On Colour-Kinematics Duality and Double Copy

Leron Borsten

Maxwell Institute for Mathematical Sciences & Heriot–Watt University, Edinburgh

Dublin Institute for Advanced Studies, 11 November 2021

Based on joint work 2007.13803, 2102.11390, 2108.03030 and 21xx.xxxxx with Branislav Jurčo, Hyungrok Kim, Tommaso Macrelli, Christian Saemann and Martin Wolf

Gravity and gauge theory

- Gravity as a gauge theory:
  - Gauge theory of Lorentz, (super) Poincaré or de Sitter symmetries
    [Utiyama '56; Kibble '61; MacDowell-Mansouri '77; Chamseddine-West '77; Stelle-West 79]
  - Holographic principle - AdS/CFT correspondence
    ['t Hooft '93; Susskind '94; Maldacena '97]

Gravity and gauge theory

- Gravity as a gauge theory:
  - Gauge theory of Lorentz, (super) Poincaré or de Sitter symmetries
    [Utiyama '56; Kibble '61; MacDowell-Mansouri '77; Chamseddine-West '77; Stelle-West 79]
  - Holographic principle - AdS/CFT correspondence
    ['t Hooft '93; Susskind '94; Maldacena '97]

- Here, we appeal to a third and (superficially) independent perspective:

$$\text{Gravity} = \text{Gauge} \times \text{Gauge}$$

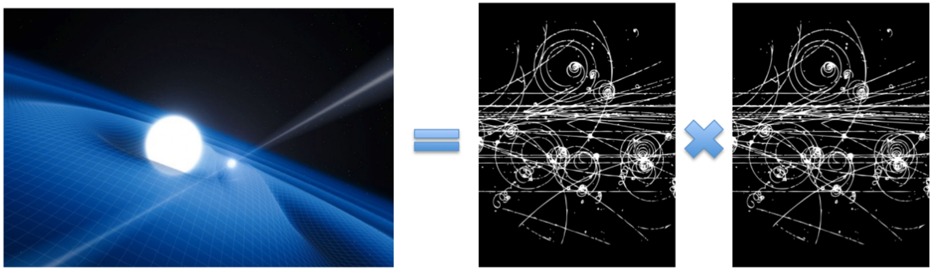
- The theme of gravity as the "square" of Yang-Mills has appeared in a variety of guises going back to the KLT relations of string theory [Kawai-Lewellen-Tye '85] Cf. Field theory [Feynman–Morinigo–Wagner; Papini '65]

- Bern-Carrasco-Johansson colour-kinematic (CK) duality and double-copy of (super) Yang-Mills (plus matter) scattering amplitudes
  [Bern-Carrasco-Johansson '08, '10; Bern-Dennen-Huang-Kiermaier '10]

Gravity = Gauge × Gauge

Longstanding open questions

- Does CK duality (in some appropriate sense) hold to all orders?
- Does the double copy hold: is Einstein really the square of Yang–Mills?
- Is this restricted to the S-matrix or more general?



Gravity = Gauge × Gauge

Off-shell field theory approach

- CK duality is property of the Yang–Mills Batalin–Vilkovisky (BV) action, up to *Jacobian counter terms* [BJKMSW '21]

$$S^{\text{YM}}_{\text{BRST-CK}} = \int C_{ij} c_{ab} A^{ia} \Box A^{ja} + F_{ijk} f_{abc} A^{ia} A^{jb} A^{kc}$$

Gravity = Gauge × Gauge

Off-shell field theory approach

- CK duality is property of the Yang–Mills Batalin–Vilkovisky (BV) action, up to *Jacobian counter terms* [BJKMSW '21]

$$S^{\text{YM}}_{\text{BRST-CK}} = \int C_{ij}\, c_{ab} A^{ia} \Box A^{ja} + F_{ijk}\, f_{abc} A^{ia} A^{jb} A^{kc}$$

- Natural, but non-standard notion of CK duality:
  - Infinite dimensional symmetry of the BV action
  - Loop amplitude integrands CK dual automatically
  - Anomalous - broken by Jacobian counterterms
  - Generalised unitarity proof of double copy doesn't straightforwardly apply

Gravity = Gauge × Gauge

Off-shell field theory approach

- CK duality is property of the Yang–Mills Batalin–Vilkovisky (BV) action, up to *Jacobian counter terms* [BJKMSW '21]

$$S^{\text{YM}}_{\text{BRST-CK}} = \int C_{ij}\, c_{ab} A^{ia} \Box A^{ja} + F_{ijk}\, f_{abc} A^{ia} A^{jb} A^{kc}$$

- Natural, but non-standard notion of CK duality:
  - Infinite dimensional symmetry of the BV action
  - Loop amplitude integrands CK dual automatically
  - Anomalous - broken by Jacobian counterterms
  - Generalised unitarity proof of double copy doesn't straightforwardly apply
- Double copy of BV action is manifestly valid → double copy to all loops

Gravity = Gauge × Gauge

Off-shell field theory approach

- CK duality is property of the Yang–Mills Batalin–Vilkovisky (BV) action, up to *Jacobian counter terms* [BJKMSW '21]

$$S^{\text{YM}}_{\text{BRST-CK}} = \int C_{ij}\, c_{ab} A^{ia} \Box A^{ja} + F_{ijk}\, f_{abc} A^{ia} A^{jb} A^{kc}$$

- Natural, but non-standard notion of CK duality:
  - Infinite dimensional symmetry of the BV action
  - Loop amplitude integrands CK dual automatically
  - Anomalous - broken by Jacobian counterterms
  - Generalised unitarity proof of double copy doesn't straightforwardly apply
- Double copy of BV action is manifestly valid $\rightarrow$ double copy to all loops
- Perturbative quantum Einstein–Hilbert gravity coupled to a Kalb–Ramond 2-form and dilaton *is* the square Yang–Mills theory [BJKMSW '20, '21]

Gravity = Gauge × Gauge

Homotopy algebra of CK duality

- BV quantised Yang-Mills → L_∞-algebra that factorises:

Bi-adjoint ϕ^3 theory		YM theory		$\mathcal{N}=0$ supergravity
$\mathfrak{col} \otimes \mathfrak{col} \otimes \mathfrak{scal}$	$\longleftarrow$	$\mathfrak{col} \otimes \mathfrak{kin} \otimes \mathfrak{scal}$	$\longrightarrow$	$\mathfrak{kin} \otimes \mathfrak{kin} \otimes \mathfrak{scal}$

- CK duality ↔ $\mathrm{BV}^{\square}_{\infty}$-algebra $\mathfrak{Kin} = \mathfrak{kin} \otimes_\tau \mathfrak{scal}$
- Homotopy relations ↔ kinematic Jacobi relations
- Only tree relations → potentially dramatic computational speed-up

Order of Events

1. Review: BCJ CK Duality and Double-Copy

2. CK Duality Redux

3. BV Lagrangian Syngamy

4. Generalisations

5. Homotopy CK Duality and Double Copy

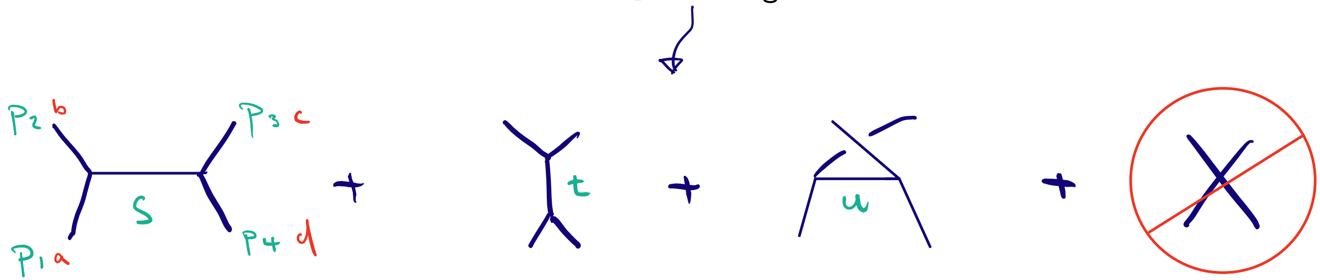
§1.

