TUNNELLING AND CONVEXITY IN QFT

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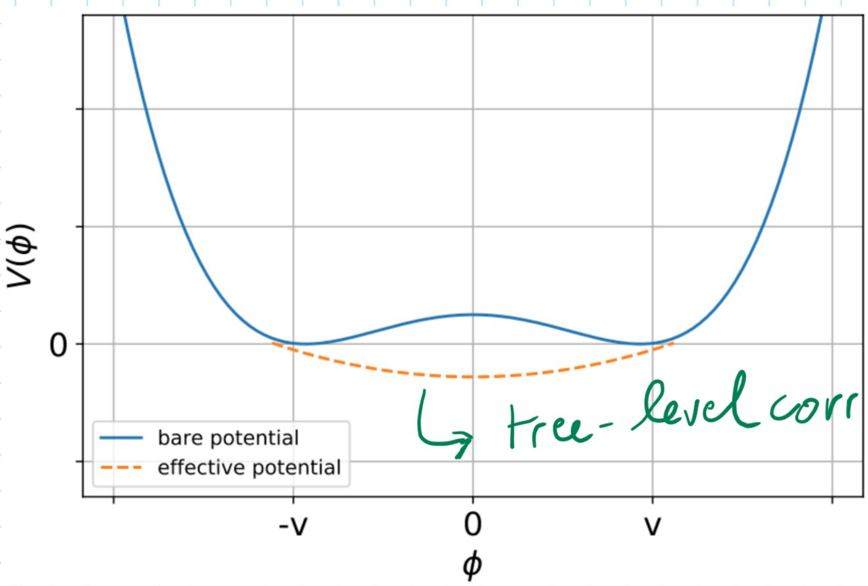
Quantum Mechanics — QFT

Tunnelling — Spontaneous Symmetry Breaking

intermediate regime?

Symmetry restoration $\longrightarrow$ true vacuum $= 0$



↳ tree-level corrections

(I) Tunnelling versus SSB

(II) Effective potential derivation

(III) FLRW spacetime

+ additional comments

(I) Tunnelling versus SSB

Quantum Mechanics
Finite number of d.o.f.

QFT
Infinite number of d.o.f.
$\Downarrow$
no tunnelling between degenerate vacua

- Standard Model partition function Z
  - $\rightarrow$ based on ONE vacuum
  - $\rightarrow$ partial partition function

decay rate (non-degenerate vacua)
$\rightarrow$ imaginary part of Z Weinberg, Wu 1987

- Several vacua taken into account
  - $\rightarrow$ Convex effective potential

Symanzik 1970; Iliopoulos, Itzykson, Martin 1975; Haymaker, Perez-Mercader 1983; ...

- $\rightarrow$ assumes tunnelling between vacua
- $\rightarrow$ Full partition function Z_{full}

Plascencia, Tamarit 2016

Convexity (one real scalar field)

- $$Z[j] = \int \mathcal{D}[\phi] \exp\left(-S[\phi] - \int j\phi\right) = e^{-W[j]}$$

- $$\phi_c = -\frac{\delta W}{\delta j}$$ classical field $\downarrow$ 1-to-1 mapping $j \leftrightarrow \phi_c$

- $$\frac{\delta^2 W}{\delta j\,\delta j} = \phi_c\phi_c - \langle\phi\phi\rangle \leq 0$$ (opposite of variance)

- $$\Gamma[\phi_c] = W[j] - \int j\phi_c$$ 1-particle-irreducible effective action

  $\uparrow$ Legendre transform

- $$\frac{\delta\Gamma}{\delta\phi_c} = -j$$

- $$\frac{\delta^2\Gamma}{\delta\phi_c\,\delta\phi_c} = -\left(\frac{\delta^2 W}{\delta j\,\delta j}\right)^{-1} \geq 0 \Rightarrow \text{convexity}$$

