Lattice nonlinear Schrödinger: history and open problems

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The model has many names: Lieb-Liniger, Bose gas with delta interaction, Tonks-Girardeau, nonlinear Schödinger. It has application in quantum optics. It was build in optical lattice by N. J. van Druten https://arxiv.org/pdf/0709.1899.pdf

The Hamiltonian is:

$$H = \int dx(\partial_x \psi^\dagger \partial_x \psi + \kappa \psi^\dagger \psi^\dagger \psi \psi),$$

$$\{\psi(x), \psi^\dagger(y)\} = i\delta(x - y)$$

It is integrable both in classical and quantum cases. It has infinitely many conservation laws and Lax representation https://en.wikipedia.org/wiki/Lax_pair.

$$\partial_t L_n = M_{n+1}(\lambda) L_n(\lambda) - L_n(\lambda) M_n(\lambda)$$

An integer n is a discrete space variable $x = n\Delta$ and Δ is lattice spacing.

R matrix

We shall discuss the quantum case. Lax representation follows from Yang-Baxter equation. https://arxiv.org/pdf/0910.0295.pdf

$$R(\lambda,\mu)\left(L_n(\lambda)\bigotimes L_n(\mu)\right)=\left(L_n(\mu)\bigotimes L_n(\lambda)\right)R(\lambda,\mu)$$

The $R(\lambda,\mu)$ solves the Yang-Baxter equation :
C.N. Yang Phys. Rev. Lett. 19, (1967) 1312-1314
https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.19.1312
Yang found the following solution:

$$R(\lambda,\mu)=i\mathbb{1}+(\mu-\lambda)\Pi$$

This is the same R matrix, which describes spin 1/2 XXX Heisenberg chain. Here Π is permutation. C.N. Yang Phys. Rev. 168, (1968) 1920
https://journals.aps.org/pr/abstract/10.1103/PhysRev.168.1920

Note

Note that for $\lambda-\mu=i$ the $R(\lambda,\mu)=i\,(\mathbb{1}-\Pi)$ is degenerate. It is 1D projector.

The model was embedded into quantum inverse scattering method https://en.wikipedia.org/wiki/Quantum_inverse_scattering_method.
We shall use algebraic Bethe Ansatz https://arxiv.org/abs/hep-th/9605187
The orthogonality and completeness of the Bethe Ansatz eigenstates was proved by Teunis C. Dorlas in 1993.
An approximate L operator on a dense lattice was discovered by Faddeev , Sklyanin and Takhtajan in 1979

$$L_n(\lambda) = \begin{pmatrix} 1 - \frac{i\lambda\Delta}{2} & -i\sqrt{\kappa}\chi_n^\dagger \\ i\sqrt{\kappa}\chi_n & 1 + \frac{i\lambda\Delta}{2} \end{pmatrix} + O(\Delta^2),$$

An integer n is a discrete space variable $x = n\Delta$ and Δ is lattice spacing.
The χ_n is the quantum field:

$$[\chi_n, \chi_m^\dagger] = \Delta\, \delta_{nm}$$

$\chi = \psi\Delta$ in the limit $\Delta \to 0$.
The λ is the spectral parameter and κ is a coupling constant.

Exact lattice Lax operator (all orders in Δ) was constructed by A.G. Izergin and V. E. Korepin in Doklady Akademii Nauk, 1981
https://arxiv.org/pdf/0910.0295.pdf
see also Nuclear Physics B 205 [FS5], 401, 1982

$$L_j(\lambda) = \begin{pmatrix} 1 - \frac{i\lambda\Delta}{2} + \frac{\kappa}{2}\chi_j^\dagger \chi_j & -i\sqrt{\kappa}\chi_j^\dagger \varrho_j \\ i\sqrt{\kappa}\varrho_j \chi_j & 1 + \frac{i\lambda\Delta}{2} + \frac{\kappa}{2}\chi_j^\dagger \chi_j \end{pmatrix}.$$

$$[\chi_j, \chi_l^\dagger] = \Delta \delta_{j,l} \quad \text{and} \quad \varrho_j = (1 + \frac{\kappa}{4}\chi_j^\dagger \chi_j)^{\frac{1}{2}},$$

here $\kappa > 0$, and $\Delta > 0$.

The same R matrix.