BCJ CK Duality and Double-Copy

Amplitudes and cubic diagrams

- Can write n-point L-loop gluon amplitude in terms of only cubic diagrams:

$$A_{\text{YM}}^{n,L} = \sum_{i \in \text{cubic diag}} \int_L \frac{c_i n_i}{S_i d_i}$$



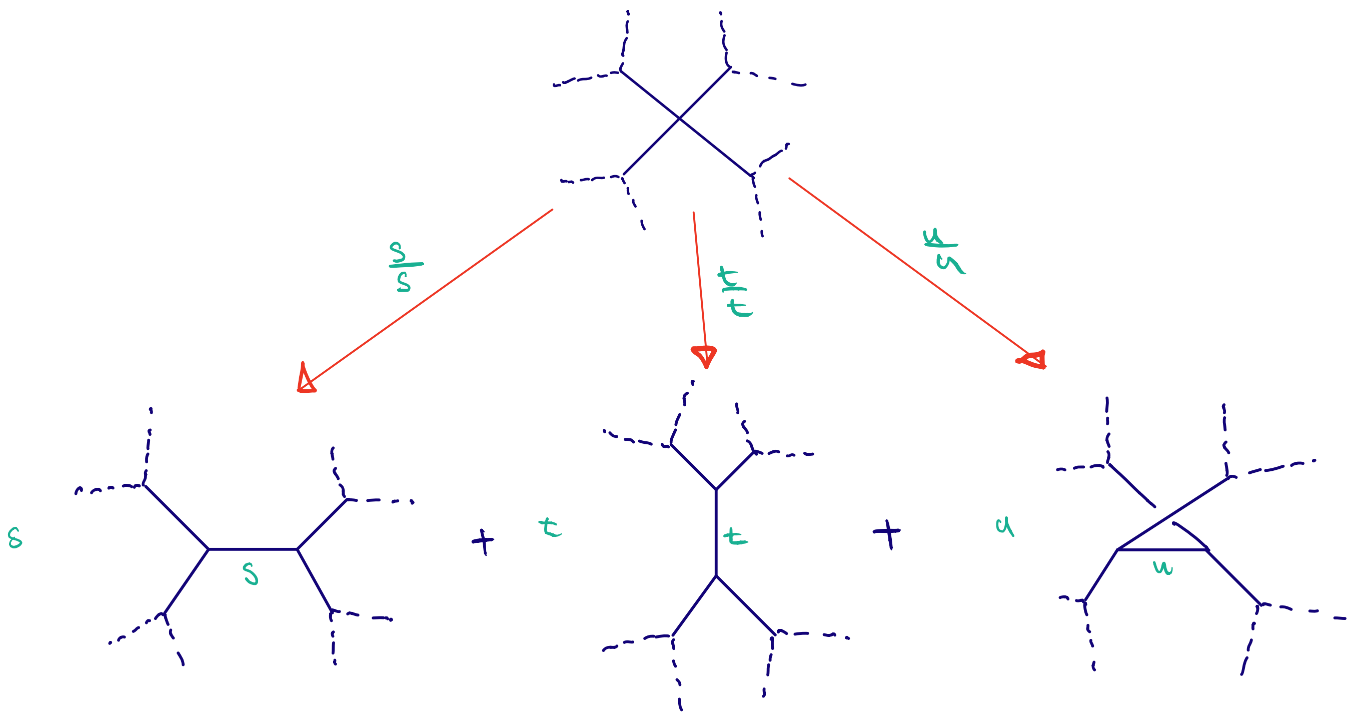
- c_i: colour numerator, built from f^{abc}, read off diagram i
- n_i: kinematic numerator, built from p, ε ← Non-unique
- d_i: propagator, $\prod_{\text{int. lines}} p^2$, read off diagram i

$s = (p_1 + p_2)^2 \qquad c_s = f_{ab}{}^{x} f_{xcd}$

Amplitudes and cubic diagrams

- Can write n-point L-loop gluon amplitude in terms of only cubic diagrams:

$$A_{\mathrm{YM}}^{n,L} = \sum_{i \in \text{cubic diag}} \int_L \frac{c_i n_i}{S_i d_i}$$

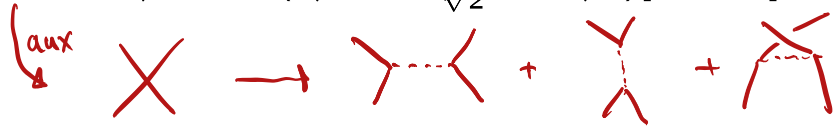


Amplitudes and cubic diagrams

- Can be realised in the YM Lagrangian through auxiliary fields:

$$g^2[A_\mu, A_\nu][A^\mu, A^\nu] \;\rightarrow\; \tfrac{1}{2}B^{\mu\nu\kappa}\,\Box\, B_{\mu\nu\kappa} - g(\partial_\mu A_\nu + \tfrac{1}{\sqrt{2}}\partial^\kappa B_{\kappa\mu\nu})[A^\mu, A^\nu]$$

[Bern-Dennen-Huang-Kiermaier '10]

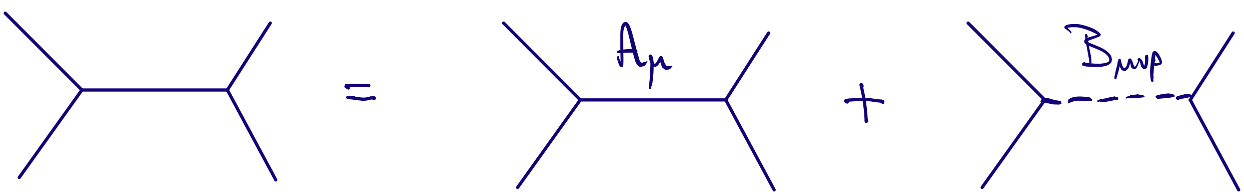


- Feynman diagrams give 'cubic' amplitudes directly:

$$A^{n,L}_{\text{YM}} = \sum_{\alpha\in\text{Feynman diag}} \int_L \frac{c_\alpha n_\alpha}{S_\alpha d_\alpha} = \sum_{i\in\text{cubic diag}} \int_L \frac{c_i n_i}{S_i d_i}$$

$$\rightarrow \sum_\varphi n_i^\varphi$$

- Example: 4-point s-channel diagram



$$n_s = n_s^A + n_s^B$$

BCJ colour-kinematic duality conjecture

- There is an organisation of the n-point L-loop gluon amplitude:

$$A_{\text{YM}}^{n,L} = \sum_{i \in \text{cubic diag}} \int_L \frac{c_i n_i}{S_i d_i}$$

such that

$$\boxed{\begin{array}{ccc} c_i + c_j + c_k = 0 & \Rightarrow & n_i + n_j + n_k = 0 \\ c_i \longrightarrow -c_i & \Rightarrow & n_i \longrightarrow -n_i \end{array}}$$

[Bern-Carrasco-Johansson '08]

BCJ colour-kinematic duality conjecture

- There is an organisation of the n-point L-loop gluon amplitude:

$$A_{\mathrm{YM}}^{n,L} = \sum_{i \in \text{cubic diag}} \int_L \frac{c_i n_i}{S_i d_i}$$

such that

$$\boxed{\begin{array}{ccc} c_i + c_j + c_k = 0 & \Rightarrow & n_i + n_j + n_k = 0 \\ c_i \longrightarrow -c_i & \Rightarrow & n_i \longrightarrow -n_i \end{array}}$$

[Bern-Carrasco-Johansson '08]

- CK duality established at tree-level
[Stieberger '09, Bjerrum-Bohr–Damgaard–Vanhove '09. . . Mizera '19; Reiterer '19]

- Significant evidence up to 4 loops in various (super)YM theories
[Carrasco–Johansson '11; Bern–Davies–Dennen–Huang–Nohle '13; Bern-Davies-Dennen '14. . .]

- Quickly becomes difficult to check: remains conjectural at the loop level
[Bern–Carrasco–Chen–Edison–Johansson–Parra-Martinez–Roiban–Zeng '18]

BCJ double-copy prescription

- Given CK dual amplitude of pure Yang-Mills

$$A_{\text{YM}}^{n,L} = \int_L \sum_{i \in \text{cubic diag}} \frac{c_i n_i}{S_i d_i}$$

$$S_{\text{YM}} = \frac{1}{2g^2} \int \text{tr} F \wedge \star F$$

BCJ double-copy prescription

- Given CK dual amplitude of pure Yang-Mills

$$A_{\text{YM}}^{n,L} = \int_L \sum_{i \in \text{cubic diag}} \frac{c_i n_i}{S_i d_i}$$

$$S_{\text{YM}} = \frac{1}{2g^2} \int \mathrm{tr} F \wedge \star F$$

- Double-copy:

$$\boxed{c_i \quad \longrightarrow \quad n_i}$$

BCJ double-copy prescription

- Given CK dual amplitude of pure Yang-Mills

$$A^{n,L}_{\text{YM}} = \int_L \sum_{i \in \text{cubic diag}} \frac{c_i n_i}{S_i d_i}$$

$$S_{\text{YM}} = \frac{1}{2g^2} \int \mathrm{tr} F \wedge \star F$$

- Double-copy:

$$\boxed{c_i \quad \longrightarrow \quad n_i}$$

- Gives an amplitude of $\mathcal{N} = 0$ supergravity

$$A^{n,L}_{\mathcal{N}=0} = \sum_{i \in \text{cubic diag}} \int_L \frac{n_i n_i}{S_i d_i}$$

$$S_{\mathcal{N}=0} = \frac{1}{2\kappa^2} \int \star R - \frac{1}{d-2} d\varphi \wedge \star d\varphi - \frac{1}{2} \mathrm{e}^{-\frac{4}{d-2}\varphi} dB \wedge \star dB$$

where B is the Kalb-Ramond 2-form, φ is the dilaton

[Bern-Carrasco-Johansson '08, '10; Bern-Dennen-Huang-Kiermaier '10]

Implications and applications

- Conceptually compelling and computationally powerful: $\mathcal{N} = 8$ supergravity four-point to 5 loops! (finite)
  [Bern–Carrasco–Chen–Edison–Johansson–Parra-Martinez–Roiban–Zeng '18]

Implications and applications

- Conceptually compelling and computationally powerful: $\mathcal{N} = 8$ supergravity four-point to 5 loops! (finite)
  [Bern–Carrasco–Chen–Edison–Johansson–Parra-Martinez–Roiban–Zeng '18]

- Can be explained by supersymmetry and $E_{7(7)}$ U-duality [Bjornsson–Green '10, Bossard–Howe–Stelle '11; Elvang–Freedman–Kiermaier '11; Bossard–Howe–Stelle–Vanhove '11]