⚠ assumes Z = full partition function

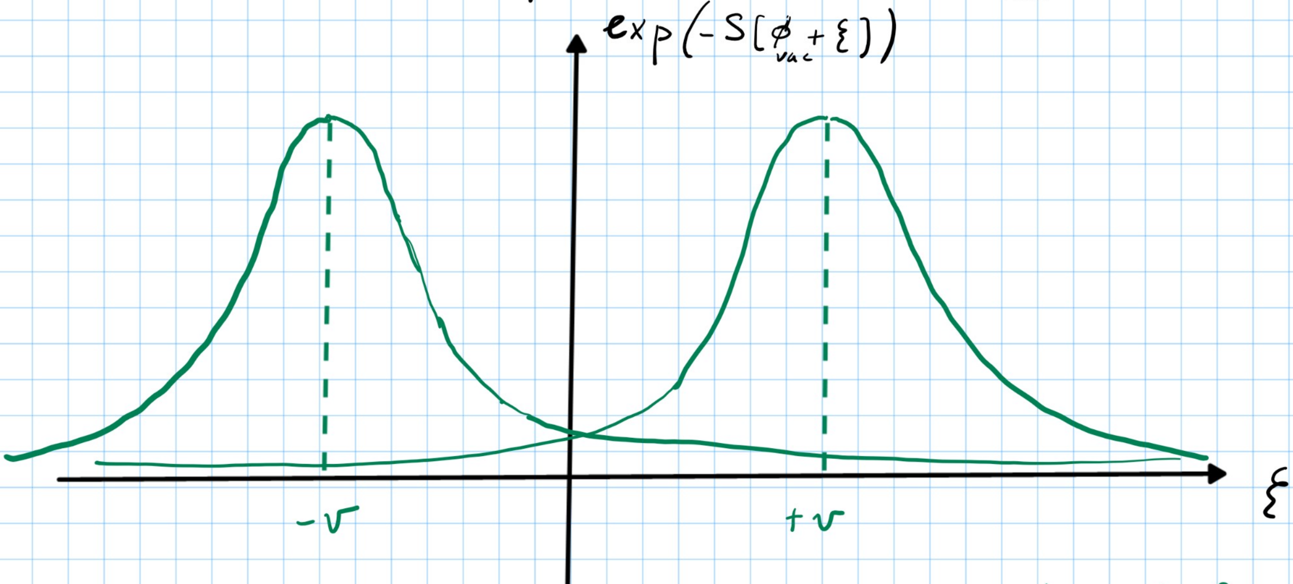
→ not valid for Standard Model

Condition for tunnelling between degenerate vacua

J.A., Polonyi 2021

Field theory argument:

- potential $\frac{\lambda}{24}(\phi^2 - v^2)^2$
- Euclidean metric + $O(4)$ symmetry
- overlap between fluctuations over $\phi^{\pm}_{vac} = \pm v$



$$\boxed{A \equiv \frac{\lambda}{24} v^4 V^{(4)} \simeq 1}$$

typical length $\simeq 10^{-17}$ m

$V^{(4)}$ = 4-d spacetime volume

Quantum Mechanics argument:

- potential barrier $\lambda v^4/24$
- 3-d volume ℓ^3
- energy gap ℓ^{-1}
- energy-time uncertainty

$\longrightarrow \frac{\lambda}{24} v^4 \ell^4 \simeq 1$

One-particle-irreducible effective potential

- $O(N)$-symmetric scalar field
- Several saddle points contributing to Z

J.A., Tsapalis 2013

$$\Gamma[\phi_c] = V^{(4)} U_{eff}(\phi_c, V^{(4)})$$

constant classical field

Non-extensive effective action

- Consequence : violation of Null Energy Condition

$$\rho + p = \frac{\Gamma}{V^{(4)}} - \frac{\partial \Gamma}{\partial V^{(4)}} = -V^{(4)} \frac{\partial U_{eff}}{\partial V^{(4)}} \neq 0$$

tunnelling regime $A \simeq 1 \Rightarrow \rho + p < 0$

- Extension to finite temperature J.A. 2016

$\longrightarrow$ Non-extensive thermodynamics

Free energy not proportional to 3-volume

(II) Effective Potential derivation

Semiclassical approximation

$$Z \simeq \sum_{\substack{\text{saddle}\\ \text{points } k}} F_k \exp\left(-S[\phi_k] - \int j\phi_k\right)$$