Equivalence of NLS and XXX chain with negative spin

We can rewrite the L operator as XXX Heisenberg chain
Zeitschrift für Physik 49, (1928) 619-636
https://link.springer.com/article/10.1007/BF01328601:

$$L_j^{XXX} = -\sigma^z L_j = i\lambda + S_j^k \otimes \sigma^k$$

$$S_j{}^+ = -i\sqrt{\kappa}\chi_j^\dagger \varrho_j, \quad S_j{}^- = i\sqrt{\kappa}\varrho_j \chi_j, \quad S_j{}^z = (1 + \frac{\kappa}{2}\chi_j^\dagger \chi_j).$$

representation of $SU(2)$ algebra with negative spin $s = -\frac{2}{\kappa\Delta}$.

The σ are Pauli matrices.

Antipode and quantum determinant

The monodromy matrix and transfer matrix:

$$T_0(\lambda) \equiv L_{0\mathcal{L}}(\lambda) \dots L_{01}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}, \qquad \tau(\lambda) = tr_0\, T_0(\lambda).$$

$R(\lambda,\mu)\,(T_0(\lambda) \bigotimes T_0(\mu)) = (T_0(\mu) \bigotimes T_0(\lambda))\,R(\lambda,\mu)$

The antipode of quantum monodromy matrix is:

$$T(\lambda)^{-1} = {d_q}^{-L}(\lambda)\sigma^y T^t(\lambda + i)\sigma^y$$

$$d_q(\lambda) = \Delta^2(\lambda - \nu)(\lambda - \nu + i)/4 \qquad \nu = -2i/\Delta$$

This is similar to Cramer's formula

http://pi.math.cornell.edu/~andreim/Lec17.pdf

The difference is a shift of the spectral parameter by i.

The denominator is the *quantum determinant*:

$\det_q T(\lambda) = A(\lambda)D(\lambda + i) - B(\lambda)C(\lambda + i) = {d_q}^L(\lambda)$

This was discovered in 1981 in https://arxiv.org/pdf/0910.0295.pdf

Remark

When $\lambda - \mu = i$ the R matrix turns into one dimensional projector.

The Hamiltonian is: $\mathcal{H} = \sum_{k=1}^{L} H_{k,k+1}$
The density of the Hamiltonian is

$$H_{k,k+1} = -\psi(-J_{k,k+1}) - \psi(J_{k,k+1} + 1) + \psi(2).$$

Here $\psi(x) = d \ln \Gamma(x)/dx$ and

$$J_{k,k+1}(J_{k,k+1} + 1) = 2\vec{S}_k \otimes \vec{S}_{k+1}$$

Can be solved by Vieta's formulas.
QCD describes deep inelastic scattering. The scattering of a lepton on a baryon is a sum of Feynman diagrams. In leading logarithmic approximation ladder diagrams dominate. Quarks exchange gluons. The Hamiltonian describing interactions of the gluons is this spin chain.
L. Lipatov https://arxiv.org/pdf/hep-th/9311037.pdf
L. Alvarez-Gaume, https://arxiv.org/pdf/0804.1464.pdf
Our paper: https://arxiv.org/pdf/1909.00800.pdf
This is a part of larger trend: spin chains are related to 4D Yang-Mills
N. A. Nekrasov, S. L. Shatashvili https://arxiv.org/pdf/0908.4052.pdf

An example is XXX chain with spin $s = -1$.
The Bethe equations are https://en.wikipedia.org/wiki/Bethe_ansatz:

$$\left(\frac{\lambda_k + is}{\lambda_k - is}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{N} \frac{\lambda_k - \lambda_j + i}{\lambda_k - \lambda_j - i} \xrightarrow{s=-1} \left(\frac{\lambda_k - i}{\lambda_k + i}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{N} \frac{\lambda_k - \lambda_j + i}{\lambda_k - \lambda_j - i}$$

$$k = 1, \cdots, N$$

These are periodic boundary conditions. The energy is:

$$E \equiv \sum_{j=1}^{N} \frac{1}{i} \frac{d}{d\lambda_j} \ln \frac{\lambda_j + i}{\lambda_j - i} = \sum_{j=1}^{N} \frac{-2}{\lambda_j^2 + 1},$$

Theorem 1. All solutions of Bethe equations for $s = -1$ are **real numbers**.

$$\left(\frac{\lambda_k - i}{\lambda_k + i}\right)^L = \prod_{j=1,\ j\neq k}^{N} \frac{\lambda_k - \lambda_j + i}{\lambda_k - \lambda_j - i}, \qquad k = 1, \cdots, N.$$