- At 7 loops any would-be cancellations are "not consequences of supersymmetry in any conventional sense" [Bjornsson–Green '10]

Implications and applications

- Conceptually compelling and computationally powerful: $\mathcal{N} = 8$ supergravity four-point to 5 loops! (finite)
  [Bern–Carrasco–Chen–Edison–Johansson–Parra-Martinez–Roiban–Zeng '18]

- Can be explained by supersymmetry and $E_{7(7)}$ U-duality [Bjornsson–Green '10, Bossard–Howe–Stelle '11; Elvang–Freedman–Kiermaier '11; Bossard–Howe–Stelle–Vanhove '11]

- At 7 loops any would-be cancellations are "not consequences of supersymmetry in any conventional sense" [Bjornsson–Green '10]

- $D = 4, \mathcal{N} = 5$ supergravity finite to 4 loops, contrary to expectations:

  "Enhanced" cancellations

  [Bern-Davies-Dennen '14]

- Such cancellations not seen for $\mathcal{N} = 8$ at 5 loops: implications unclear

Implications and applications

- Classical (non)perturbative solutions and gravity wave astronomy
  [Monteiro–O'Connell–White '14; Cardoso–Nagy–Nampuri '16;
  Luna–Monteiro–Nicholson–Ochirov–O'Connell–Westerberg–White '16;
  Berman–Chacón–Luna–White '18; Kosower–Maybee–O'Connell '18;
  Bern–Cheung–Roiban–Shen–Solon–Zeng '19; Bern–Luna–Roiban–Shen–Zeng '20;
  Chacón-Nagy-White '21. . .]

- Geometric/world-sheet picture: ambitwistor string theories theories and scattering equations, e.g. non-trivial gluon and spacetime backgrounds
  [Cachazo–He–Yuan '13 '14; Mason–Skinner '13; Adamo–Casali–Skinner '13;
  Adamo–Casali–Mason–Nekovar '17 '18; Geyer–Monteiro '18; Geyer–Mason '19;
  Geyer-Monteiro-Stark–Muchão '21. . .]

- Surprising applications: gauge structure of the conjectured $(4, 0)$ phase of M-theory [LB '18] and twin non-Lagrangian S-folds theories [LB-Duff-Marrani '19]

Off-shell BRST-Lagrangian double-copy

Two key ideas:

- Can CK duality and the double-copy be realised at the level of field theory?

Off-shell BRST-Lagrangian double-copy

Two key ideas:

- Can CK duality and the double-copy be realised at the level of field theory?

1. CK duality manifesting actions and kinematic algebras
   [Bern–Dennen–Huang–Kiermaier '10; Tolotti–Weinzierl '13; Cheung–Shen '16; Luna–Monteiro–Nicholson–Ochirov–O'Connell–Westerberg–White '16]
   [Monteiro–O'Connell '11, '13; Bjerrum–Bohr–Damgaard–Monteiro–O'Connell '12; Fu–Krasnov '16; Chen–Johansson–Teng–Wang 19; Campiglia-Nagy '21. . .]

2. Field theory product of BRST gauge theories and Lagrangian double-copy
   [Bern–Dennen–Huang–Kiermaier '10; Anastasiou–LB-Duff–Hughes–Nagy '14; LB '17; Anastasiou–LB–Duff-Nagy–Zoccali '18; LB–Jubb–Makwana–Nagy '20; LB-Nagy '20]

Off-shell BRST-Lagrangian double-copy

Two key ideas:

- Can CK duality and the double-copy be realised at the level of field theory?

    1. CK duality manifesting actions and kinematic algebras
       [Bern–Dennen–Huang–Kiermaier '10; Tolotti–Weinzierl '13; Cheung–Shen '16; Luna–Monteiro–Nicholson–Ochirov–O'Connell–Westerberg–White '16]
       [Monteiro–O'Connell '11, '13; Bjerrum–Bohr–Damgaard–Monteiro–O'Connell '12; Fu–Krasnov '16; Chen–Johansson–Teng–Wang 19; Campiglia-Nagy '21. . .]

    2. Field theory product of BRST gauge theories and Lagrangian double-copy
       [Bern–Dennen–Huang–Kiermaier '10; Anastasiou–LB-Duff–Hughes–Nagy '14; LB '17; Anastasiou–LB–Duff-Nagy–Zoccali '18; LB–Jubb–Makwana–Nagy '20; LB-Nagy '20]

- Covariant Color-Kinematics Duality: a closed-form, analytic expression for all tree-level BCJ numerators in YM theory! [Cheung-Mangan '21]

Off-shell BRST-Lagrangian double-copy

Two key ideas:

- Can CK duality and the double-copy be realised at the level of field theory?

  1. CK duality manifesting actions and kinematic algebras
     [Bern–Dennen–Huang–Kiermaier '10; Tolotti–Weinzierl '13; Cheung–Shen '16; Luna–Monteiro–Nicholson–Ochirov–O'Connell–Westerberg–White '16]
     [Monteiro–O'Connell '11, '13; Bjerrum–Bohr–Damgaard–Monteiro–O'Connell '12; Fu–Krasnov '16; Chen–Johansson–Teng–Wang 19; Campiglia-Nagy '21. . .]

  2. Field theory product of BRST gauge theories and Lagrangian double-copy
     [Bern–Dennen–Huang–Kiermaier '10; Anastasiou–LB-Duff–Hughes–Nagy '14; LB '17; Anastasiou–LB–Duff-Nagy–Zoccali '18; LB–Jubb–Makwana–Nagy '20; LB-Nagy '20]

- Covariant Color-Kinematics Duality: a closed-form, analytic expression for all tree-level BCJ numerators in YM theory! [Cheung-Mangan '21]

- Today: the YM BV action admits a natural form of 'anomalous' CK duality that immediately implies the double copy to all orders

Lighting overview

Step 1. Cubic tree-level off-shell CK duality manifesting Yang-Mills BRST-action:

$$S^{\text{YM}}_{\text{BRST-CK}} = \int C_{ij} c_{ab} A^{ia} \Box A^{ja} + F_{ijk} f_{abc} A^{ia} A^{jb} A^{kc}$$

Step 2. BRST-action double-copy:

$$S_{\text{DC}} = \int C_{ij} C_{\tilde{\imath}\tilde{\jmath}} A^{i\tilde{\imath}} \Box A^{j\tilde{\jmath}} + F_{ijk} F_{\tilde{\imath}\tilde{\jmath}\tilde{k}} A^{i\tilde{\imath}} A^{j\tilde{\jmath}} A^{k\tilde{k}}$$

Step 3. Double-copy BRST operator:

$$(Q_{\text{YM}}, \tilde{Q}_{\text{YM}}) \longrightarrow Q_{\text{DC}} = Q_{\text{diffeo}} + Q_{\text{2-form}} + \text{trivial symmetries}$$

Step 4. Assuming tree-level physical CK duality, perturbative quantum equivalence:

$$Q_{\text{DC}}{}^2 = Q_{\text{DC}} S_{\text{DC}} = 0 \quad \Rightarrow \quad S_{\text{DC}} \cong S^{\mathcal{N}=0}_{\text{BRST}}$$

Corollary: Loop amplitude (integrands) computed from Feynman diagrams of $S^{\text{YM}}_{\text{BRST-CK}}$ manifest CK duality, *up to counterterms needed for unitarity*, and double-copy correctly to give amplitudes of $\mathcal{N} = 0$ supegravity

Colour-Kinematics Duality Redux

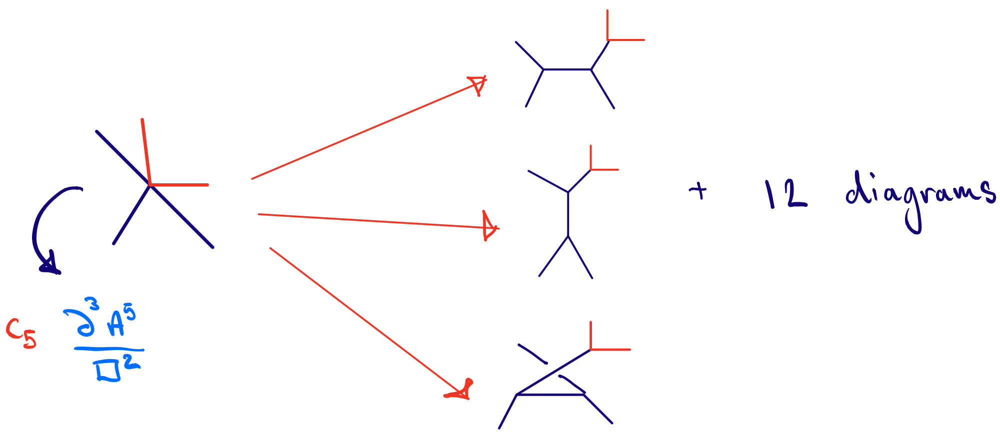
Colour-Kinematic Duality Redux

Manifest physical tree-level CK duality

- There is a YM action such that the Feynman diagrams yield amplitudes manifesting CK duality for tree-level amplitudes:

$$S^{\text{YM}}_{\text{on-shell CK}} = \sum_{n=2}^{\infty} \int \mathcal{L}^{(n)}_{\text{YM}} \sim A\Box A + \partial AAA + \frac{\Box}{\Box} AAAA + \underbrace{\frac{\partial^3}{\Box^2} AAAAA}_{} + \cdots$$

