TOPOLOGICAL ORDERS FROM CENTERS IN (3+1)D

ALEX BULLIVANT
MAYNOOTH

Topological Phases of Matter

Topological Phases of Matter

- Quantum phases of matter

  Quantum effects dominate physics not thermal as in classical case

- Gapped

  Excited states have finite energy gap from G.S. in ∞ - volume limit

- Local

Topological Phases of Matter

Defn: Path connected component in space of gapped, local QFT's

continuous family of gapped QFT's given by varying parameters of model w/o

Topological Phases of Matter

Defn: Path connected component in space of gapped, local QFT's

* more generally can require further properties eg: symmetry and ask for continuous family preserving

– symmetry enriched top phase

continuous family of gapped QFT's given by varying parameters of model w/o

Topological Phases of Matter

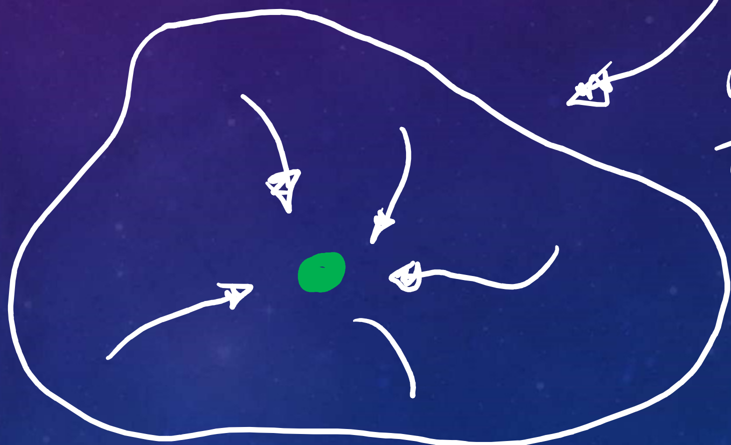
Defn: Path connected component in space of gapped, local QFT's

* Universal properties (low energy limit)

effective field theory ⇒

Topological Quantum Field Theory

TQFT



taking renormalisation group flow to low energy limit

TQFT is fixed point

TQFT

Defn: QFT whose correlation functions independent of metric structure of space-time

* partition function on closed space-time = diffeomorphism invariant

* TQFT describing phase defines the Topological Order

Algebraic Model of Topological Order

Conjecture (Kong + Wen 1405.5858)

(n+1)D topological orders admit an algebraic description via non-degenerate braided fusion (n-1)-categories

Algebraic Model of Topological Order

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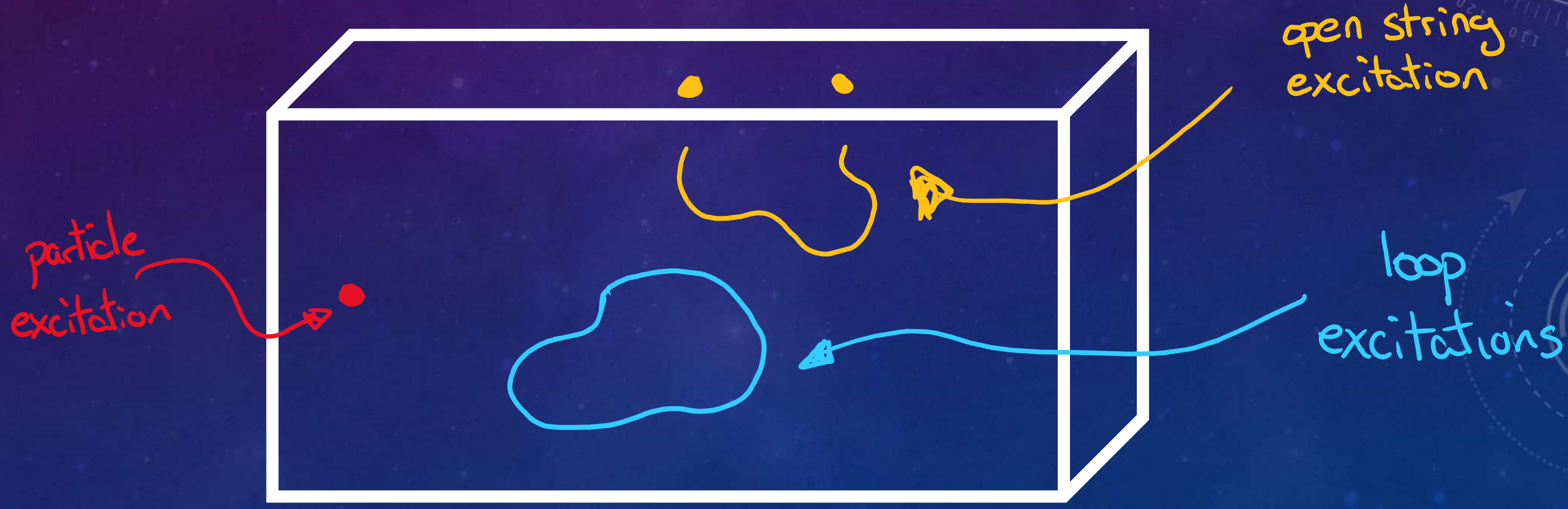
(3+1)D topological orders admit an algebraic description via non-degenerate braided fusion 2-categories

* In this talk we will consider n=3 example!

Rules: * Assign f.d. Hilbert space to 3-disk D^3 w. massive topological excitations

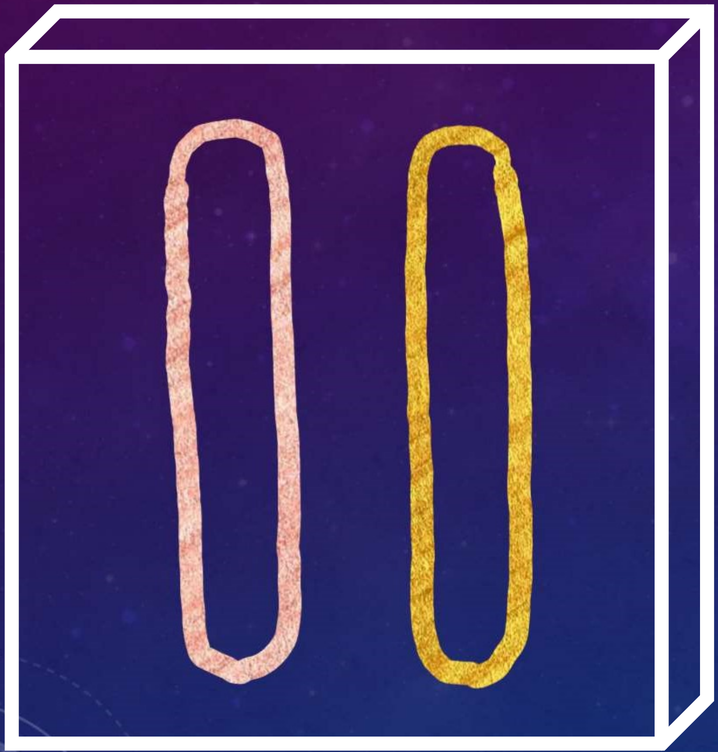
- Top excitations ~ measurable at all length scales + properties protected topologically/non-locally.