F_k: fluctuation determinant

$$\downarrow$$

$$Z[j, \underline{\underline{V^{(4)}}}]$$

$$\downarrow$$

$$\phi_c = \frac{-1}{Z}\frac{\delta Z}{\delta j} = -\frac{1}{V^{(4)}Z}\frac{\partial Z}{\partial j}$$

for constant source j

$\Downarrow$

constant ϕ_c

$$\downarrow$$

ϕ_c as a series in j

$\updownarrow$ one-to-one mapping for finite $V^{(4)}$

j as a serie in ϕ_c

$$\downarrow$$

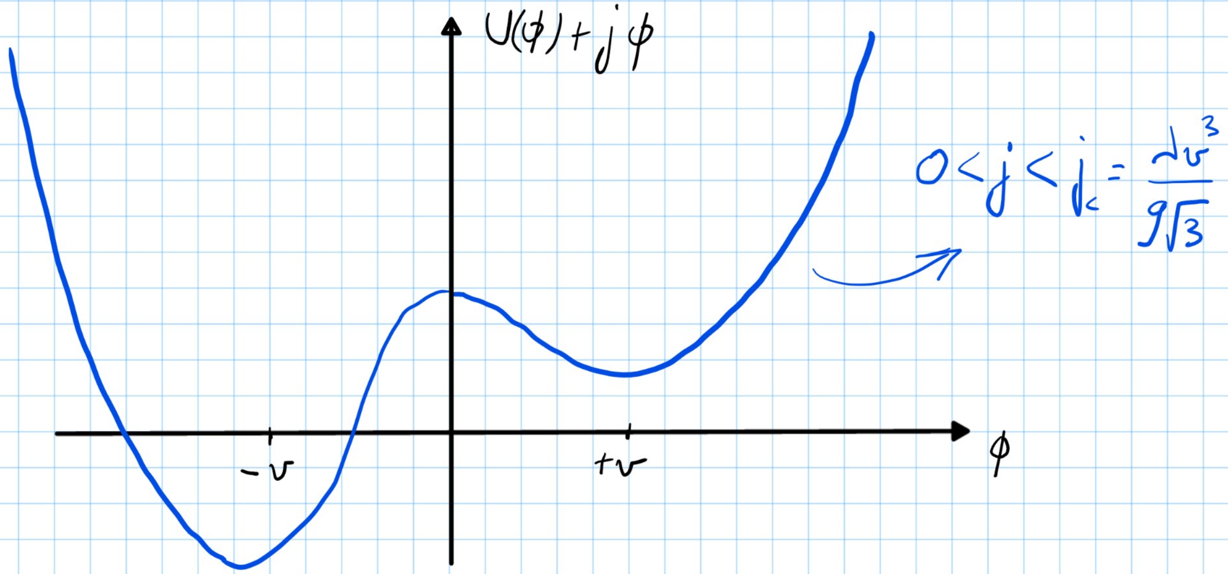
$$\frac{\partial \Gamma}{V^{(4)}\partial \phi_c} = -j = \text{series in } \phi_c$$

$$\downarrow$$

$\frac{\partial}{\partial \phi_c} U_{eff}$ = series in ϕ_c, <u>coefficients depend on $V^{(4)}$</u>

Saddle points ?