$f^{abc} f_c{}^{de} f_e{}^{fg} + \cdots = 0$ by Jacobi



[Bern–Dennen–Huang–Kiermaier 1004.0693; Tolotti–Weinzierl 1306.2975]

Colour-Kinematic Duality Redux

Manifest physical tree-level CK duality

- This can be "strictified" to have only cubic interactions through infinite tower of auxiliaries [BJKMSW '21]

$$
\begin{aligned}
S^{\text{YM}}_{\text{on-shell CK}} = \operatorname{tr}\int & d^D x \tfrac{1}{2} A_\mu \Box A^\mu + \tfrac{1}{2} g \partial_\mu A_\nu [A^\mu, A^\nu] \\
& \tfrac{1}{2} B^{\mu\nu\kappa} \Box B_{\mu\nu\kappa} - g(\partial_\mu A_\nu + \tfrac{1}{\sqrt{2}} \partial^\kappa B_{\kappa\mu\nu})[A^\mu, A^\nu] \\
& + \tfrac{1}{2} B^{\mu\nu\kappa} \Box B_{\mu\nu\kappa} - g(\partial_\mu A_\nu + \tfrac{1}{\sqrt{2}} \partial^\kappa B_{\kappa\mu\nu})[A^\mu, A^\nu] \\
& + C^{\mu\nu} \Box \bar{C}_{\mu\nu} + C^{\mu\nu\kappa} \Box \bar{C}_{\mu\nu\kappa} + C^{\mu\nu\kappa\lambda} \Box \bar{C}_{\mu\nu\kappa\lambda} + \\
& + g C^{\mu\nu}[A_\mu, A_\nu] + g \partial_\mu C^{\mu\nu\kappa}[A_\nu, A_\kappa] - \tfrac{g}{2} \partial_\mu C^{\mu\nu\kappa\lambda}[\partial_{[\nu} A_{\kappa]}, A_\lambda] \\
& + g \bar{C}^{\mu\nu} \left(\tfrac{1}{2} [\partial^\kappa \bar{C}_{\kappa\lambda\mu}, \partial^\lambda A_\nu] + [\partial^\kappa \bar{C}_{\kappa\lambda\nu\mu}, A^\lambda] \right) + \cdots
\end{aligned}
$$

4-point aux. field

5-point aux. fields

[Bern–Dennen–Huang–Kiermaier '10]

- Purely cubic Feynman diagrams $\longrightarrow$

$$
A_n^{\text{tree}} = \sum_i \frac{c_i n_i}{d_i} \quad \text{s.t.} \quad c_i + c_j + c_k = 0 \Rightarrow n_i + n_j + n_k = 0
$$

Colour-Kinematic Duality Redux

Generalise to off-shell BRST CK duality

- Does not imply loop-level CK duality, e.g. unphysical off-shell modes propagate in the loops
- To lift to loop-level we should include off-shell unphysical/ghost modes in the external states so that we can glue trees into loops:

Colour-Kinematic Duality Redux

Generalise to off-shell BRST CK duality

- Does not imply loop-level CK duality, e.g. unphysical off-shell modes propagate in the loops
- To lift to loop-level we should include off-shell unphysical/ghost modes in the external states so that we can glue trees into loops:

  1. Longitudinal gluons - gauge choice
  2. Ghosts - BRST Ward identities
  3. Off-shell - nonlocal field redefinitions (invisible on-shell)

- 3. $\Rightarrow$ induces Jacobian counterterms that cancel spurious modes

[BJKMSW '21]

Colour-Kinematic Duality Redux

Tree-level CK duality for longitudinal gluons

- Relax transversality $p_n \cdot \varepsilon_n \neq 0 \Rightarrow$ tree CK duality fails
- By analogy can compensate with new *non-zero* vertices [BJKMSW '20]:
- We can add them to the action without changing the physics [BJKMSW '20]

Colour-Kinematic Duality Redux

Tree-level onn-shell CK duality for longitudinal gluons and ghosts

- Using Lagrangian perspective, all CK failures can simultaneously be compensated by terms of the form

$$(\partial \cdot A) Y[A]$$

Colour-Kinematic Duality Redux

Tree-level onn-shell CK duality for longitudinal gluons and ghosts

- Using Lagrangian perspective, all CK failures can simultaneously be compensated by terms of the form

$$(\partial \cdot A) Y[A]$$

- Can add through the gauge-fixing functional

$$\begin{aligned} \text{Gauge-fixing func. } G[A]: \quad & \partial \cdot A & \mapsto \quad & G'[A] & = \quad & \partial \cdot A - 2\xi Y \\ \text{Nakanishi-Lautrup } b: \quad & b & \mapsto \quad & b' & = \quad & b + Y \end{aligned}$$

Colour-Kinematic Duality Redux

Tree-level onn-shell CK duality for longitudinal gluons and ghosts

- Using Lagrangian perspective, all CK failures can simultaneously be compensated by terms of the form

$$(\partial \cdot A) Y[A]$$

- Can add through the gauge-fixing functional

$$\begin{aligned} \text{Gauge-fixing func. } G[A]: \quad & \partial \cdot A && \mapsto && G'[A] && = && \partial \cdot A - 2\xi Y \\ \text{Nakanishi-Lautrup } b: \quad & b && \mapsto && b' && = && b + Y \end{aligned}$$

- Longitudinal CK duality $\Leftrightarrow$ gauge choice [BJKMSW '20, '21]

Colour-Kinematic Duality Redux

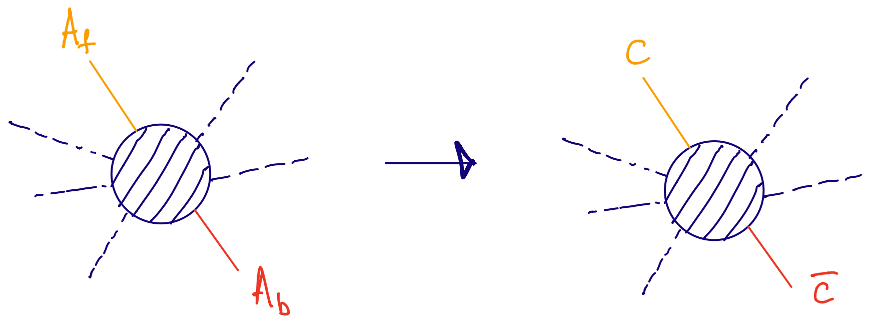
Tree-level CK duality for ghosts

- Use on-mass-shell BRST Ward identities

$$Q_{\rm YM}^{\rm lin} A_{\sf phys} = 0, \quad Q_{\rm YM}^{\rm lin} A_{\sf f} = c, \quad Q_{\rm YM}^{\rm lin} b = \bar{c}$$

- Analogous to global SUSY Ward identities

$$0 \; = \; \langle 0 | [Q_{\rm YM}^{\rm lin}, O_1 \cdots O_n] | 0 \rangle$$



- Transfers CK duality onto ghosts through

$$\mathcal{L}_{\sf ghost}^{\sf YM} = \bar{c} Q_{\rm YM} (\partial^\mu A_\mu - 2\xi Y)$$

Colour-Kinematic Duality Redux

On-shell tree-level CK manifesting BRST action

- Introduce new auxiliary gluons and ghosts [BJKMSW '20, '21]:

$$
\begin{aligned}
\mathcal{L}^{\text{YM}}_{\text{BRST CK-dual}} = {} & \tfrac{1}{2} A_{a\mu}\Box A^{\mu a} - \bar{c}_a \Box c^a + \tfrac{1}{2} b_a \Box b^a + \xi\, b_a \sqrt{\Box}\, \partial_\mu A^{\mu a} \\
& - K^{\mu}_{1a}\Box \bar{K}^{1a}_{\mu} - K^{\mu}_{2a}\Box \bar{K}^{2a}_{\mu} - g f_{abc} \bar{c}^a \partial^\mu (A^b_\mu c^c) \\
& - \tfrac{1}{2} B^{\mu\nu\kappa}_{a} \Box B^{a}_{\mu\nu\kappa} + g f_{abc} \Big(\partial_\mu A^a_\nu + \tfrac{1}{\sqrt{2}} \partial^\kappa B^a_{\kappa\mu\nu} \Big) A^{\mu b} A^{\nu c} \\
& - g f_{abc} \Big\{ K^{a\mu}_{1} (\partial^\nu A^b_\mu) A^c_\nu + [(\partial^\kappa A^a_\kappa) A^{b\mu} + \bar{c}^a \partial^\mu c^b] \bar{K}^{1c}_{\mu} \Big\} \\
& + g f_{abc} \Big\{ K^{a\mu}_{2} \Big[(\partial^\nu \partial_\mu c^b) A^c_\nu + (\partial^\nu A^b_\mu) \partial_\nu c^c \Big] + \bar{c}^a A^{b\mu} \bar{K}^{2c}_{\mu} \Big\} + \cdots
\end{aligned}
$$

long. aux. fields

ghost aux. fields

- Cubic Feynman diagrams yield CK dual tree amplitudes for physical gluons and unphysical longitudinal modes and ghosts (on-shell)

Colour-Kinematic Duality Redux

Lifting to off-shell CK duality

- Relaxing on-shell to off-shell momenta CK duality violated by terms

$$p_i^2 F_i$$

for each external momentum p_i (unphysical gluons and ghosts)