* Linear map to diffeomorphism class of D^4 space-time cobordism



Fusion 2-Category Roughly

* Book keeping device for physics of 3+1D topological order

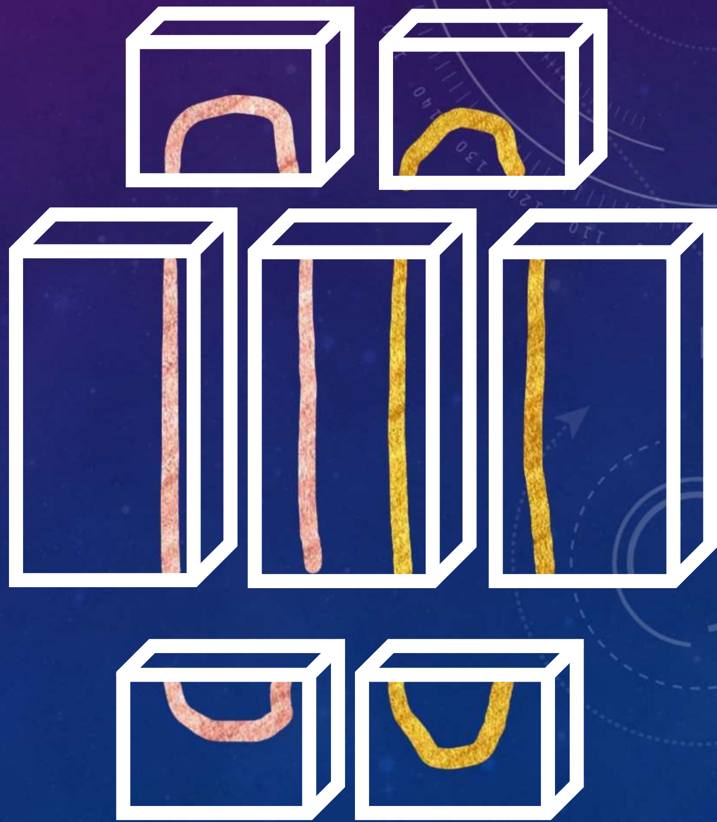


collection of rules to define Hilbert space

$\simeq$

for 3-Disk w. excit from "elementary" building blocks

"cut" Hilbert space into tensor factors using locality



Motion Group Representations

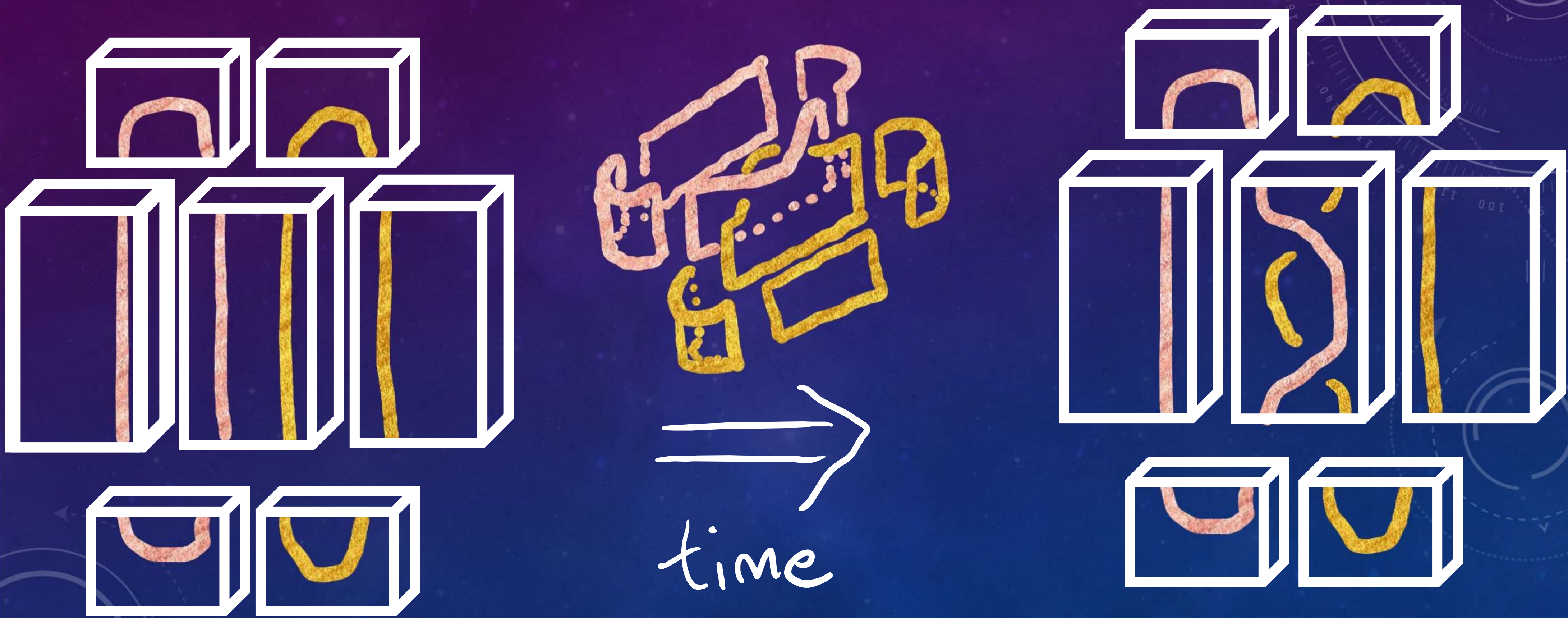
* TQFT $\Rightarrow$ $H|\psi\rangle = 0$ trivial time evolution on $M^n \times I$

__However__ Topologically non-trivial space-time leads to dynamics. (Topological Quantum Computing)

Amplitude depends on ISOTOPY class of loop world-sheet in $D^4 = [0,1]^4$ space-time

Braided Fusion 2-category Roughly

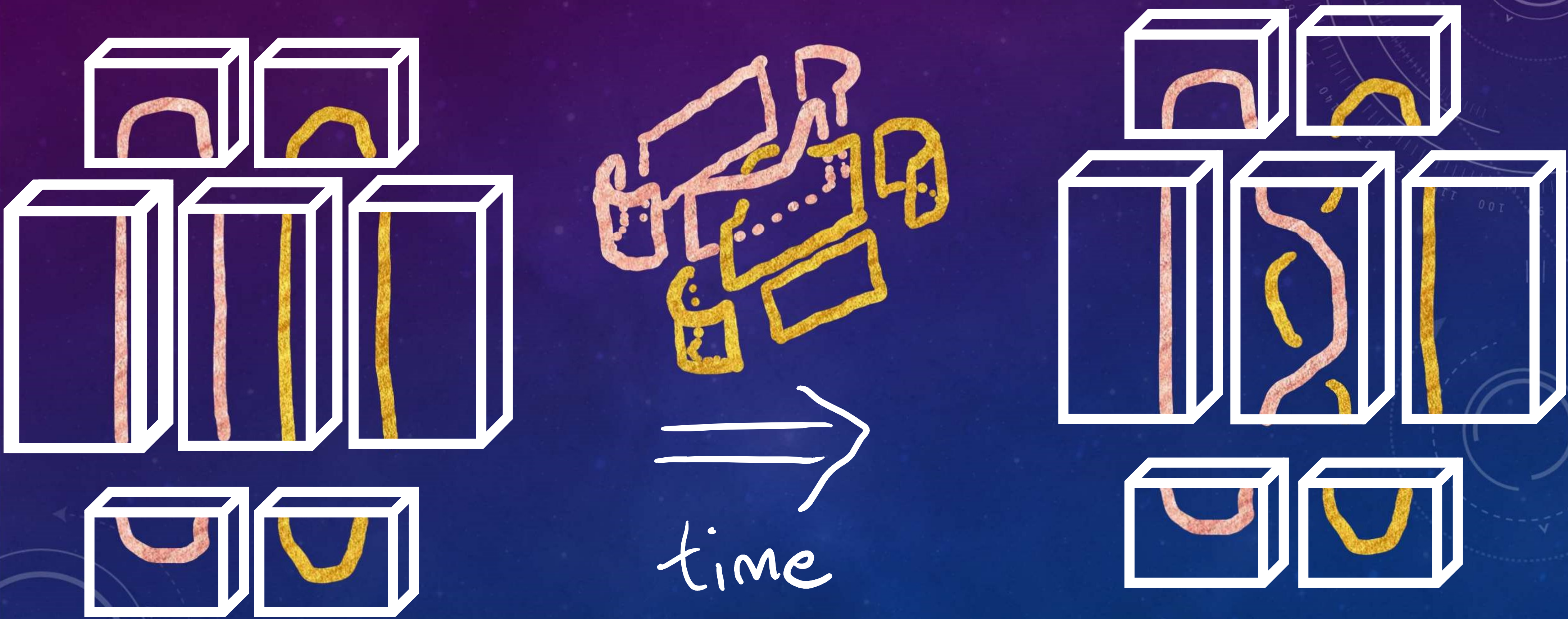
* Allows us to discuss "under and over crossing" ≡ braiding
* Non-degeneracy – braiding allows us to distingush all excitations!



