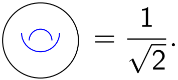


Topological Relations

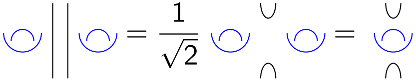


(Last two only for qubits)

String-Genus Relation:

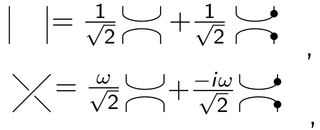


Neutrality:



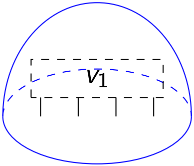
(We will list additional bulk relations later.)

Linear Relations on the Boundary



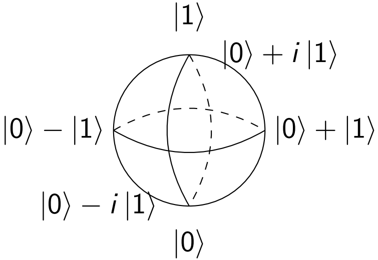
where $\omega^2 = \frac{1+i}{\sqrt{2}}$.

The Hilbert space $\mathcal{H}_{2n}$ is given by linear sums of diagrams with $2n$ boundary points and even charges. Modulo these relations, the Hilbert space $\mathcal{H}_4$ is 2 dimensional. We simulate the 1-qubit space by $\mathcal{H}_4$, pictorially



Bloch Sphere and XYZ Basis

Bloch sphere: $\alpha|0\rangle + \beta|1\rangle \to \beta/\alpha$

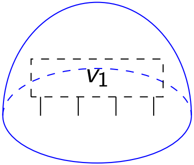


$$|0_Z\rangle = |0\rangle, \qquad |1_Z\rangle = |1\rangle;$$

$$|0_Y\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle), \qquad |1_Y\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle);$$

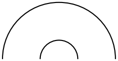
$$|0_X\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad |1_X\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle);$$

XYZ basis



$\sqrt{2}\ |0\rangle_Z =$ 

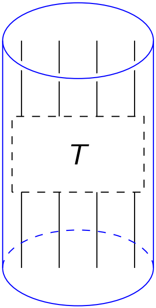
$\sqrt{2}\ |0\rangle_Y =$ 

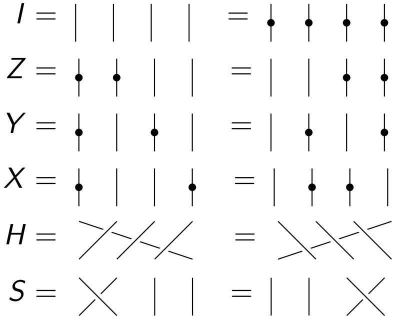
$\sqrt{2}\ |0\rangle_X =$ 

We obtain $|1\rangle_Z, |1\rangle_Y, |1\rangle_X$ by adding a pair of (opposite) charges to the pair of strings of $|0\rangle_Z, |0\rangle_Y, |0\rangle_X$, respectively.

1-qubit gates

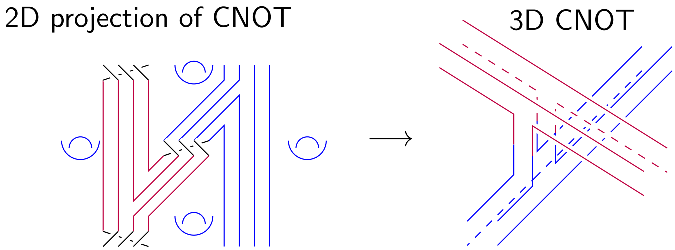
We represent a 1-qubit transformation T by four strings in a cylinder.





The last diagrammatic relation implies the others. They are additional bulk relations.

2D CNOT to 3D CNOT

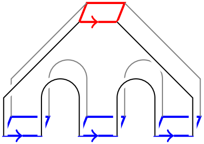


Convention: For 2D pictures, we use the genus to indicate the shape of the surface.

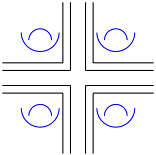
Repetition Map and Identity Tensor

3D Repetition map:

$$\iota(\alpha |0\rangle + \beta |1\rangle) = \alpha |000\rangle + \beta |111\rangle .$$



By reversing the input to an output, we obtain the identity tensor $|0000\rangle + |1111\rangle$. It has the following 2D pictorial representation:



Simulation of Clifford Gates

Clifford tensor network: X, Y, Z, H, S and the identity tensor $|000\rangle + |111\rangle$,.

Theorem (Gao-Jaffe-L-Ren-Wang)

Using the bulk relations above, we can efficiently simulate/evaluate a Clifford circuit/tensor network.

Actually these bulk relations implies all tensor network relations in ZX calculus. (See Bob and Duncan NJP 13 for ZX calculus.)

Universality

Clifford gates are not universal for quantum computing. Clifford gates and phase gates $e^{i\theta X}$ are universal.

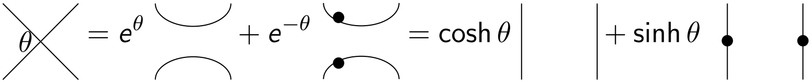
Question: Is there any topological feature for phase gates?

Universality

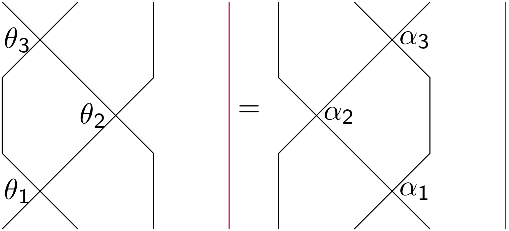
Clifford gates are not universal for quantum computing. Clifford gates and phase gates $e^{i\theta X}$ are universal.