- Can compensate with terms $\propto F\Box\Phi$ with non-local field redefinition

$$\Phi \mapsto \Phi + F, \qquad \Phi\Box\Phi \mapsto \Phi\Box\Phi + F\Box\Phi + \cdots$$

so that off-shell tree-level BRST CK duality is manifest $\to$ loop CK duality [BJKMSW '21]

Colour-Kinematic Duality Redux

Lifting to off-shell CK duality

- Relaxing on-shell to off-shell momenta CK duality violated by terms

$$p_i^2 F_i$$

  for each external momentum p_i (unphysical gluons and ghosts)

- Can compensate with terms $\propto F\Box\Phi$ with non-local field redefinition

$$\Phi \mapsto \Phi + F, \qquad \Phi\Box\Phi \mapsto \Phi\Box\Phi + F\Box\Phi + \cdots$$

  so that off-shell tree-level BRST CK duality is manifest $\rightarrow$ loop CK duality [BJKMSW '21]

- Price to pay: Jacobian determinants $\rightarrow$ counterterms ensuring unitarity

- In this sense, this manifest loop CK duality is anomalous on the physical Hilbert space (but is exact on the complete pre-Hilbert space)

Colour-Kinematic Duality Redux

Perfect off-shell 'BRST-Lagrangian CK duality'

- BV YM action with manifest *off-shell* CK duality

Anti fields

$$S^{\text{YM}}_{\text{BV CK-dual}} = \int C_{ij}\, c_{ab} A^{ia} \Box A^{ja} + F_{ijk}\, f_{abc} A^{ia} A^{jb} A^{kc} + A^{+}_{ia} \underbrace{\left(Q^{i}{}_{j} A^{ja} + Q^{i}_{jk} f^{a}_{bc} A^{jb} A^{kc} \right)}_{Q_{BV} A}$$

- Rendered cubic with infinite tower of aux. fields

$$A^{ia} = (A_{\mu}{}^{a}, b^{a}, \bar{c}^{a}, c^{a}, \underbrace{G_{\mu\nu\rho}{}^{a}, \bar{K}_{\mu}{}^{a}, \ldots}_{\text{auxiliaries}})$$

- c_{ab}, f^{abc} gauge group Killing form and structure constants

- C_{ij}, F^{ijk} are differential operators that satisfy the same identities as c_{ab}, f^{abc} as operator equations

$$c_{ab} = c_{(ab)} \qquad f_{abc} = f_{[abc]} \qquad c_{a(b} f^{a}_{c)d} = 0 \qquad f_{[ab|d} f^{d}_{c]e} = 0$$

$$C_{ij} = C_{(ij)} \qquad F_{ijk} = F_{[ijk]} \qquad C_{i(j} F^{i}_{k)l} = 0 \qquad F_{[ij|l} F^{l}_{|k]m} = 0$$

Colour-Kinematic Duality Redux

Some comments

- Action has manifest CK duality
- The F_{ijk} are the structure constants of a *kinematic Lie algebra* mirroring the usual colour structure constants f_{abc}. Cf. [Monteiro–O'Connell '11, '13; Bjerrum–Bohr–Damgaard–Monteiro–O'Connell '12; Fu–Krasnov '16; Chen–Johansson–Teng–Wang 19; Campiglia-Nagy '21. . .]
- Corollary: loop amplitude integrands are CK dual automatically
- *Anomalous, in a controlled manner, due to Jacobian counterterms* that ensure (generalised) unitarity

Colour-Kinematic Duality Redux

Some comments

- Action has manifest CK duality
- The F_{ijk} are the structure constants of a *kinematic Lie algebra* mirroring the usual colour structure constants f_{abc}. Cf. [Monteiro–O'Connell '11, '13; Bjerrum–Bohr–Damgaard–Monteiro–O'Connell '12; Fu–Krasnov '16; Chen–Johansson–Teng–Wang 19; Campiglia-Nagy '21. . .]
- Corollary: loop amplitude integrands are CK dual automatically
- *Anomalous, in a controlled manner, due to Jacobian counterterms* that ensure (generalised) unitarity
- Shift in point of view:
    - A consistent field theory formulation of CK duality
    - Anomaly: generalised unitarity proof of loop double copy doesn't go through, at least not straightforwardly
    - Departure from standard articulation of loop integrand CK duality: all desiderata *except* generalised unitarity
    - Latter replaced with off-shell CK duality of BV action (without Jacobian counterterms): alternative proof of double copy

§2.

BV Lagrangian Syngamy

BV Lagrangian Syngamy

Syngamatic reproduction of factorable theories

Parent theories

$c_{IJ}\phi^I \Box \phi^J + f_{IJK}\phi^I \phi^J \phi^K$

$\tilde{c}_{\tilde{I}\tilde{J}}\tilde{\phi}^{\tilde{I}} \Box \tilde{\phi}^{\tilde{J}} + \tilde{f}_{\tilde{I}\tilde{J}\tilde{K}}\tilde{\phi}^{\tilde{I}} \tilde{\phi}^{\tilde{J}} \tilde{\phi}^{\tilde{K}}$

Factorise (meiosis)

Factorisation

$c_{ab}C_{ij}\Phi^{ai}\Box\Phi^{aj} + f_{abc}F_{ijk}\Phi^{ai}\Phi^{bj}\Phi^{ck}$

$\tilde{c}_{\tilde{a}\tilde{b}}\tilde{C}_{\tilde{\imath}\tilde{\jmath}}\Phi^{\tilde{a}\tilde{\imath}}\Box\tilde{\Phi}^{\tilde{a}\tilde{\jmath}} + \tilde{f}_{\tilde{a}\tilde{b}\tilde{c}}\tilde{F}_{\tilde{\imath}\tilde{\jmath}\tilde{k}}\tilde{\Phi}^{\tilde{a}\tilde{\imath}}\tilde{\Phi}^{\tilde{b}\tilde{\jmath}}\tilde{\Phi}^{\tilde{c}\tilde{k}}$

Fusion (Syngamy)

Daughter theories

Double copy

$C_{ij}\tilde{C}_{\tilde{\imath}\tilde{\jmath}}\Phi^{i\tilde{\imath}}\Box\Phi^{j\tilde{\jmath}} + F_{ijk}\tilde{F}_{\tilde{\imath}\tilde{\jmath}\tilde{k}}\Phi^{i\tilde{\imath}}\Phi^{j\tilde{\jmath}}\Phi^{k\tilde{k}}$

$\tilde{c}_{\tilde{a}\tilde{b}}C_{ij}\Phi^{\tilde{a}i}\Box\Phi^{\tilde{a}j} + \tilde{f}_{\tilde{a}\tilde{b}\tilde{c}}F_{ijk}\Phi^{\tilde{a}i}\Phi^{\tilde{b}j}\Phi^{\tilde{c}k}$

$c_{ab}\tilde{C}_{ij}\Phi^{a\tilde{\imath}}\Box\Phi^{a\tilde{\jmath}} + f_{abc}\tilde{F}_{\tilde{\imath}\tilde{\jmath}\tilde{k}}\Phi^{a\tilde{\imath}}\Phi^{b\tilde{\jmath}}\Phi^{c\tilde{k}}$

$c_{ab}\tilde{c}_{\tilde{a}\tilde{b}}\Phi^{a\tilde{a}}\Box\Phi^{a\tilde{b}} + f_{abc}\tilde{f}_{\tilde{a}\tilde{b}\tilde{c}}\Phi^{a\tilde{a}}\Phi^{b\tilde{b}}\Phi^{c\tilde{c}}$

Zeroth copy

BV Lagrangian Syngamy

Yang–Mills squared

- $S^{\text{YM}}_{\text{BRST-CK}} \otimes \tilde{S}^{\text{YM}}_{\text{BRST-CK}} \to \mathcal{N} = 0$ supergravity

$$A^{ia} = (A_\mu{}^a, \text{ghosts}, \text{auxiliaries}) \qquad S^{\text{YM}}_{\text{CK}} = \int C_{ij} c_{ab} A^{ia} \Box A^{ja} + F_{ijk} f_{abc} A^{ia} A^{jb} A^{kc}$$

$$A^{i\tilde{\imath}} = (h_{\mu\nu}, B_{\mu\nu}, \varphi, \text{ghosts}, \text{auxiliaries}) \quad S^{\mathcal{N}=0}_{\text{DC}} = \int C_{ij} C_{\tilde{\imath}\tilde{\jmath}} A^{i\tilde{\imath}} \Box A^{j\tilde{\jmath}} + F_{ijk} F_{\tilde{\imath}\tilde{\jmath}\tilde{k}} A^{i\tilde{\imath}} A^{j\tilde{\jmath}} A^{k\tilde{k}}$$

- $G \times \tilde{G}$ bi-adjoint scalar theory,

$$S^{\text{bi-adj}}_{\text{DC}} = c_{ab} \tilde{c}_{\tilde{a}\tilde{b}} \Phi^{a\tilde{a}} \Box \Phi^{a\tilde{b}} + f_{abc} \tilde{f}_{\tilde{a}\tilde{b}\tilde{c}} \Phi^{a\tilde{a}} \Phi^{b\tilde{b}} \Phi^{c\tilde{c}}$$

- Cf. scattering equation formalism [Hodges '11; Cachazo–He–Yuan '13 '14]