Braided Fusion 2-category Roughly

* Allows us to discuss "under and over crossing" ≡ braiding
* Non-degeneracy – braiding allows us to distinguish all excitations!

possible tension:
data in finite region D^3
but gapped defn requires ∞-volume?



Categorification

"the process of replacing sets with categories"

Categories

$$\begin{tikzcd}
X \arrow[loop left, "1_x"] \arrow[r, bend left, "a"] \arrow[r, "m"] \arrow[r, bend right, "n"'] & Y \arrow[loop below, "1_y"] \arrow[r, "b"] & Z \arrow[loop right, "1_z"]
\end{tikzcd}$$

Categories

$$\begin{tikzcd}
X \arrow[loop left, "1_X"] \arrow[r, bend left, "a"] \arrow[r, "m"] \arrow[r, bend right, "n"'] & Y \arrow[loop below, "1_Y"] \arrow[r, "b"] & Z \arrow[loop right, "1_Z"]
\end{tikzcd}$$

Set of OBJECTS $\{X, Y, Z, \ldots\}$

Categories

$$\begin{tikzcd} X \arrow[loop left, "1_X"] \arrow[r, bend left=50, "a"] \arrow[r, "m"] \arrow[r, bend right=50, "n"'] & Y \arrow[loop below, "1_Y"] \arrow[r, "b"] & Z \arrow[loop right, "1_Z"] \end{tikzcd}$$

Set of OBJECTS $\{X, Y, Z, \dots\}$

For each pair of objects, a set $\hom(X,Y) = \{X \xrightarrow{n} Y, X \xrightarrow{a} Y, \dots\}$ of MORPHISMS

Categories

$$
\begin{tikzcd}
X \arrow[loop left, "1_X"] \arrow[rr, bend left=20, "a"] \arrow[rr, "m"] \arrow[rr, bend right=20, "n"'] \arrow[rrrr, bend left=45, "ab"] & & Y \arrow[loop below, "1_Y"] \arrow[rr, "b"] & & Z \arrow[loop right, "1_Z"]
\end{tikzcd}
$$

Set of OBJECTS $\{X, Y, Z, \dots\}$

For each pair of objects, a set $\hom(X,Y) = \{X \xrightarrow{n} Y, X \xrightarrow{a} Y, \dots\}$ of MORPHISMS

For each triple of objects, a COMPOSITION function

$$\circ : \hom(X,Y) \times \hom(Y,Z) \longrightarrow \hom(X,Z)$$

Categories

$$
\begin{tikzcd}
X \arrow[loop left, "1_X"] \arrow[r, bend left, "a"] \arrow[r, "m"] \arrow[r, bend right, "n"'] & Y \arrow[loop below, "1_Y"] \arrow[r, "b"] & Z \arrow[loop right, "1_Z"]
\end{tikzcd}
$$

Such that

1) + 2)

are satisfied

1) Composition is ASSOCIATIVE

$$X \xrightarrow{a} Y \xrightarrow{b} Z \xrightarrow{c} W = X \xrightarrow{a(bc)} W = X \xrightarrow{(ab)c} W$$

2) $\forall X \in$ objects, $\exists\, X \xrightarrow{1_X} X \in hom(X, X)$ called UNIT

$$X \xrightarrow{1_X} X \xrightarrow{a} Y = X \xrightarrow{a} Y = X \xrightarrow{a} Y \xrightarrow{1_Y} Y$$

Examples:

1) Given a set $S = \{x, y, z, \dots\}$ $\Rightarrow$ "discrete category"

$$\begin{tikzcd} x \arrow[loop above, "1_x"] & y \arrow[loop above, "1_y"] & z \arrow[loop above, "1_z"] & \cdots \end{tikzcd}$$

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- a category is a set with "relations" between elements

Examples:

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- a category is a set with "relations" between elements

2) Given a finite dim k-alg A $\Rightarrow$ "$\mathrm{Mod}(A)$", objects $\equiv$ (right) A-modules, morphisms $\equiv$ intertwiners

* here $\hom_{\mathrm{Mod}(A)}(\rho, \sigma)$ is not just a <u>set</u> but <u>k-vector space</u>

$$\forall\ \rho \xrightarrow{f} \sigma \ \ \&\ \ \rho \xrightarrow{g} \sigma, \quad \rho \xrightarrow{a \cdot f + b \cdot g} \sigma$$

* We say $\mathrm{Mod}(A)$ is k-linear

Categorification

Defn: The process of generating new mathematical structures by replacing sets $\longmapsto$ categories

functions $\longmapsto$ functors

Functors

Given a pair of categories $\mathcal{C}$ and $\mathcal{D}$ a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$

consists of : a function $F^0 : ob(\mathcal{C}) \longrightarrow ob(\mathcal{D})$

a function $F^1 : hom(\mathcal{C}) \longrightarrow hom(\mathcal{D})$

s.t. $F(x \xrightarrow{f} y \xrightarrow{g} z) = F(x \xrightarrow{f} y) \circ_{\mathcal{D}} F(y \xrightarrow{g} z)$

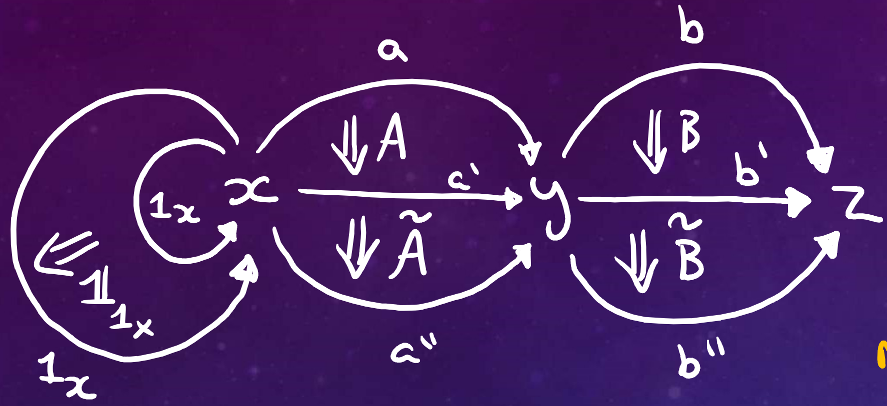
2-Categories (categorified categories)

$$\begin{tikzcd}[column sep=huge]
x \arrow[loop, in=120, out=240, distance=8mm, "1_x"'] \arrow[loop, in=100, out=260, distance=25mm, "1_x"'] \arrow[r, bend left=60, "a"] \arrow[r, "a'" description] \arrow[r, bend right=60, "a''"'] \arrow[r, phantom, bend left=30, "\Downarrow A"] \arrow[r, phantom, bend right=30, "\Downarrow \tilde{A}"] & y \arrow[r, bend left=60, "b"] \arrow[r, "b'" description] \arrow[r, bend right=60, "b''"'] \arrow[r, phantom, bend left=30, "\Downarrow B"] \arrow[r, phantom, bend right=30, "\Downarrow \tilde{B}"] & z
\end{tikzcd}$$