Question: Is there any topological feature for phase gates? Yes!

Euler Formula and Yang-Baxter equation



Yang-Baxter equation:

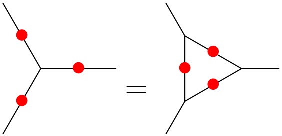


Euler formula: Any 1-qubit unitary

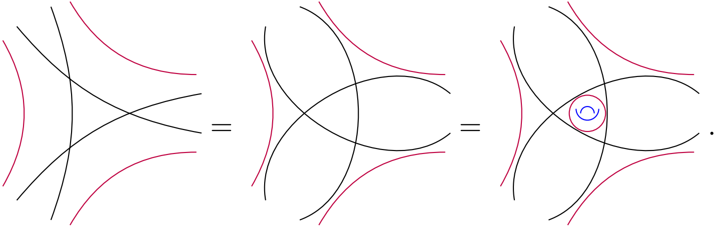
$$U = e^{i\theta_1 Z} e^{i\theta_2 X} e^{i\theta_3 Z} = e^{i\alpha_1 X} e^{i\alpha_2 Z} e^{i\alpha_3 X}.$$

Star-Triangle Equation

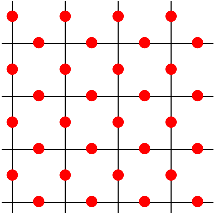
Star-triangle equation in tensor network:
(Recall that a red bullet represents a tensor $a(|00\rangle + |11\rangle) + b(|01\rangle + |10\rangle)$.)



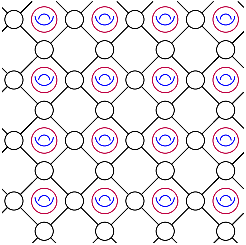
Yang-Baxter equation + string genus relations in quon language: (The crossings have parameters.)



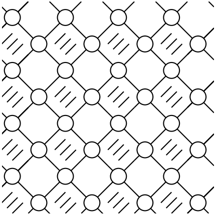
Tensor Network



Quon Language



Spin Model



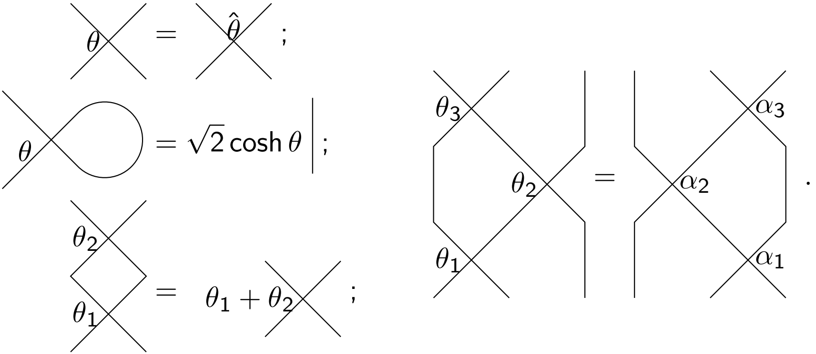
Recall that the red bullet in a tensor $a(|00\rangle + |11\rangle) + b(|01\rangle + |10\rangle)$, corresponding to a parameterized crossing of the second and the third string in quon language.

Kramer-Wannier duality: switch the position of the string-genus pairs.

Simulation of Matchgates

Matchgate tensor network: $e^{\theta X}$, $\forall \theta \in \mathbb{C}$, identity tensor $|000\rangle + |111\rangle$.
Matchgate tensor network reduces to a planar 4-valent graph without genus, like a 2D projection of a link with parameters at crossings.

It can be evaluated efficiently using the Yang-Baxter relation (Liu 15):

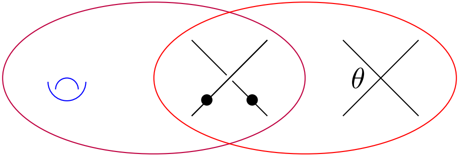


The last equation holds for $\theta_1, \theta_2, \theta_3 \in \mathbb{C}$ with probability 1.

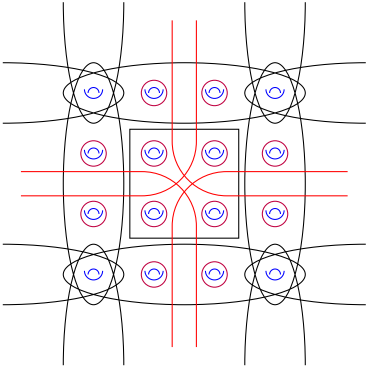
Topological Complexity: Unified Framework

(1.1) planar strings
(1.2) charges
(1.3) braids
(2a) surface/genus
(2b) parameterized crossings

Clifford gates: (1)+(2a)
Matchgates: (1)+(2b)
Universal gates: (1)+(2a)+(2b)



New Efficiently Simulable Tensor Networks



Red crossings have arbitrary parameters;
The other crossings are braids;
Black strings are above red strings.

This quon picture represents the local fractional data of a tensor network, neither Clifford nor matchgate, on a square lattice.
Interpretation: Ising model for interacting fermions.
Its partition function is the sum of two partition functions for free fermions.
Chain Reaction: Replacing one genus by a sum of two terms using boundary relations, then the rest genus can be removed using bulk relations.

Outlook

Topological Design of

- Communication Protocols
- Quantum Error Correcting Codes
- Exactly solvable models
- Simulable tensor networks

Quantum circuit complier for simplifying circuits
Polynomial algorithms (for combinatorial/graphical problems)
Exactly solvable models with physics implementations
. . .

Thank you!