BV Lagrangian Syngamy

BRST-Lagrangian CK duality $\Rightarrow$ consistent syngamy

- No mention of CK duality - overly general?
- How do we know $S_{\text{DC}}^{\mathcal{N}=0}$ is equivalent to $S_{\text{BRST}}^{\mathcal{N}=0}$?
- Semi-classical equivalence of $S_{\text{DC}}^{\mathcal{N}=0}$ (requires on-shell tree-level CK duality)

$$F_{ijk} f_{abc} A^{ia} A^{jb} A^{kc} \quad \rightarrow \quad F_{ijk} F_{\tilde{\imath}\tilde{\jmath}\tilde{k}} A^{i\tilde{\imath}} A^{j\tilde{\jmath}} A^{k\tilde{k}}$$

$$\sum \frac{nc}{d} \quad \rightarrow \quad \sum \frac{n\tilde{n}}{d}$$

- $\Rightarrow$ physical (h, B, φ) tree-level amplitudes of $\mathcal{N} = 0$ supergravity
- Cf. [Bern-Dennen-Huang-Kiermaier 1004.0693] for gravitons up to 6 points

BV Lagrangian Syngamy

BRST-Lagrangian CK duality ⇒ consistent syngamy

- No mention of CK duality - overly general?
- How do we know $S_{\text{DC}}^{\mathcal{N}=0}$ is equivalent to $S_{\text{BRST}}^{\mathcal{N}=0}$?
- Semi-classical equivalence of $S_{\text{DC}}^{\mathcal{N}=0}$ (requires on-shell tree-level CK duality)

$$F_{ijk} f_{abc} A^{ia} A^{jb} A^{kc} \quad \to \quad F_{ijk} F_{\tilde{\imath}\tilde{\jmath}\tilde{k}} A^{i\tilde{\imath}} A^{j\tilde{\jmath}} A^{k\tilde{k}}$$

$$\sum \frac{nc}{d} \quad \to \quad \sum \frac{n\tilde{n}}{d}$$

- ⇒ physical (h, B, φ) tree-level amplitudes of $\mathcal{N} = 0$ supergravity
- Cf. [Bern-Dennen-Huang-Kiermaier 1004.0693] for gravitons up to 6 points
- Quantum consistency: how do we we know that there exists some BRST Q such that:

$$QS_{\text{DC}} = 0, \qquad Q^2 = 0$$

BV Lagrangian Syngamy

BRST-Lagrangian CK duality $\Rightarrow$ consistent syngamy

- No mention of CK duality - overly general?
- How do we know $S_{\text{DC}}^{\mathcal{N}=0}$ is equivalent to $S_{\text{BRST}}^{\mathcal{N}=0}$?
- Semi-classical equivalence of $S_{\text{DC}}^{\mathcal{N}=0}$ (requires on-shell tree-level CK duality)

$$F_{ijk} f_{abc} A^{ia} A^{jb} A^{kc} \quad \rightarrow \quad F_{ijk} F_{\tilde{\imath}\tilde{\jmath}\tilde{k}} A^{i\tilde{\imath}} A^{j\tilde{\jmath}} A^{k\tilde{k}}$$

$$\sum \frac{nc}{d} \quad \rightarrow \quad \sum \frac{n\tilde{n}}{d}$$

- $\Rightarrow$ physical (h, B, φ) tree-level amplitudes of $\mathcal{N} = 0$ supergravity
- Cf. [Bern-Dennen-Huang-Kiermaier 1004.0693] for gravitons up to 6 points
- Quantum consistency: how do we we know that there exists some BRST Q such that:

$$QS_{\text{DC}} = 0, \qquad Q^2 = 0$$

Answer: double-copy operator Q_{DC} (requires off-shell BRST CK duality)

BV Lagrangian Syngamy

Double copy of BRST charge

- Double copy of BV action implies double copy BRST operator Q_{DC}

$$S^{\text{YM}}_{\text{BV CK-dual}} = \int C_{ij} c_{ab} A^{ia} \Box A^{ja} + F_{ijk} f_{abc} A^{ia} A^{jb} A^{kc} + A^{+}_{ia} \underbrace{\left(Q^{i}{}_{j} A^{ja} + Q^{i}_{jk} f^{a}_{bc} A^{jb} A^{kc} \right)}_{QA \text{ and sim. for } \tilde{Q}\tilde{A}}$$

$$QA^{ia} = Q^{i}{}_{j} A^{ja} + Q^{i}{}_{jk} f^{a}{}_{bc} A^{jb} A^{kc} \qquad \tilde{Q}\tilde{A}^{\tilde{a}\tilde{\imath}} = Q^{\tilde{\imath}}{}_{\tilde{\jmath}} \tilde{A}^{\tilde{b}\tilde{\jmath}} + \tilde{f}^{\tilde{a}}{}_{\tilde{b}\tilde{c}} \tilde{Q}^{\tilde{\imath}}{}_{\tilde{\jmath}\tilde{k}} \tilde{A}^{\tilde{b}\tilde{\jmath}} \tilde{A}^{\tilde{c}\tilde{k}}$$

$$\underbrace{\underbrace{Q^{i}{}_{j} A^{j\tilde{\imath}} + Q^{i}{}_{jk} F^{\tilde{\imath}}{}_{\tilde{\jmath}\tilde{k}} A^{j\tilde{\jmath}} A^{k\tilde{k}}}_{Q_L} + \underbrace{Q^{\tilde{\imath}}{}_{\tilde{\jmath}} A^{i\tilde{\jmath}} + F^{i}{}_{jk} Q^{\tilde{\imath}}{}_{\tilde{\jmath}\tilde{k}} A^{j\tilde{\jmath}} A^{k\tilde{k}}}_{Q_R}}_{Q_{\text{DC}}}$$

- Yang-Mills gauge ⇒ diffeomorphisms and 2-form gauge symmetries:

$$Q_{\text{DC}} = Q_{\text{diffeo}} + Q_{\text{2-form}} + \text{trivial symmetries}$$

Cf. [Anastasiou-LB-Duff-Hughes-Nagy '14]

BV Lagrangian Syngamy

All order double copy

- Since F^{ijk} satisfy the same identities as f^{abc}

$$Q_{\text{DC}} S_{\text{DC}} = 0, \qquad Q_{\text{DC}}^2 = 0$$

BV Lagrangian Syngamy

All order double copy

- Since F^{ijk} satisfy the same identities as f^{abc}

$$Q_{\mathrm{DC}} S_{\mathrm{DC}} = 0, \qquad Q^2_{\mathrm{DC}} = 0$$

- Semi-classical equivalence + $Q_{\mathrm{DC}} \Rightarrow$ quantum equivalence

- Einstein *is* the square of Yang–Mills (at least perturbatively)

BV Lagrangian Syngamy

All order double copy

- Since F^{ijk} satisfy the same identities as f^{abc}

$$Q_{\text{DC}} S_{\text{DC}} = 0, \qquad Q^2_{\text{DC}} = 0$$

- Semi-classical equivalence $+$ Q_{DC} $\Rightarrow$ quantum equivalence

- Einstein *is* the square of Yang–Mills (at least perturbatively)

- Double copy of symmetries generalises, e.g.

$$\text{global susy} \quad \times \quad \text{gauge} \quad \rightarrow \quad \text{local susy}$$

- Straightforward supersymmetric completion

§4.

Generalisations

Generalisations

The double copy to all orders

- Given CK duality of the tree-level physical S-matrix we can run our argument:
  - Non-linear sigma model [Chen-Du '13] $\to$ special Galileon
  - Fundamental couplings [Johansson-Ochirov '14] $\to$ plethora of supergravity theories
  - Bagger–Lambert–Gustavsson [Bargheer-He-McLoughlin '12; Huang-Johansson '12]$\to$ $D = 3$ maximal supergravity

Super Yang–Mills and Supergravity

BRST-Lagrangian CK duality for super Yang–Mills

- Irreducible super Yang–Mills multiplets are CK duality respecting
  Cf. [Bjerrum-Bohr–Damgaard–Vanhove '09]

- Susy Ward identities: CK gluons + susy $\Rightarrow$ CK gluini
  (Caveat: higher order operators can spoil this argument, since there are superamplitudes with vanishing all-gluon component)

Super Yang–Mills and Supergravity

BRST-Lagrangian CK duality for super Yang–Mills

- Irreducible super Yang–Mills multiplets are CK duality respecting
  Cf. [Bjerrum-Bohr–Damgaard–Vanhove '09]

- Susy Ward identities: CK gluons + susy $\Rightarrow$ CK gluini
  (Caveat: higher order operators can spoil this argument, since there are superamplitudes with vanishing all-gluon component)

- CK dual BRST-Lagrangian then follows with (essentially) no new ideas

Super Yang–Mills and Supergravity

BRST-Lagrangian double copy

- (Type I super Yang–Mills)^2 = Type IIA/B supergravity

$$A^{ia} = (A_\mu{}^a, \psi_\alpha{}^a, \text{ghosts}, \text{aux})$$

$$A^{i\tilde{\jmath}} = (h_{\mu\nu}, B_{\mu\nu}, \phi, \Psi_{\alpha\nu}, \Psi_{\mu\beta}, F_{\alpha\beta}, \text{ghosts}, \text{aux})$$

gluino

R-R field strengths

gravitini

- Local NS-R sector susy follows from super Yang–Mills factors

$$\mathcal{Q}_\alpha A_\mu{}^a = \delta^a{}_b \gamma_{\mu\alpha}{}^\beta \psi_\beta{}^b + \cdots \quad \longrightarrow \quad \mathcal{Q}_\alpha h_{\mu\nu} = \gamma_{(\mu\alpha}{}^\beta \Psi_{\beta\nu)} + \cdots$$