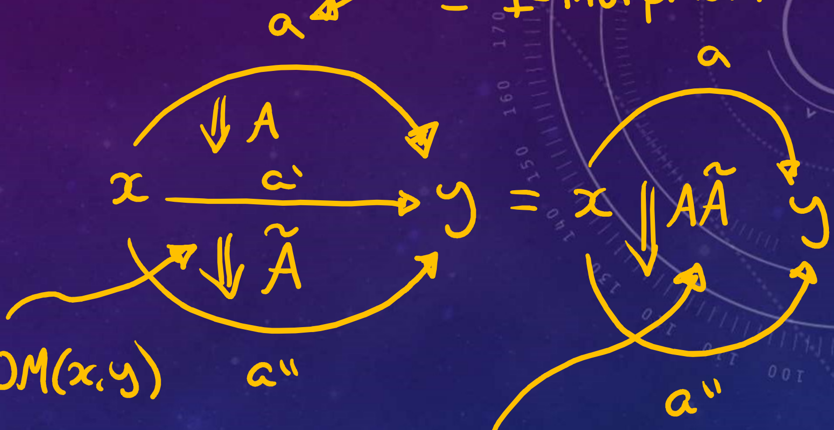
$$\Leftarrow \mathbb{1}_{1_x}$$

Set of OBJECTS $\{x, y, z, \ldots\}$

2-Categories (categorified categories)



object of HOM(x,y)
≡ 1-morphism



Morphism of HOM(x,y)
≡ 2-morphism

composition in HOM(x,y)
≡ vertical composition

Set of OBJECTS $\{x, y, z, \ldots\}$

For each pair of objects, a category HOM(x, y)

2-Categories (categorified categories)

$$
\begin{tikzcd}[column sep=huge]
x \arrow[loop left, "1_x"] \arrow[r, bend left=60, "a"] \arrow[r, "a'" description] \arrow[r, bend right=60, "a''"'] \arrow[r, phantom, shift left=4, "\Downarrow A"] \arrow[r, phantom, shift right=4, "\Downarrow \tilde{A}"] & y \arrow[r, bend left=60, "b"] \arrow[r, "b'" description] \arrow[r, bend right=60, "b''"'] \arrow[r, phantom, shift left=4, "\Downarrow B"] \arrow[r, phantom, shift right=4, "\Downarrow \tilde{B}"] & z
\end{tikzcd}
$$

$\Leftarrow \mathbb{1}_{1_x}$ $\quad 1_x$

Horizontal composition

$$
\otimes : \begin{tikzcd}
x \arrow[r, bend left=50, "a"] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow A"] & y \arrow[r, bend left=50, "b"] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow B"] & z
\end{tikzcd}
\longrightarrow
\begin{tikzcd}
x \arrow[r, bend left=50, "a \otimes b"] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow A\otimes B"] & z
\end{tikzcd}
$$

Set of OBJECTS $\{x, y, z, \ldots\}$

For each pair of objects, a category $HOM(x, y)$

For each triple of objects, a functor $\otimes : HOM(x,y) \times HOM(y,z) \longrightarrow HOM(x,z)$

More generally : Such a categorification can be iterated to form n-categories which have an n-dim'l like structure

(sets can be thought of as 0-categories)

Algebra A (unital)

f.d. Vector space V

Composition linear map

$\circ : V \otimes V \longrightarrow V$

Module $\triangleright : M \otimes V \longrightarrow V$

Modules of A form a category

1-Algebra A

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Modules of A form a category

2-Algebra $\underline{A}$

f.d. 2-Vector space

$\underline{V} = \mathrm{Mod}(A)$

for separable 1-alg

Composition linear functor

$\otimes : \underline{V} \boxtimes \underline{V} \longrightarrow \underline{V}$

(monoidal structure)

Module category

$\odot : \underline{M} \boxtimes \underline{V} \longrightarrow \underline{V}$

Module cats of $\underline{A}$

form a 2-cat $2\mathrm{Mod}(\underline{A})$

separable if A-bimodule intertwines

$A \xrightarrow{\Delta} A \otimes A \xrightarrow{\circ} A = A \xrightarrow{id} A$

1-Algebra A

f.d. Vector space V

Composition linear map

$\circ : V \otimes V \longrightarrow V$

Module $\triangleright : M \otimes V \longrightarrow V$

Modules of A form a category

2-Algebra $\underline{A}$

f.d. 2-Vector space

$\underline{V} = \mathrm{Mod}(A)$

for separable 1-alg

Composition linear functor

$\otimes : \underline{V} \boxtimes \underline{V} \longrightarrow \underline{V}$

(monoidal structure)

Module category

$\odot : \underline{M} \boxtimes \underline{V} \longrightarrow \underline{V}$

Module cats of $\underline{A}$ form a 2-cat $2\mathrm{Mod}(\underline{A})$

3-Algebra $\underline{\underline{A}}$

f.d. 3-Vector space

$\underline{\underline{V}} \equiv 2\mathrm{Mod}(\underline{A})$

for separable 2-alg

Composition linear 2-functor

$\boxtimes : \underline{\underline{V}} \mathbin{\Diamond\!\!\!\!\times} \underline{\underline{V}} \longrightarrow \underline{\underline{V}}$

(2-monoidal structure)

Module 2-category

$\Diamond\!\!\!\!\cdot \; : \underline{\underline{M}} \boxtimes \underline{\underline{V}} \longrightarrow \underline{\underline{V}}$

Module 2-cats of $\underline{\underline{A}}$ form a 3-cat $3\mathrm{Mod}(\underline{\underline{A}})$

Defn: A multifusion n-category is an $(n+1)$-algebra which is fully dualisable

For a detailed account see:

Gaiotto + Johnson-Freyd 1905.09566

Kong + Zheng 2011.02859

Defn: A multifusion n-category is an $(n+1)$-algebra which is fully dualisable

Corollary: Multifusion n-categories are semisimple

$\stackrel{?}{=}$ all modules are equivalent to direct sum of simple modules (no non-zero proper submodules)

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Defn: A multifusion n-category is an (n+1)-algebra which is fully dualisable