Proof: Let us use the following properties:

$$\text{LHS: } \left|\frac{\lambda - i}{\lambda + i}\right| \leq 1, \quad \text{when Im}\lambda \geq 0; \quad \left|\frac{\lambda - i}{\lambda + i}\right| \geq 1, \quad \text{when Im}\lambda \leq 0;$$

$$\text{RHS: } \left|\frac{\lambda + i}{\lambda - i}\right| \geq 1, \quad \text{when Im}\lambda \geq 0; \quad \left|\frac{\lambda + i}{\lambda - i}\right| \leq 1, \quad \text{when Im}\lambda \leq 0;$$

If we denote the one with maximal imaginary part as $\lambda_{max} \in \{\lambda_j\}$, then

$$\text{Im } \lambda_{max} \geq \text{Im } \lambda_j, \quad j = 1, \cdots, N.$$

For $\lambda_k = \lambda_{max}$

$$\left|\frac{\lambda_{max} - i}{\lambda_{max} + i}\right|^L = \left|\prod_{j=1}^{N} \frac{\lambda_{max} - \lambda_j + i}{\lambda_{max} - \lambda_j - i}\right| \geq 1.$$

Due to LHS, this results in: $\text{Im } \lambda_j \leq \text{Im } \lambda_{max} \leq 0$

Similarly, one has $0 \leq \text{Im } \lambda_{min} \leq \text{Im } \lambda_j \to$ so $\text{Im } \lambda_j = 0$

The logarithm of **Bethe equations** for the model $s = -1$,

$$2\pi n_k = \sum_{j=1}^{N} \theta(\lambda_k - \lambda_j) + L\,\theta\left(\lambda_k\right),$$

Here n_k are different integer (or half integer) numbers: Pauli principle
http://insti.physics.sunysb.edu/~korepin/PDF_files/Pauli.pdf

$$\theta(\lambda) = -\theta(-\lambda) = i \ln\left(\frac{i\kappa + \lambda}{i\kappa - \lambda}\right); \quad -\pi < \theta(\lambda) < \pi, \quad \mathrm{Im}\,\lambda = 0$$

$$\theta'(\lambda - \mu) = K(\lambda, \mu) = \frac{2\kappa}{\kappa^2 + (\lambda - \mu)^2}, \qquad K(\lambda) = K(\lambda, 0).$$

Theorem 2. The solutions of the logarithmic form **Bethe equations** exist. Logarithmic Bethe equations are the extremums of the Yang action:

$$S = L \sum_{k=1}^{N} \theta_1(\lambda_k) + \frac{1}{2} \sum_{k,j}^{N} \theta_1(\lambda_k - \lambda_j) - 2\pi \sum_{k=1}^{N} n_k \lambda_k,$$

$\theta_1(\lambda) = \int_0^\lambda \theta(\mu) d\mu$. Bethe equations: $\partial S / \partial \lambda_j = 0$.

$$\frac{\partial^2 S}{\partial \lambda_j \partial \lambda_l} = \delta_{jl}[L\, K(\lambda_j) + \sum_{m=1}^{N} K(\lambda_j, \lambda_m)] - K(\lambda_j, \lambda_l)$$

Consider some real vector v_j . The quadratic form is positive:

$$\sum_{j,l} \frac{\partial^2 S}{\partial \lambda_j \partial \lambda_l} v_j v_l = \sum_{j=1}^{N} L\, K(\lambda_j) v_j^2 + \sum_{j>l}^{N} K(\lambda_j, \lambda_l)(v_j - v_l)^2 \geq 0$$

The $K(\lambda_j)$ are positive. The action is convex: it has unique minimum. Solution of Bethe equation exists and unique.