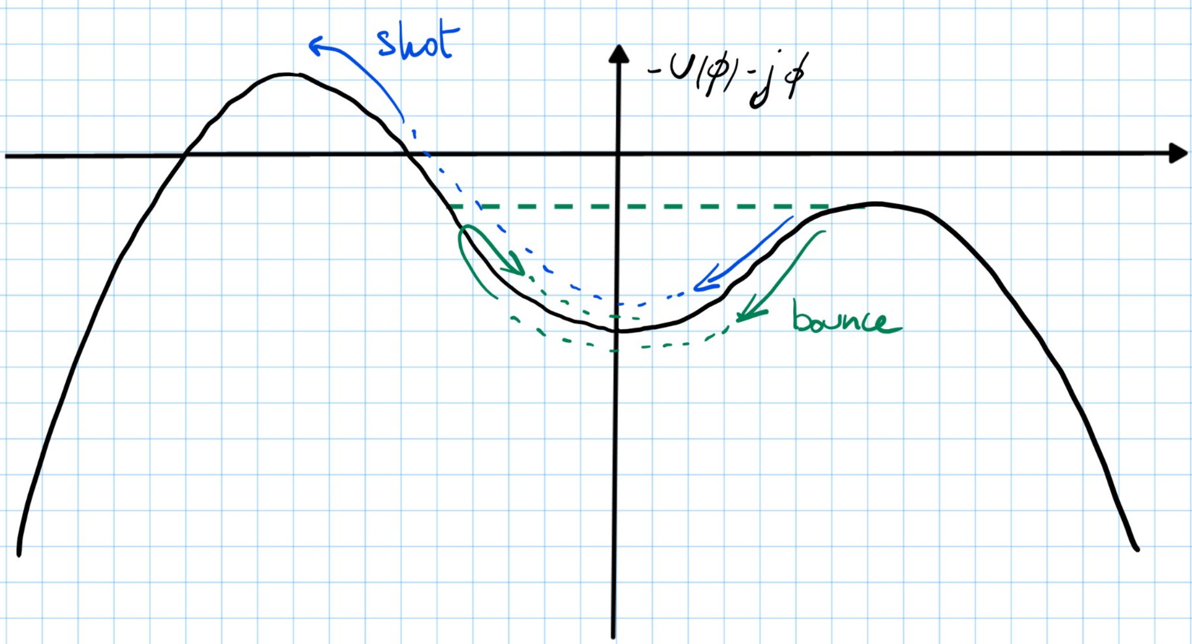
$$\frac{\delta S}{\delta \phi} + j = 0$$



Description in 0+1 dimensions (Quantum Mechanics)

- Constant saddle points $\phi_{1,2}(j)$ contribute at finite $V^{(4)}$
- Bounce Coleman 1977 $\dot{\phi}_B(0) = 0$
- Shot Andreassen, Farhi, Frost, Schwartz 2017 $\dot{\phi}_S(0) \neq 0$

Euclidean metric $\longleftrightarrow$ real-time motion in potential $-U(\phi) - j\phi$



(back to 3+1 dim) **Fate of the bounce in finite volume** 8

Coleman bounce → bubble of true vacuum, radius R

($O(4)$ symmetry)

$$U(\phi) = \frac{\lambda}{24}(\phi^2 - v^2)^2$$

Action : volume terms + surface term

competition ⟹ $R \simeq \frac{3v^2}{2j}\sqrt{\frac{3}{\lambda}}$

Tunnelling condition $\frac{\lambda}{24} v^4 \ell^4 \lesssim 1$

$R \leq \ell$

$$j \gtrsim \frac{3v^2}{\ell}\sqrt{\frac{3}{\lambda}}$$

True vacuum $\phi_c = 0$ ⟺ $j = 0$

no contribution of bounce in the vicinity of true vacuum

Shot → similar to constant saddle points (spend most time in true vacuum)

Andreassen, Farhi, Frost, Schwartz 2017

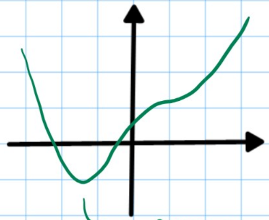
→ to be studied further

Effective potential for constant saddle points

J.A., Polonyi 2021

Semiclassical approximation based on constant modes

- One saddle point for $j > j_c = \frac{\lambda v^3}{g\sqrt{3}}$



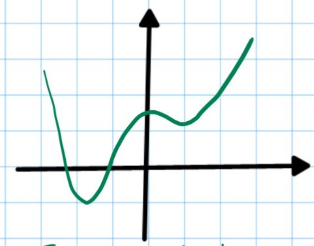
$\hookrightarrow$ ONE minimum

$\longrightarrow$ usual perturbative corrections for $|\phi_c| \geq \frac{2v}{\sqrt{3}}$

- Two saddle points for $j < j_c = \frac{\lambda v^3}{g\sqrt{3}}$ $\qquad |\phi_c| < \frac{2v}{\sqrt{3}}$

$$\phi_1 = \frac{2v}{\sqrt{3}} \cos\left(\frac{\pi}{3} - \frac{1}{3}\cos^{-1}(j/j_c)\right)$$

$$\phi_2 = \frac{2v}{\sqrt{3}} \cos\left(\pi - \frac{1}{3}\cos^{-1}(j/j_c)\right)$$