Corollary: Multifusion n-categories are semisimple

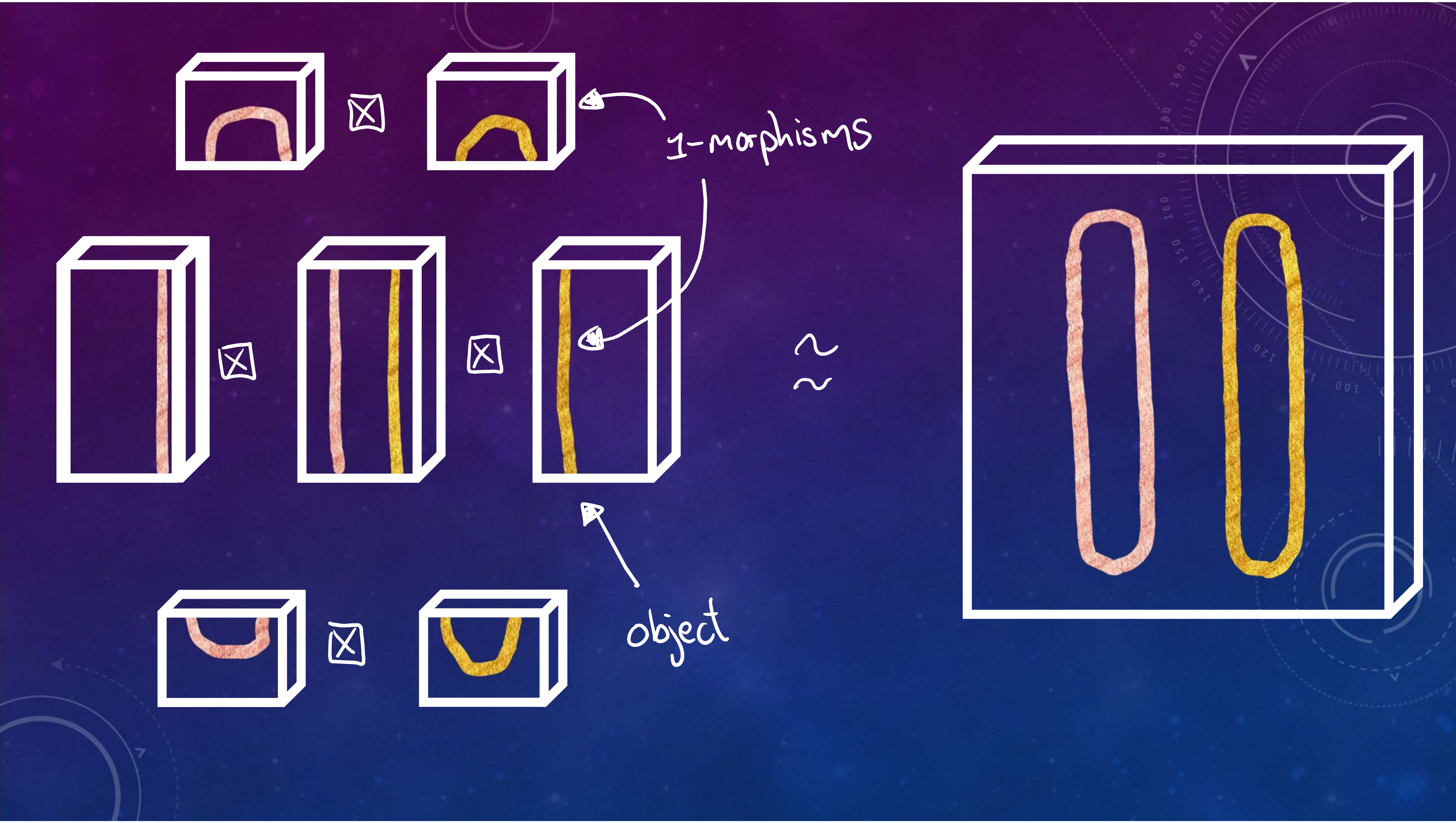
$\simeq$ all modules are equivalent to direct sum of simple modules (no non-zero proper submodules)

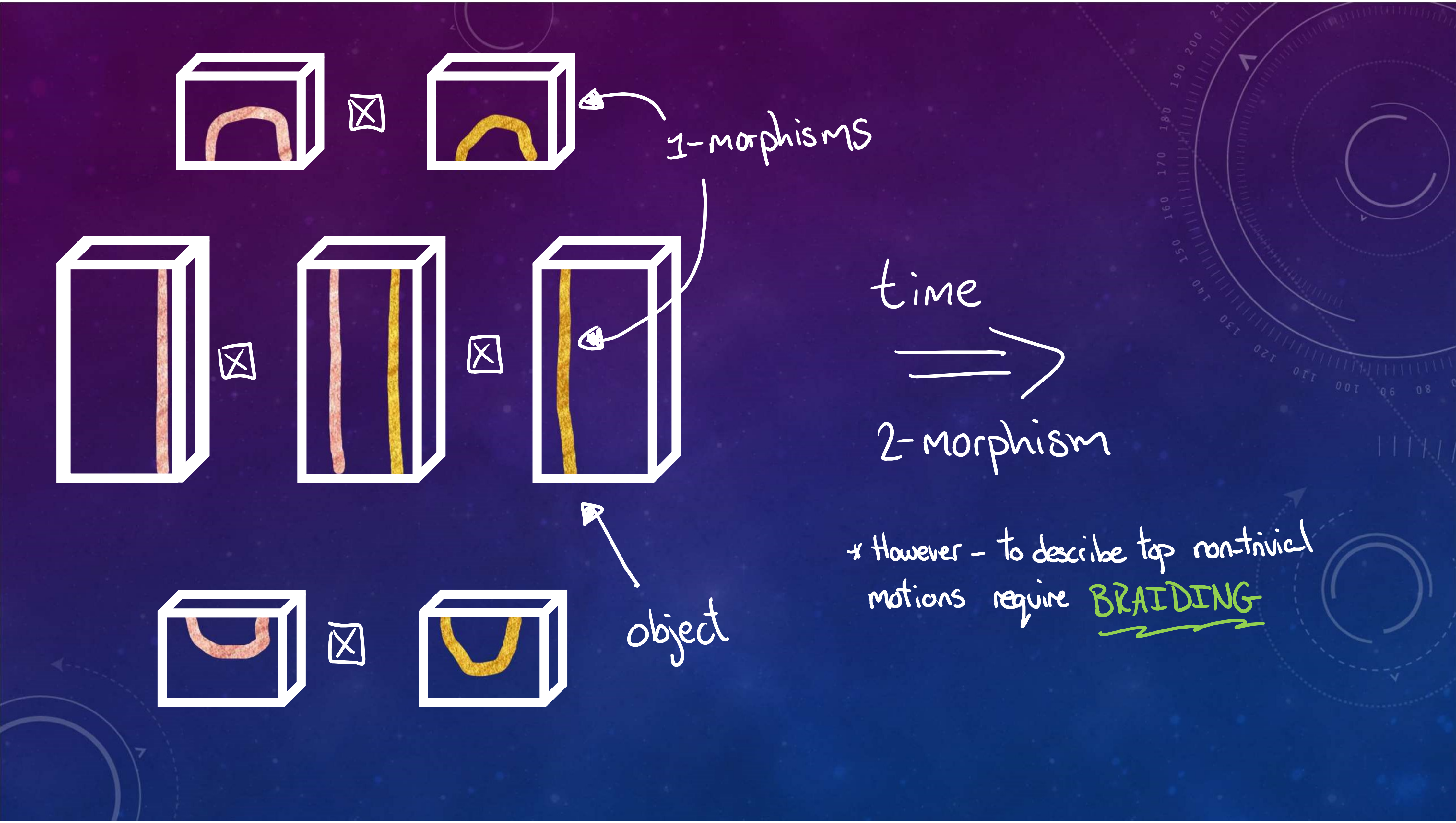
Defn: Multifusion n-category is fusion if unit object is simple

For a detailed account see:

Gaiotto + Johnson-Freyd 1905.09566

Kong + Zheng 2011.02859


1-morphisms
object



1-morphisms
time
2-morphism
object
* However - to describe top non-trivial motions require BRAIDING

CENTERS

Center of an Algebra

Recall: Given an algebra A, its centre $Z(A) \subseteq A$ the subalgebra consisting of $\{ z \in A \mid a \circ z = z \circ a \ \forall a \in A \}$

Center of a 3-Algebra

* 3-Algebra consists of: 3-Vector space $\underline{\underline{A}}$ (2-cat)
  product $\boxtimes : \underline{\underline{A}} \boxtimes \underline{\underline{A}} \longrightarrow \underline{\underline{A}}$ (2-monoidal structure)

* Center $Z(\underline{\underline{A}})$ is the commutative sub 3-algebra (braided monoidal 2-category)

* center $Z(C)$ for fusion 2-category C $\Rightarrow$ non-degenerate, braided fusion 2-category
  $\underset{K-W}{\simeq}$ 3+1D Top order

- center defined generally for monoidal 2-category

See: Baez + Neuchl q-alg/9511013

Crans Adv. Math 136 (1998), 183-223

Kong + Tian + Zhou 1905.04644

Objects in $Z(\underline{\underline{A}})$

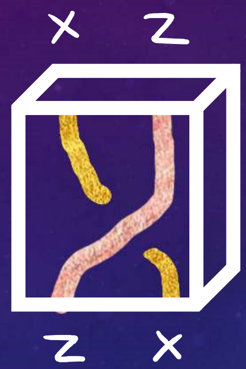
$Z \in ob(\underline{\underline{A}})$ is in $Z(\underline{\underline{A}})$ if there exist:

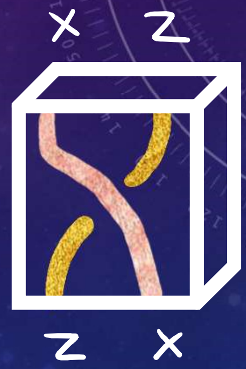
$\forall X \in ob(\underline{\underline{A}})$

half-braiding

$$Z \boxtimes X \underset{R^{-1}_{Z,X}}{\overset{R_{Z,X}}{\rightleftarrows}} X \boxtimes Z$$

relation w. diagrams see also Saito + Carter "Combinatorial description of knotted surfaces and their isotopies"

$R_{Z,X} =$ 

$R^{-1}_{Z,X} =$ 

categorifies

$Z \otimes X = X \otimes Z \quad \forall X \in A$

Objects in $Z(\underline{\underline{A}})$

$Z \in ob(\underline{\underline{A}})$ is in $Z(\underline{\underline{A}})$ if there exist:

$\forall X \in ob(\underline{\underline{A}})$

$$\begin{tikzcd}
 & X \boxtimes Z \arrow[dr, "R^{-1}_{Z,X}"] & \\
Z \boxtimes X \arrow[ur, "R_{Z,X}"] \arrow[rr, "1_{Z \boxtimes X}"'] \arrow[rr, phantom, shift left=4, "\cong \Uparrow \varepsilon_{Z,X}"] & & Z \boxtimes X
\end{tikzcd} \simeq$$

categorifies

$Z \circ X = X \circ Z \quad \forall X \in A$

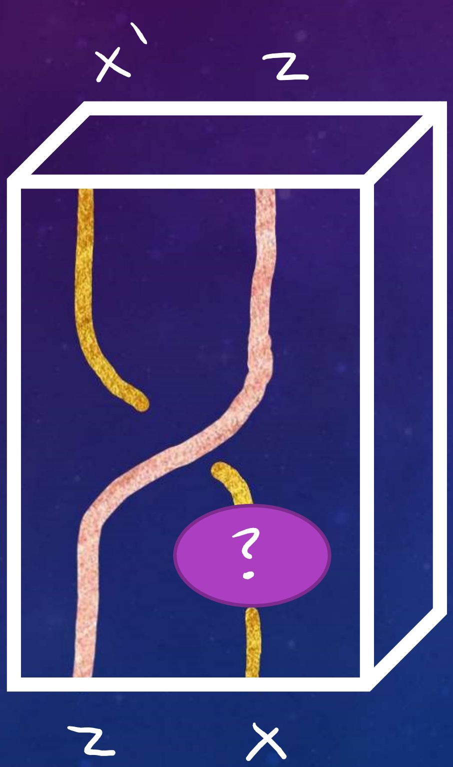
$\forall\ x \xrightarrow{?} x'$

a 1-morphism in $\underline{\underline{A}}$

$$\begin{tikzcd} z\boxtimes x' \arrow[r, "R_{z,x'}"] & x'\boxtimes z \\ z\boxtimes x \arrow[u, "1_z\boxtimes ?"] \arrow[r, "R_{z,x}"'] \arrow[from=1-1, to=2-2, Rightarrow, "R_{z,?}"] & x\boxtimes z \arrow[u, "?\boxtimes 1_z"'] \end{tikzcd}$$