The same second derivative appears later in the theory.
The square of the norm of the Bethe wave function is a determinant:

$$\langle \Phi_N | \Phi_N \rangle = \det{}_N \left(\frac{\partial^2 S}{\partial \lambda_j \partial \lambda_l} \right)$$

M. Gaudin conjectured in 1972. V.E. Korepin proved in 1982
http://insti.physics.sunysb.edu/~korepin/PDF_files/norm.PDF

$$\frac{\partial^2 S}{\partial \lambda_j \partial \lambda_l} = \delta_{jl}[L\, K(\lambda_j) + \sum_{m=1}^{N} K(\lambda_j, \lambda_m)] - K(\lambda_j, \lambda_l)$$

The thermodynamic limit at zero temperature

For positive κ all λ_j has to be different: Pauli principle in the momentum space is valid.
A.G. Izergin and V.E. Korepin; Letters in Mathematical Physics 1982:
http://insti.physics.sunysb.edu/~korepin/PDF_files/Pauli.pdf
In the limit $L \to \infty$ and $N \to \infty$, the λ_j are condensed into Fermi sphere $[-q, q]$.
The distribution function $\rho_p(\lambda_j) = \frac{1}{L(\lambda_{j+1} - \lambda_j)}$ satisfy:

$$2\pi\rho_p(\lambda) = \int_{-q}^{q} K(\lambda, \mu)\rho_p(\mu)d\mu + K(\lambda)$$

$$K(\lambda, \mu) = \frac{2\kappa}{\kappa^2 + (\lambda - \mu)^2}, \quad K(\lambda) = K(\lambda, 0), \quad \int_{-q}^{q} \rho_p(\lambda)d\lambda = D = \frac{N}{L}$$

For $\kappa = 0$ Fermi sphere collapse: the ground state is Bose-Einstein condensate.

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Open problem:

Is to analyze the integral equation in the limit of $\kappa \to 0$
The $K(\lambda, \mu) \to 2\pi\delta(\lambda - \mu)$: the integral cancel the LHS ...?

In the continuous case of nonlinear Schrödinger the integral equation is:

$$2\pi\rho_p(\lambda) = \int_{-q}^{q} K(\lambda,\mu)\rho_p(\mu)d\mu + 1.$$

In the limit $\kappa \to 0$ the $K(\lambda,\mu) \to 2\pi\delta(\lambda-\mu)$.The integral cancel the LHS. The limit was studied by

S. Prolhac https://arxiv.org/pdf/1610.08912.pdf

G. Lang https://arxiv.org/pdf/1907.04410.pdf

C. Tracy, H. Widom https://arxiv.org/pdf/1609.07793.pdf

The decomposition is in $\sqrt{\kappa}$ and $\log\kappa$. Coefficients are objects of number theory.

In the lattice case the limit is an open problem.

Energy of elementary excitation

Return to the lattice case: special case XXX with spin $s = -1$. Considering the grand canonical ensemble, the energy spectrum becomes

$$E_h = \sum_{j=1}^{N} (\frac{-2}{\lambda_j^2 + 1} - h).$$

The h is the chemical potential. In thermodynamic limit the energy of elementary excitation $\varepsilon(\lambda)$ satisfies the linear integral equation

$$\varepsilon(\lambda) - \frac{1}{2\pi} \int_{-q}^{+q} K(\lambda, \mu) \varepsilon(\mu) d\mu = \frac{-2}{\lambda^2 + 1} - h \equiv \varepsilon_0(\lambda),$$

$$\varepsilon(q) = \varepsilon(-q) = 0$$

Remark

The elementary excitation has a topological charge: it does not fit into periodical boundary conditions, we have to change the boundary conditions into anti-periodic.

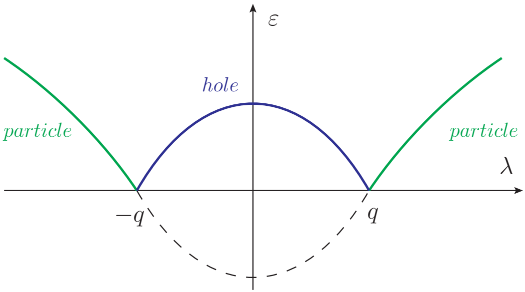

Figure 1: The energy of the elementary excitation as a function of λ.
For $-q < \lambda < q$ elementary excitation is a hole, but it is the particle for other values of λ.

In the infinite volume limit any energy level is a scattering state of several elementary excitations with different momenta.

The momentum of the particle $\mathbf{k}(\lambda_p)$ is

$$\mathbf{k}(\lambda_p) = p_0(\lambda_p) + \int_{-q}^{q} \theta(\lambda_p - \mu)\rho_p(\mu) d\mu, \quad \theta(\lambda) = p_0(\lambda) = i \ln\left(\frac{i+\lambda}{i-\lambda}\right).$$

The momentum $\mathbf{k}_h(\lambda_h)$ of elementary hole excitation is

$$\mathbf{k}_h(\lambda_h) = -p_0(\lambda_h) - \int_{-q}^{q} \theta(\lambda_h - \mu)\rho_p(\mu) d\mu.$$

where $-q < \lambda_h < q$. At zero temperature all the observables are described by a linear integral equation.