- Super $\eta, \bar{\eta}$ and Nielsen–Kallosh χ ghosts

$$\bar{c} \otimes \psi \sim \bar{\eta}, \quad c \otimes \psi \sim \eta, \quad b \otimes \psi \sim \chi$$

- Similar for R–NS sector

Super Yang–Mills and Supergravity

Ramond–Ramond sector

- Double copy $\psi_\alpha \otimes \psi_\beta$ gives *field strengths* $F_{\alpha\beta}$, not potentials:
  - Representation theory

$$\begin{aligned} \text{IIA:} \quad \overline{16} \otimes 16 &= 1 \oplus 45 \oplus 210 \\ \text{IIB:} \quad 16 \otimes 16 &= 10 \oplus 120 \oplus 126 \end{aligned}$$

  - The BRST transformation of the gluino has no linear contribution, $Q_{\text{BRST}}\psi = [c, \psi]$, so $\psi \otimes \psi$ cannot transform as a potential
  - R-R background fields couple to worldsheet through field strengths

Super Yang–Mills and Supergravity

Ramond–Ramond sector

- Double copy $\psi_\alpha \otimes \psi_\beta$ gives *field strengths* $F_{\alpha\beta}$, not potentials:
  - Representation theory

$$\begin{aligned} \text{IIA:} \quad & \overline{16} \otimes 16 &= 1 \oplus 45 \oplus 210 \\ \text{IIB:} \quad & 16 \otimes 16 &= 10 \oplus 120 \oplus 126 \end{aligned}$$

  - The BRST transformation of the gluino has no linear contribution, $Q_{\text{BRST}}\psi = [c, \psi]$, so $\psi \otimes \psi$ cannot transform as a potential
  - R-R background fields couple to worldsheet through field strengths

- Type IIA/B action can be written in terms of field strengths, e.g.

$$F_2 \wedge \star F_2 + \tilde{F}_4 \wedge \star F_4 + B_2 \wedge \tilde{F}_4 \wedge \tilde{F}_4 + B_2 \wedge B_2 \wedge F_2 \wedge \tilde{F}_4 - \tfrac{1}{3} B_2 \wedge B_2 \wedge B_2 \wedge F_2 \wedge F_2$$

Super Yang–Mills and Supergravity

Sen's mechanism from double copy Ramond–Ramond sector

- Double copy R–R field strengths are *elementary* fields that correctly reproduce scattering amplitudes through their Feynman diagrams

$$\mathcal{L}^{\mathsf{DC}}_{\mathsf{R–R}} = \overline{F}^{\alpha\beta}\,\Box^{-1}\partial\!\!\!/_{\alpha}{}^{\alpha'}\partial\!\!\!/_{\beta}{}^{\beta'}F_{\alpha'\beta'} + \cdots$$

← direct from double copy

$F_{\alpha\beta} \sim \sum_{p=0}^{d} \frac{1}{p!}(\gamma^{\mu_1\cdots\mu_p}C)F_{\mu_1\cdots\mu_p}$ } $\rightarrow$ $-\frac{1}{2}\left(F\wedge\star F - \mathrm{d}F\wedge\star\Box^{-1}\mathrm{d}F\right) + \cdots$

Aux. $(D-p-1)$-form B } $\rightarrow$ $-\frac{1}{2}F\wedge\star F - \xi B\wedge\mathrm{d}F - \frac{1}{2}B\wedge\star\Box B + \cdots$

Undo Feynman gauge } $\rightarrow$ $-\frac{1}{2}F\wedge\star F - \xi B\wedge\mathrm{d}F + \frac{1}{2}\mathrm{d}B\wedge\star\mathrm{d}B + \cdots$

Sen's mechanism from double copy Ramond–Ramond sector

- Double copy R–R field strengths are *elementary* fields that correctly reproduce scattering amplitudes through their Feynman diagrams

$$\begin{aligned}
\mathcal{L}^{\mathsf{DC}}_{\mathsf{R\text{–}R}} &= \overline{F}^{\alpha\beta}\,\Box^{-1}\not{\partial}_{\alpha}{}^{\alpha'}\not{\partial}_{\beta}{}^{\beta'}F_{\alpha'\beta'} + \cdots \\
&\to -\tfrac{1}{2}\left(F\wedge\star F - \mathrm{d}F\wedge\star\Box^{-1}\mathrm{d}F\right) + \cdots \\
&\to -\tfrac{1}{2}F\wedge\star F - \xi B\wedge\mathrm{d}F - \tfrac{1}{2}B\wedge\star\Box B + \cdots \\
&\to -\tfrac{1}{2}F\wedge\star F - \xi B\wedge\mathrm{d}F + \tfrac{1}{2}\mathrm{d}B\wedge\star\mathrm{d}B + \cdots
\end{aligned}$$

- Sen's mechanism [Sen '15] generalized to arbitrary (as opposed to self-dual) field strengths [BJKMSW '21]

- Sen's mechanism was motivated by IIB string field theory, where the R–R sector is naturally given in terms of bispinors - natural double copy shadow

§5.

Homotopy CK Duality and Double Copy

Homotopy Algebras and BV Lagrangian Field Theories

- Homotopy algebras: generalise familiar (matrix, Lie. . .) algebras to include "higher products" satisfying "higher relations" up to homotopies

Homotopy Algebras and BV Lagrangian Field Theories

- Homotopy algebras: generalise familiar (matrix, Lie. . .) algebras to include "higher products" satisfying "higher relations" up to homotopies

- Lie algebras $\to$ L_∞-algebras, first arose in string field theory:

Vector space $\mathfrak{g} = V_0$	Graded vector space $\mathfrak{L} = \bigoplus_n V_n$
Bracket $\mu_2 = [-,-]$	Higher brackets $\mu_1 = [-],\ \mu_2 = [-,-],\ \mu_3 = [-,-,-], \ldots$
Relations $Antisymmetry + Jacobi$	Relations $Antisymmetry + homotopy Jacobi$

[Zwiebach '93; Hinich–Schechtman '93]

Homotopy Algebras and BV Lagrangian Field Theories

- Homotopy algebras: generalise familiar (matrix, Lie. . .) algebras to include "higher products" satisfying "higher relations" up to homotopies

- Lie algebras $\to$ L_∞-algebras, first arose in string field theory:

Vector space $\mathfrak{g} = V_0$	Graded vector space $\mathfrak{L} = \bigoplus_n V_n$
Bracket $\mu_2 = [-,-]$	Higher brackets $\mu_1 = [-], \ \mu_2 = [-,-], \ \mu_3 = [-,-,-], \dots$
Relations $\mathit{Antisymmetry} + \mathit{Jacobi}$	Relations $\mathit{Antisymmetry} + \mathit{homotopy Jacobi}$

[Zwiebach '93; Hinich–Schechtman '93]

- Associative algebras $\to$ A_∞-algebras [Stasheff '63]

- Commutative algebras $\to$ C_∞-algebras [Kadeishvili '88]

Homotopy Algebras and BV Lagrangian Field Theories

- Chevalley–Eilenberg formulation of Lie algebra $\mathfrak{g}$ with basis t_a:

$$\mathrm{CE}(\mathfrak{g}) = \bar{T}(\mathfrak{g}[1]^*) := \bigoplus_{p=1}^{\infty} \mathrm{Sym}^p(\mathfrak{g}[1]^*)$$

$$Qt^a = -\tfrac{1}{2} f^a{}_{bc} t^b t^c, \qquad Q^2 = 0 \Leftrightarrow \text{Jacobi}$$

Homotopy Algebras and BV Lagrangian Field Theories

- Chevalley–Eilenberg formulation of Lie algebra $\mathfrak{g}$ with basis t_a:

$$\mathrm{CE}(\mathfrak{g}) = \bar{T}(\mathfrak{g}[1]^*) := \bigoplus_{p=1}^{\infty} \mathrm{Sym}^p\,(\mathfrak{g}[1]^*)$$

$$Qt^a = -\tfrac{1}{2} f^a{}_{bc} t^b t^c, \qquad Q^2 = 0 \Leftrightarrow \text{Jacobi}$$

- Chevalley–Eilenberg formulation of L_∞-algebra $\mathfrak{L}$ with basis t_a:

$$\mathrm{CE}(\mathfrak{L}) = \bar{T}(\mathfrak{L}[1]^*)$$

$$Qt^a = -\sum_n \tfrac{1}{n!} \mu_n{}^a{}_{a_1 \cdots a_n} t^{a_1} \cdots t^{a_n}, \qquad Q^2 = 0 \Leftrightarrow \text{homotopy Jacobi}$$

- Any BV field theory with operator Q_{BV} corresponds to an L_∞-algebra in the CE picture, see e.g. [Jurco-Raspollini-Saemann-Wolf '18]

Homotopy Algebras and BV Lagrangian Field Theories

- Yang-Mills theory $\mathfrak{L}^{\text{YM}}$

$$
\begin{array}{ccccccc}
\mathfrak{L}_0^{\text{YM}} & \oplus & \mathfrak{L}_1^{\text{YM}} & \oplus & \mathfrak{L}_2^{\text{YM}} & \oplus & \mathfrak{L}_3^{\text{YM}} \\
c & \xrightarrow{d} & A & \xrightarrow{d^\dagger d} & A^+ & \xrightarrow{d^\dagger} & c^+ \\
 & & b & \xrightarrow{\text{Id}} & \bar{c} & & \\
 & & \bar{c}^+ & \xrightarrow{-\text{Id}} & b^+ & &
\end{array}
$$