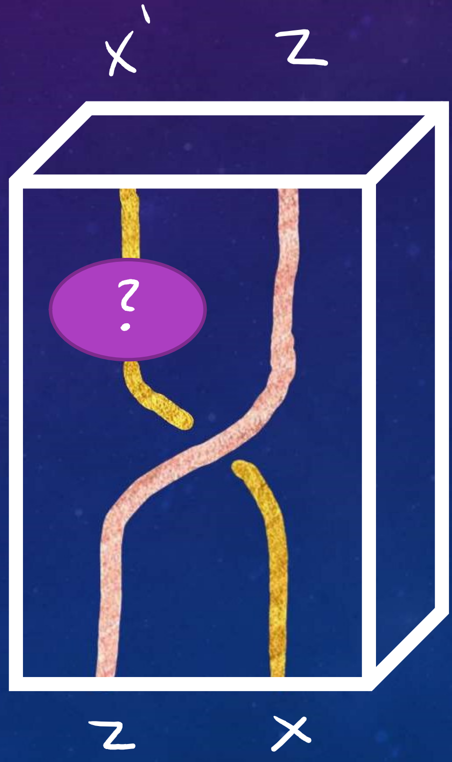
no analogue for algebras

$(1_z \boxtimes ?) \otimes R_{z,x'}$



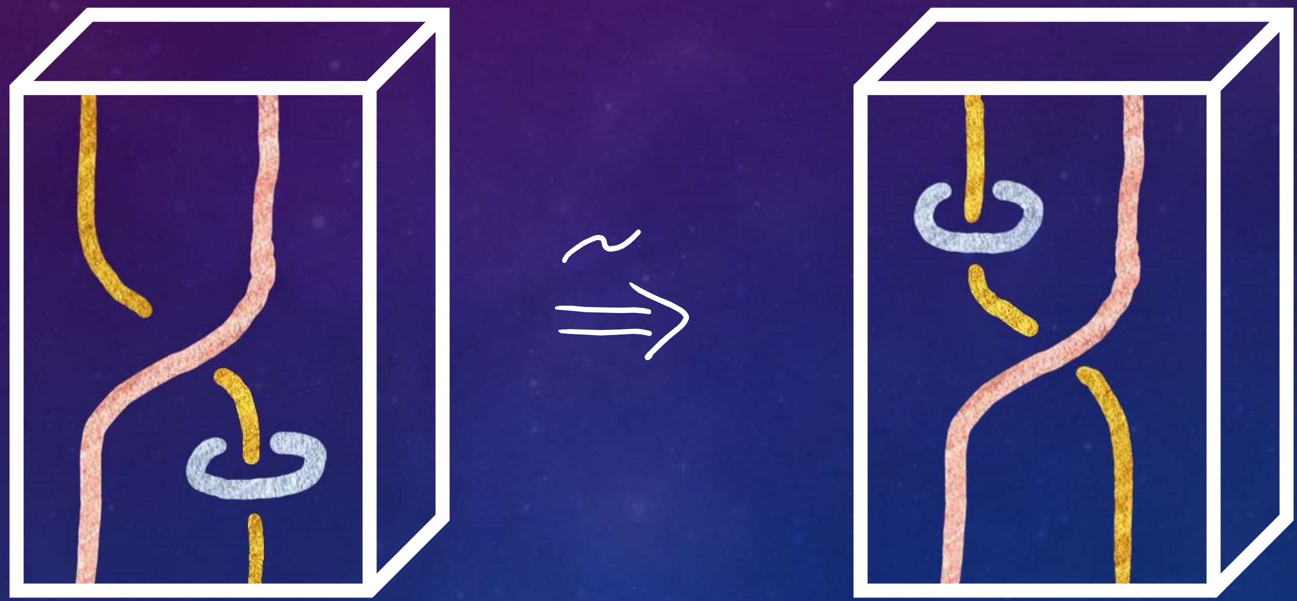
$R_{z,?}$

$\Rrightarrow$



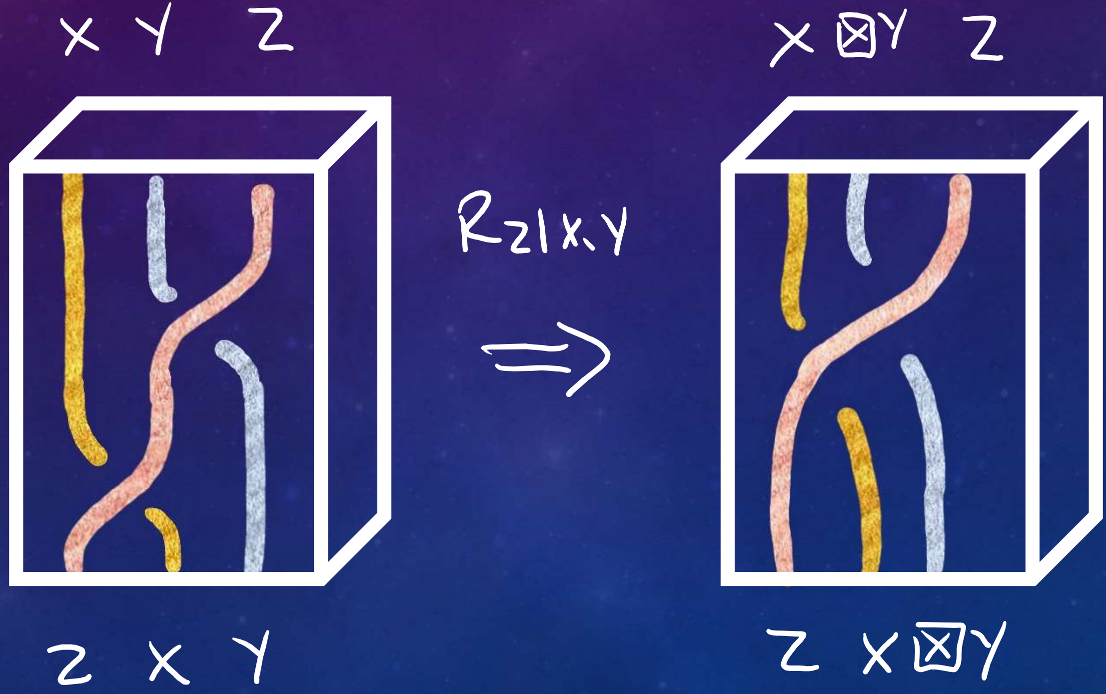
$R_{z,x} \otimes (? \boxtimes 1_z)$

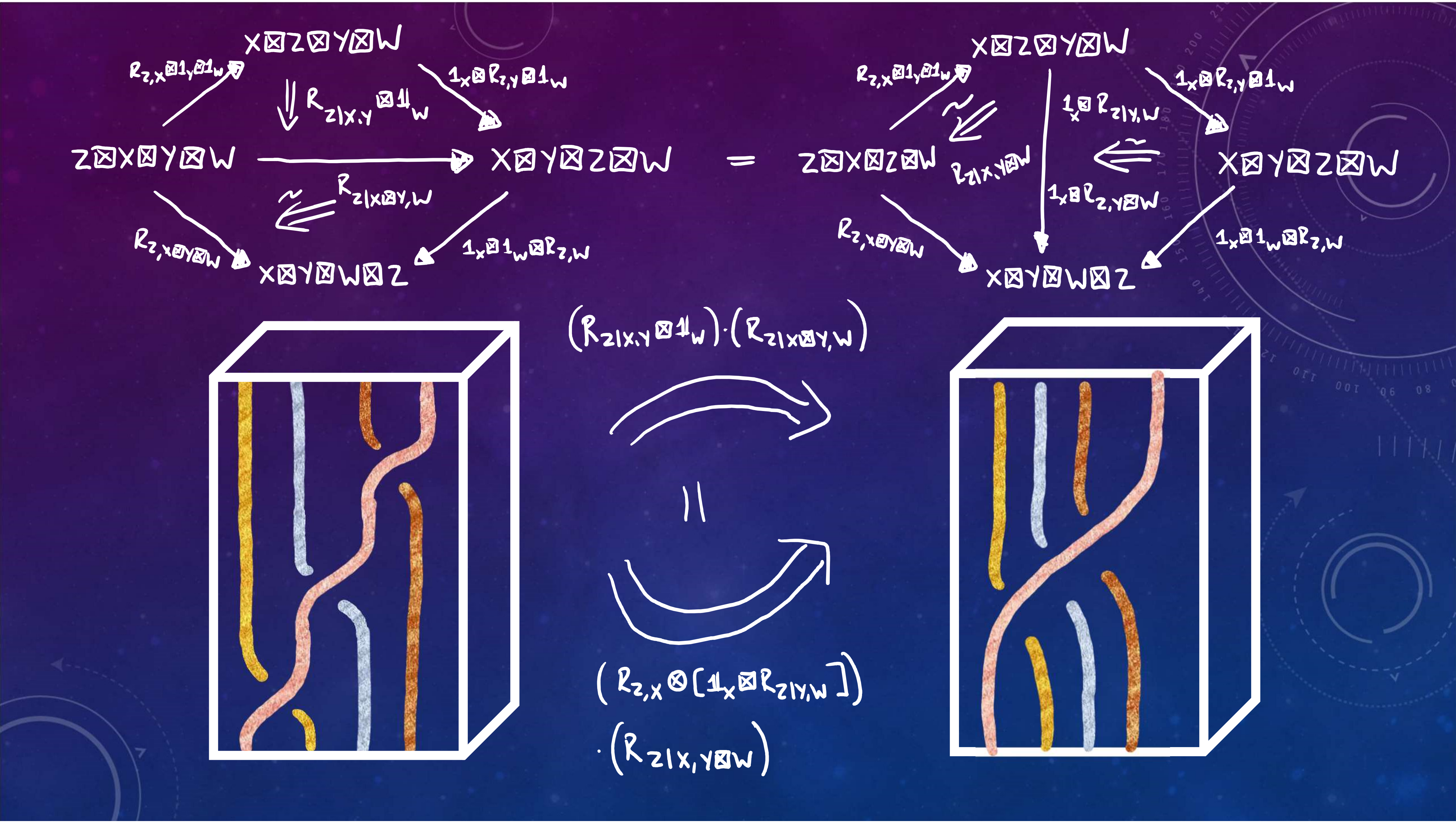
eg:



$$
\begin{tikzcd}
 & X \boxtimes Z \boxtimes Y \arrow[dr, "1_X \boxtimes R_{Z,Y}"] \arrow[d, phantom, "\Downarrow R_{Z|X,Y}"] & \\
Z \boxtimes X \boxtimes Y \arrow[ur, "R_{Z,X} \boxtimes 1_Y"] \arrow[rr, "R_{Z,X\boxtimes Y}"'] & \, & X \boxtimes Y \boxtimes Z
\end{tikzcd}
$$

Categorifies if $Z \circ X = X \circ Z$
$Z \circ Y = Y \circ Z$
$\Rightarrow Z \circ X \circ Y = X \circ Y \circ Z$





X⊠Z⊠Y⊠W
R_{Z,X}⊠1_Y⊠1_W
1_X⊠R_{Z,Y}⊠1_W
R_{Z|X,Y}⊠1_W
Z⊠X⊠Y⊠W
X⊠Y⊠Z⊠W
R_{Z|X⊠Y,W}
R_{Z,X⊠Y⊠W}
1_X⊠1_W⊠R_{Z,W}
X⊠Y⊠W⊠Z
=
X⊠Z⊠Y⊠W
R_{Z,X}⊠1_Y⊠1_W
1_X⊠R_{Z,Y}⊠1_W
1_X⊠R_{Z|Y,W}
Z⊠X⊠Z⊠W
R_{Z|X,Y⊠W}
X⊠Y⊠Z⊠W
1_X⊠R_{Z,Y⊠W}
R_{Z,X⊠Y⊠W}
1_X⊠1_W⊠R_{Z,W}
X⊠Y⊠W⊠Z
(R_{Z|X,Y}⊠1_W)·(R_{Z|X⊠Y,W})
||
(R_{Z,X}⊗[1_X⊠R_{Z|Y,W}])
·(R_{Z|X,Y⊠W})