Scattering matrix

The scattering matrix of two elementary excitation is a transition coefficient:

$$S = \exp\{-i\phi(\lambda_p, \lambda_h)\},$$

the scattering phase satisfies the integral equation:

$$\phi(\lambda_p, \lambda_h) - \frac{1}{2\pi}\int_{-q}^{+q} K(\lambda_p, \nu)\phi(\nu, \lambda - \lambda_h)d\nu = \theta(\lambda_p - \lambda_h).$$

$$\theta(\lambda) = -\theta(-\lambda) = i\ln\left(\frac{i\kappa + \lambda}{i\kappa - \lambda}\right); \quad -\pi < \theta(\lambda) < \pi, \quad \mathrm{Im}\,\lambda = 0$$

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Remark

Many body scattering matrix is a product of pairwise scattering matrices. This can be used as a definition of complete integrability in many body quantum mechanics.

The Yang-Yang equation describes thermodynamics

$$\varepsilon(\lambda) = \frac{-2}{\lambda^2+1} - h - \frac{T}{2\pi}\int_{-\infty}^{+\infty} K(\lambda,\mu)\ln(1+e^{-\varepsilon(\mu)/T})d\mu,$$

$$\frac{\rho_h(\lambda)}{\rho_p(\lambda)} = e^{\varepsilon(\lambda)/T}, \qquad D = \frac{N}{L} = \int_{-\infty}^{\infty} \rho_p(\lambda)d\lambda.$$

The rigorous proof was given by Tony Dorlas.

Remark

Stable excitation exists even at positive temperature, because of the conservation laws. The $\varepsilon(\lambda)$ is the energy of this excitaiton.

The free energy is:

$$\mathcal{F} = Nh - \frac{LT}{2\pi}\int_{-\infty}^{+\infty} K(\mu)\ln(1+\exp(-\varepsilon(\mu)/T))d\mu.$$

The pressure is:

$$\mathcal{P} = -\left(\frac{\partial \mathcal{F}}{\partial L}\right)_T = \frac{T}{2\pi}\int_{-\infty}^{+\infty} K(\mu)\ln(1+e^{-\varepsilon(\mu)/T})d\mu.$$

Thermal entropy is:

$$S = -\frac{\partial \mathcal{F}}{\partial T} = \frac{L}{2\pi}\int_{-\infty}^{+\infty} K(\mu)\left[\ln(1+e^{-\varepsilon(\mu)/T}) + \frac{\varepsilon(\mu)}{T(e^{\varepsilon(\mu)/T}+1)}\right]d\mu.$$

Entanglement entropy

At zero temperature the ground state $|gs\rangle$ is unique. The entropy of the ground state is zero. Let us consider a block of x sequential lattice cites. We interpret the rest of the lattice as an environment. We trace away the environment: this will give us the density matrix of the block $\rho = tr_E\left(|gs\rangle\langle gs|\right)$. The entropy of the block is a complicated function of x, but for large x it scales logarithmically

$$-tr(\rho \log \rho) \to \frac{1}{3}\log(x) \quad \text{as} \quad x \to \infty$$

similar to the continuous case
https://arxiv.org/pdf/cond-mat/0311056.pdf
Open problem is to study time evolution of entanglement entropy.

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The Renyi entropy is defined as

$$S_R = \frac{tr\rho^{\alpha}}{1-\alpha}$$

with $\alpha > 0$. It also scales logarithmically

$$S \rightarrow \frac{(1+\alpha^{-1})\log x}{6}$$

as in XX spin chain https://arxiv.org/pdf/quant-ph/0304108.pdf

Can we calculate correlation functions?
First at zero temperature, time independent in the infinite volume.

At spin 1/2 correlation functions in XXX chain can be expressed as polynomials [with rational coefficients] of the values of Riemann zeta function with odd arguments
H.E. Boos, V.E. Korepin https://arxiv.org/pdf/hep-th/0104008.pdf
T. Miwa, F. Smirnov https://arxiv.org/pdf/1802.08491.pdf

The values of Riemann zeta function with odd arguments are celebrated object of number theory. They are conjectured to be transcendental numbers, algebraically independent over the field of rational numbers, see wikipedia: Apery's theorem.