Two minima

- Fluctuation determinants for constant modes

$$F_k = \frac{v}{\sqrt{U''(\phi_k)}} \qquad k = 1, 2$$

$$\hookrightarrow \quad \Gamma(\phi_c) = \frac{A}{g} + \frac{24A^2\left[(\phi_c/v)^2 - 4/3\right]}{\frac{117}{16} - 12A + 48A^2} + \mathcal{O}\left(\phi_c/v\right)^4$$

$$\left(A = \frac{\lambda}{24} v^4 V^{(4)}\right)$$

$\searrow$ valid for $|\phi_c/v| \leq \frac{2v}{\sqrt{3}}$

$$\underline{\rho + p} = \lambda v^4 \frac{64A(39 - 256A^2)}{g(39 - 64A + 256A^2)^2} \qquad \text{in vacuum } \phi_c = 0$$

$A \simeq 1 \implies \rho + p < 0$ $\qquad$ NEC violated

(III) FLRW spacetime

J.A. Clough 2019

How to define volume?

(spatially flat time-dependent metric)

$\longrightarrow$ periodic space boundary conditions

cell volume ℓ^3

comoving volume $a^3(t)\ell^3$

Di Tucci, Feldbrugge, Lehners, Turok 2019

spacetime volume

$$\int d^4x\sqrt{|g|} = \ell^3\int_{t_0}^{t_1} dt'\, a^3(t') \equiv \ell^4 \underline{\underline{a^3(t)}} \qquad t_0 \leq t \leq t_1$$

assume equilibrium during quantisation time $[t_0, t_1]$

↳ justified around a cosmological bounce (motivation)

Tunnelling condition $\quad \tilde{A}(t) \equiv \frac{\lambda}{24}(v\ell)^4 a^3(t) \simeq 1$

$\downarrow$

U_{eff} depends on time

Energy density $\quad \rho = \frac{2}{\sqrt{|g|}}\frac{\delta}{\delta g_{00}(t)}\int d^3_x\, dt'\sqrt{|g|}\; U_{eff}(t', 0)$

true vacuum

Continuity equation $\quad \rho + p = -\frac{\dot{\rho}}{3H} \qquad \left(H \equiv \frac{\dot{a}}{a}\right)$

identical expression as in flat spacetime

with $A \longrightarrow \tilde{A}(t)$

Cosmological bounce scenario

Initial condition $\begin{cases} H > 0 \longrightarrow \text{inflation} \\ H < 0 \longrightarrow \text{collapse ?} \end{cases}$

Friedmann equations for :

- cosmological constant $\rho_0 \,,\ p_0 = -\rho_0$
- scalar field discussed here $\rho(t) \,,\ p(t)$

$$\left.\begin{aligned} H^2 &= \frac{8\pi G}{3}(\rho_0 + \rho) \\ \dot{H}^2 + H^2 &= -\frac{4\pi G}{3}(\rho_0 + \rho + 3p_0 + 3p) \end{aligned}\right\} \Rightarrow \boxed{\dot{H} = -4\pi G(\rho + p)}$$

(i) comoving volume such that $\tilde{A}(t) \gg 1$

$\rightarrow \rho + p = 0$ (no tunnelling)