Given fusion 2-category C, $Z(C)$ defines TO in sense of Kong + Wen

REALISING ON THE LATTICE

Lattice Hamiltonians

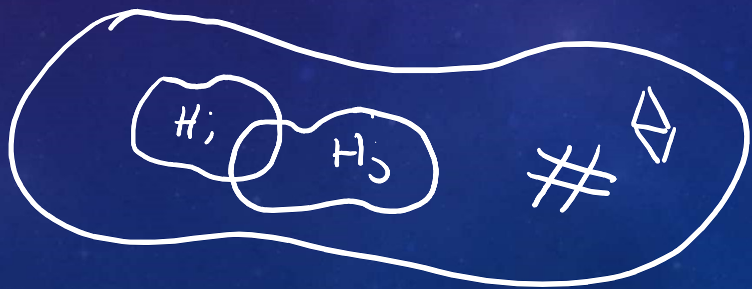
Lattice ≃ not too degenerate CW-complex homeomorphic to a manifold
– usually a triangulation

* Multifusion n-category ≡ Semisimple (n+1)-algebra
* Algebraic data of multifusion n-cat $\mathcal{C}$ define (n+2)D state-sum TQFT / Lattice Hamiltonian

(Gaiotto + Johnson-Freyd 1905.09566)

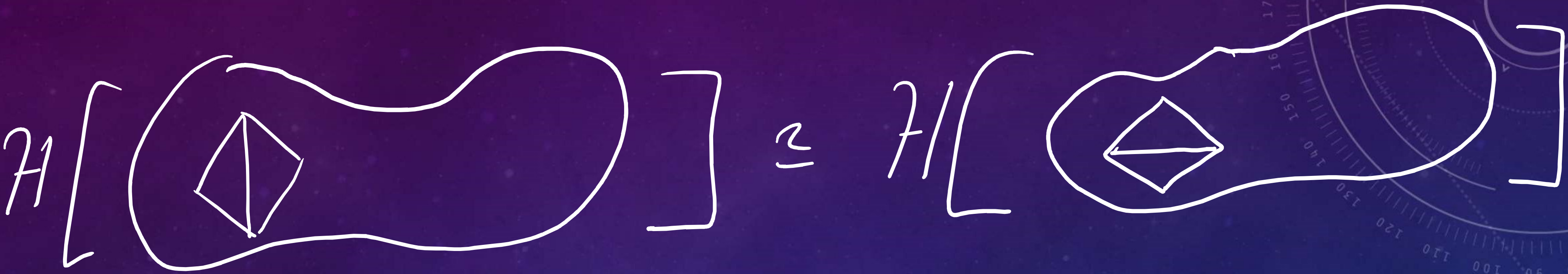
eg – 2+1D Turaev-Viro-Barrett-Westbury TQFT / Levin-Wen String-Net model
– 3+1D Crane-Yetter-Kauffman / Walker-Wang-Williamson model
+ many more ....

Local, mutually commuting Hamiltonian: $H = -\sum_i H_{\Delta^n_i}$, $H^2_{\Delta_i} = H_{\Delta_i}$, $[H_{\Delta_i}, H_{\Delta_j}] = 0$



$$\hat{M}_\Delta = \bigcup_i \Delta^n_i$$

Groundstate subspace



Given a multifusion n-category

- groundstate subspace on piecewise-linear homeomorphic CW-complexes give rise to isomorphic Hilbert spaces.

Conjecture (Bullivant 21')

Given a pivotal fusion 2-category C & corresponding 3+1D Hamiltonian model $H = -\sum_i H_i$, the physics of topological excitations in 3-disk is captured by $Z(C)$

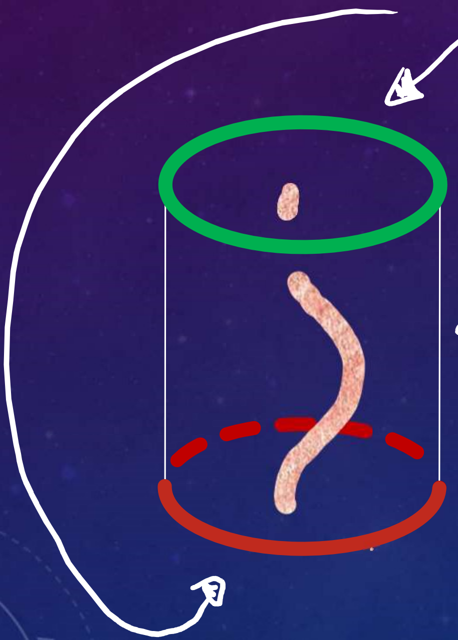
Proof - outlined in following

defn of pivotal fusion 2-cat
see: Douglas + Reutter
1812.11933

Given 3-disk, $H = -\sum_i H_i$

- groundstate $|\psi\rangle = (\prod_i H_i)|\psi\rangle$
- excitation $E \subseteq D^3$ s.t. $\forall \Delta_i \subseteq E \quad H_i|\psi\rangle = 0$

gapped boundary (data for such bdry found from C $\Rightarrow$ separable pseudomonoids in C)



open boundary condition

1) For gapped system correlation function decays exponentially (zero in our model)

2) Topological excitations are measurable at all length scales!

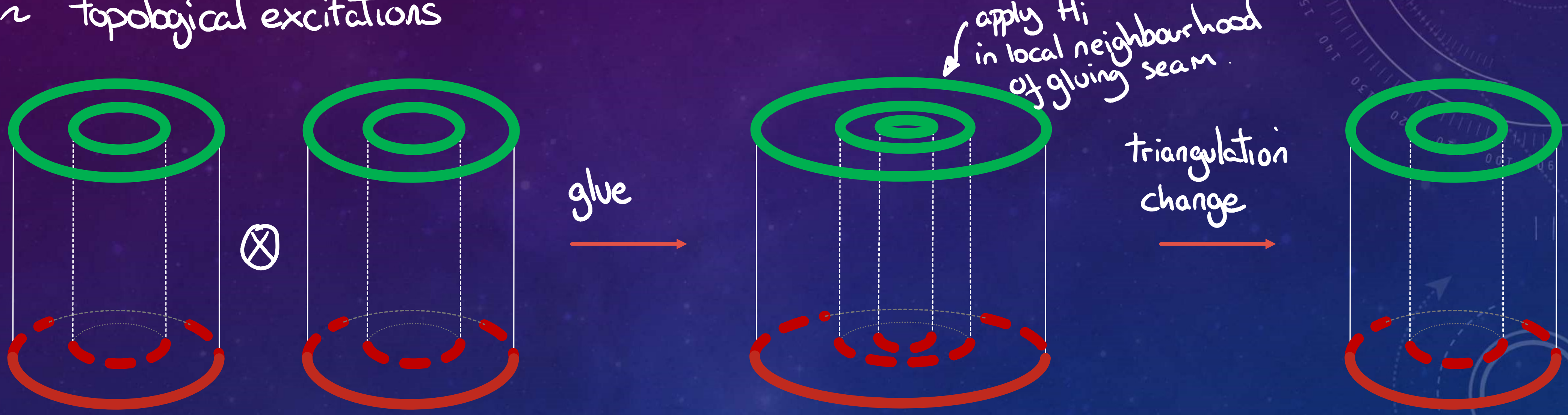
3) Excitations classified by entanglement w. G.S.

- using 1) ⇒ bdry condition for local neighbourhood of excitation

Boundary Tube Algebra

- f.d. $*$-algebra
- modules define bdry conditions invariant under "renorm group" / adding more space around excitation

$\sim$ topological excitations



$*$ For Dijkgraaf-Witten theory computation see AB + Delcamp 2006.06536

* Data for string endpoints $\rightleftharpoons$ objects of $Z(C)$
* Modules of bdry tube algebra $\rightleftharpoons$ 1-morphism of $Z(C)$
* Intertwiners $\rightleftharpoons$ 2-morphisms of $Z(C)$

- In this way we can construct $Z(C)$ from bdry tube algebra and vice versa

- This can be done formally w. Tube 2-Category
  * generalisation of Ocneanu Tube algebra

Thanks

For

Listening!