Open problems

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Open problem

Can we describe correlation functions of the XXX with negative spin by number theory?

Extra symmetry in the infinite volume is an open problem.

At spin $1/2$ the XXX chain gains an additional symmetry in the thermodynamic limit. It is the YANGIAN SYMMETRY (infinite dimensional quantum group):
https://arxiv.org/abs/hep-th/9211133v2

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Open problem

Does an additional symmetry arise in XXX with negative spin in the limit of infinitely long lattice?

Side remark

M. J. Ablowitz and J. F. Ladik in J. Math. Phys. 17, (1976) 1011 constructed a different integrable discretization of nonlinear Schoedinger. The L-operator is different. The equations of motion are simple.

$$\frac{i}{2}\frac{\partial}{\partial t}\psi(n,t) = (1+4\psi(n,t)\psi^{\dagger}(n,t))(\psi(n+1,t)+\psi(n-1,t))$$

$$-\frac{i}{2}\frac{\partial}{\partial t}\psi^{\dagger}(n,t) = (1+4\psi(n,t)\psi^{\dagger}(n,t))(\psi^{\dagger}(n+1,t)+\psi^{\dagger}(n-1,t)).$$

In classical case Tim Hoffmann proved that the L operator of Ablowiz-Ladik model is gauge equivalent to the one, which we presenting here. Physics Letters A 265 (2000) 62-67. http://insti.physics.sunysb.edu/~korepin/PDF_files/Hoff.pdf

Also Hamiltonian structures are different. The gauge matrix depends on dynamical variable ψ. The R matrices are different. Non-uniqueness of quantization.

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The non-uniqueness of quantization in integrable models was first found in massive Thiring model: Commun. Math. Phys. 76, (1980) 165-176
http://insti.physics.sunysb.edu/~korepin/PDF_files/NewEff.pdf

In classical case Tim Hoffmann constructed an integrable double discrete version of nonlinear Schoedinger and related it to geometry:
Discrete Hashimoto Surfaces and a Doubly Discrete Smoke-Ring Flow.
Discrete Differential Geometry, (2008), Vol. 38, pp 95-115.
https://link.springer.com/chapter/10.1007%2F978-3-7643-8621-4_5.

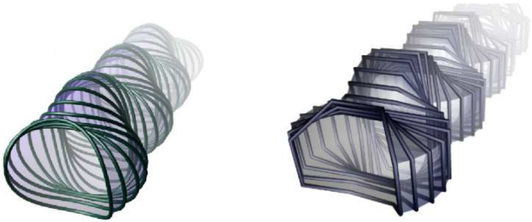

Figure 2: An oval curve under the Hashimoto flow and the discrete evolution.

Tropical geometry https://en.wikipedia.org/wiki/Tropical_geometry

Open problem

The field was developed by Alexander Bobenko and Yuri Suris.
The book: Discrete Differential Geometry, Integrable Structure
https://books.google.com/books/about/Discrete_Differential_Geometry.html?id=H1u1OanYfigC
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Open problem

Quantization of the double discrete nonlinear Schrödinger.

Cellular automata

In classical case one can also discretize the value of the unknown function [the field]. This leads us to cellular automata https://en.wikipedia.org/wiki/Cellular_automaton

Mathematicians worked on classical completely integrable cellular automata [CA]
https://scholar.google.com/scholar?q=completely+integrable+cellular+automata&hl=en&as_sdt=0&as_vis=1&oi=scholart
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Has Lax representation.

Open problem

Construction of quantum completely integrable cellular automata.

Review on quantum CA is https://arxiv.org/pdf/1904.13318.pdf
The book by Gerard 't Hooft : https://www.springer.com/gp/book/9783319412849
A special case was analytically solved case by means of MPS:
https://arxiv.org/abs/2012.12256
MPS is related to algebraic Bethe Ansatz
https://arxiv.org/pdf/1201.5627.pdf
https://arxiv.org/pdf/1201.5636.pdf
Can we relate this version of quantum CA to Yang-Baxter?

Related models

It would be important to study related models, with similar problems:
The sine-Gordon model :
A.G. Izergin, V.E. Korepin: Nuclear Physics B 205 [FS5], 401, 1982
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