(ii) comoving volume decreases $\Longrightarrow$ $\tilde{A}(\tau) \simeq 1$

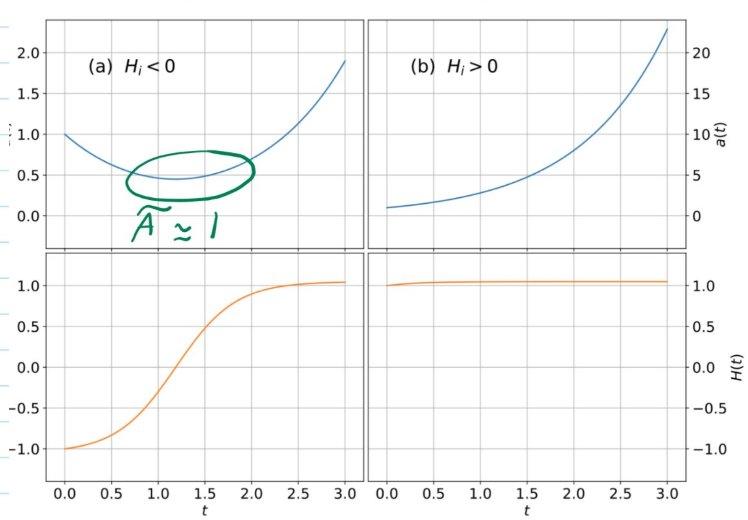
$\longrightarrow$ tunnelling switches on $\Rightarrow$ $\rho + p < 0 \longrightarrow \dot{H} > 0$

$\rightarrow$ cosmological bounce

(iii) comoving volume increases

$\rightarrow$ tunnelling suppressed

dynamical effect



Additional comments

(A) Averaged NEC

$$\int d\lambda \, T_{\mu\nu} n^\mu n^\nu \geq 0 \qquad T_{\mu\nu} n^\mu n^\nu = \rho + P$$

along null geodesic — matter — tangent to geodesic

→ weaker condition than NEC

ANEC satisfied because tunnelling temporary

(B) Casimir effect

- repulsive or attractive, depending on geometry/topology
- suppressed exponentially for massive scalar
- Cosmology: massless scalar, space = 3-torus } ⇒ expansion

Zeldovich, Starobinsky 1984

(C) Wilsonian approach

- Wilsonian effective potential
  = 1PI effective potential
  only for infinite volume

$$e^{iVU_{Wils}(\phi_0)} = \int \mathcal{D}(\phi)\, \exp\left(iS[\phi]\right)\, \delta\left(\int d^4x\, (\phi-\phi_0)\right)$$

$$= \int \mathcal{D}[\phi] \int dj\, \exp\left(iS[\phi] + i\int d^4x\, (\phi-\phi_0)\, j\right)$$

$$= \int dj\, e^{-iV^{(4)}\phi_0 j} \int \mathcal{D}[\phi]\, \exp\left(iS[\phi] + i\int d^4x\, j\phi\right)$$

$$= \int dj\, e^{iW[j] - iV^{(4)}\phi_0 j}$$

saddle point approximation $\rightarrow$ $$= e^{iV^{(4)}U_{1PI}(\phi_0)} + \mathcal{O}\left(1/V^{(4)}\right)$$

$$\rightarrow \quad U^{eff}_{Wils}(\phi_0) = U^{eff}_{1PI}(\phi_0) \quad \text{for } V^{(4)} \to \infty$$

(note: Minkowski metric)

- Exact Wilsonian functional methods
  - $\rightarrow$ all saddle points automatically included
  - $\rightarrow$ convexity

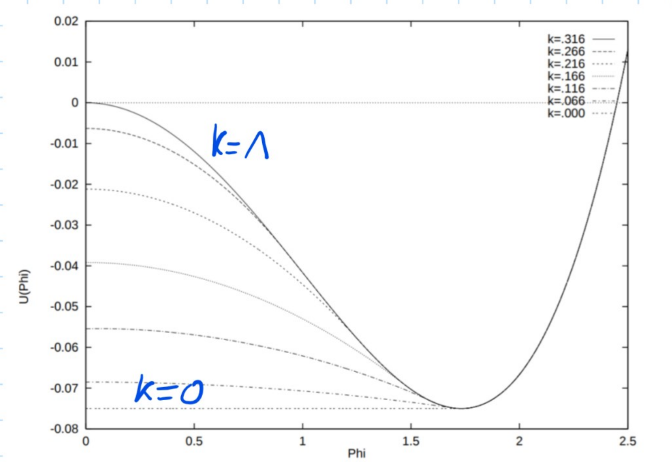
Wetterich 1991

Wegner-Houghton approach

Wegner, Houghton 1973

+ non-trivial saddle points

J.A, Branchina, Polonyi 1999



Maxwell construction

Conclusion: a lot more to do

QFT
- more thorough study of the different saddle points
- beyond $O(4)$ symmetry
- real time analysis ...

Cosmology
- additional matter content
- inhomogeneities ...