- Homotopy Maurer-Cartan theory $\longrightarrow$ field strengths $+$ gauge trans.
- Cartan-Killing form $\langle -,-\rangle_{\mathfrak{g}} \rightarrow$ cyclic structure $\langle -,-\rangle_{\text{YM}}$ on $\mathfrak{L}^{\text{YM}}$
- BV action $\sim \sum \frac{1}{(i+1)!}\langle a, \mu_i(a,\ldots,a)\rangle$

Homotopy Algebras and BV Lagrangian Field Theories

- Yang-Mills theory $\mathfrak{L}^{\mathrm{YM}}$

$$
\begin{array}{ccccccc}
\mathfrak{L}_0^{\mathrm{YM}} & \oplus & \mathfrak{L}_1^{\mathrm{YM}} & \oplus & \mathfrak{L}_2^{\mathrm{YM}} & \oplus & \mathfrak{L}_3^{\mathrm{YM}} \\
c & \xrightarrow{d} & A & \xrightarrow{d^\dagger d} & A^+ & \xrightarrow{d^\dagger} & c^+ \\
 & & b & \xrightarrow{\mathrm{Id}} & \bar{c} & & \\
 & & \bar{c}^+ & \xrightarrow{-\mathrm{Id}} & b^+ & &
\end{array}
$$

- Homotopy Maurer-Cartan theory $\longrightarrow$ field strengths + gauge trans.
- Cartan-Killing form $\langle -,-\rangle_{\mathfrak{g}} \to$ cyclic structure $\langle -,-\rangle_{\mathrm{YM}}$ on $\mathfrak{L}^{\mathrm{YM}}$
- BV action $\sim \sum \frac{1}{(i+1)!}\langle a, \mu_i(a,\dots,a)\rangle$
- L_∞ quasi-isomorphisms $\longrightarrow$ physical equivalence (field redefinitions etc)
  - Strictification: $\mu_i = 0, i > 2 \to$ cubic theory
  - Minimal model: $\mu_1 = 0 \to$ tree scattering amplitudes

Cf. [Jurčo-Raspollini-Saemann-Wolf '18; Jurčo-Macrelli-Saemann-Wolf '19]

Colour-Kinematic-Scalar Factorisation of Yang-Mills

- $\mathfrak{L}^{\mathsf{YM}}$ factorises into $\mathfrak{colour} \otimes \mathfrak{kinematics} \otimes_\tau \mathfrak{scalar}$

$$\mathfrak{L}^{\mathsf{YM}} = \underbrace{\underbrace{\mathfrak{colour}}_{L_\infty} \otimes \underbrace{\mathfrak{kinematics} \otimes_\tau \underbrace{\mathfrak{scalar}}_{A_\infty}}_{C_\infty}}_{L_\infty}$$

[BLKMSW '21]

Colour-Kinematic-Scalar Factorisation of Yang-Mills

- $\mathfrak{L}^{\mathsf{YM}}$ factorises into $\mathfrak{colour} \otimes \mathfrak{kinematics} \otimes_\tau \mathfrak{scalar}$

$$\mathfrak{L}^{\mathsf{YM}} = \underbrace{\underbrace{\mathfrak{colour}}_{L_\infty} \otimes \underbrace{\mathfrak{kinematics} \otimes_\tau \underbrace{\mathfrak{scalar}}_{A_\infty}}_{C_\infty}}_{L_\infty}$$

[BLKMSW '21]

- $\mathfrak{colour}$: gauge group Lie algebra

Colour-Kinematic-Scalar Factorisation of Yang-Mills

- $\mathfrak{L}^{\mathsf{YM}}$ factorises into $\mathfrak{colour} \otimes \mathfrak{kinematics} \otimes_\tau \mathfrak{scalar}$

$$\mathfrak{L}^{\mathsf{YM}} = \underbrace{\underbrace{\mathfrak{colour}}_{L_\infty} \otimes \underbrace{\mathfrak{kinematics} \otimes_\tau \underbrace{\mathfrak{scalar}}_{A_\infty}}_{C_\infty}}_{L_\infty}$$

[BLKMSW '21]

- $\mathfrak{colour}$: gauge group Lie algebra

- $\mathfrak{kinematics}$: graded vector space of Poincaré representations of fields

$$\begin{array}{ccccccc} \mathbb{R}[-1] & \oplus & (\mathbb{R}^d \oplus \mathbb{R}) & \oplus & \mathbb{R}[1] & \oplus & \text{Auxiliaries} \\ c & & (A_\mu, b) & & \bar{c} & & B_{\mu\nu\rho} \cdots \end{array}$$

- $\mathfrak{scalar}$: A_∞-algebra of a scalar field theory

$$\langle -, - \rangle_{\mathsf{YM}} = \langle -, - \rangle_{\mathfrak{colour}} \langle -, - \rangle_{\mathfrak{kinematics}} \langle -, - \rangle_{\mathfrak{scalar}}$$

Homotopy algebra of CK duality

Michel Reiterer [1912.03110]

- Proof of on-shell tree-level CK duality for physical gluons via $BV_\infty^\square$-algebra!
- Relies on the existence of a degree -1 unary map h on Zeitlin-Costello BV complex for Yang–Mills (think order formulation with A, F^+) satisfying

$$h^2 = 0, \quad dh + hd = \square \quad \text{(plus some other conditions)}$$

- h exists and is a second-order derivation up to homotopy $\Rightarrow$
  - $BV_\infty^\square$-algebra on Zeitlin-Costello BV complex
  - On-shell tree-level CK duality for physical gluons

Homotopy algebra of CK duality

Michel Reiterer [1912.03110]

- Proof of on-shell tree-level CK duality for physical gluons via $BV_\infty^\square$-algebra!
- Relies on the existence of a degree -1 unary map h on Zeitlin-Costello BV complex for Yang–Mills (think order formulation with A, F^+) satisfying

$$h^2 = 0, \quad dh + hd = \square \quad \text{(plus some other conditions)}$$

- h exists and is a second-order derivation up to homotopy $\Rightarrow$
  - $BV_\infty^\square$-algebra on Zeitlin-Costello BV complex
  - On-shell tree-level CK duality for physical gluons
- Very special: only $D = 4$, no loop desiderata (ghosts, gauge-fixing)
- A little mysterious: $BV_\infty^\square$-algebra generalise famous BV_∞-algebras (homotopy BV-algebras [Galvez-Carrillo–Tonks–Vallette '09]), where e.g.

$$\Delta^2\square = (\mathrm{id} + \sigma_{(123)} + \sigma^2_{(123)})(\mathrm{id} \otimes \Delta\square) - (\mathrm{id} + \sigma_{(123)} + \sigma^2_{(123)})(\mathrm{id} \otimes \mathrm{id} \otimes \square)$$

Homotopy algebra of CK duality

The homotopy algebra of CK duality [BJKMSW 'to appear 21]

Homotopy algebra of CK duality

The homotopy algebra of CK duality [BJKMSW 'to appear 21]

- BRST-Lagrangian CK duality $\Leftrightarrow$ $BV^{\square}$-algebra, cf. [Getzler '93]

$$\mathfrak{L}^{\mathsf{YM}} = \mathfrak{g} \otimes \underbrace{\mathfrak{kinematics} \otimes_\tau \mathfrak{scalar}}_{\mathfrak{Kin}\ \equiv\ BV^{\square}\text{-algebra}}$$

- $BV^{\square}$-algebra comes with two products $- \cdot -$ and $[-,-]$ and three unary operators

$$d^2 = h^2 = 0, \quad dh + hd = \square$$

Homotopy algebra of CK duality

The homotopy algebra of CK duality [BJKMSW 'to appear 21]

- BRST-Lagrangian CK duality $\Leftrightarrow$ $BV^{\square}$-algebra, cf. [Getzler '93]

$$\mathfrak{L}^{\mathsf{YM}} = \mathfrak{g} \otimes \underbrace{\mathfrak{kinematics} \otimes_{\tau} \mathfrak{scalar}}_{\mathfrak{Kin}\ \equiv\ BV^{\square}\text{-algebra}}$$

- $BV^{\square}$-algebra comes with two products $- \cdot -$ and $[-,-]$ and three unary operators

$$d^2 = h^2 = 0, \quad dh + hd = \square$$

- The homotopy $BV^{\square}$-algebra depends on the ambient category
- In the usual category of chain complexes d is privileged

Homotopy algebra of CK duality

The homotopy algebra of CK duality [BJKMSW 'to appear 21]

- BRST-Lagrangian CK duality $\Leftrightarrow$ $BV^{\square}$-algebra, cf. [Getzler '93]

$$\mathfrak{L}^{\mathsf{YM}} = \mathfrak{g} \otimes \underbrace{\mathfrak{kinematics} \otimes_\tau \mathfrak{scalar}}_{\mathfrak{Kin} \;\equiv\; BV^{\square}\text{-algebra}}$$

  - $BV^{\square}$-algebra comes with two products $- \cdot -$ and $[-,-]$ and three unary operators
$$d^2 = h^2 = 0, \quad dh + hd = \square$$
  - The homotopy $BV^{\square}$-algebra depends on the ambient category
  - In the usual category of chain complexes d is privileged
  - Introduce symmetric monoidal category of Hodge complexes (modules over twisted Hopf algebras with central element $\square$)
$$d^2 = h^2 = 0, \quad dh + hd = \square$$
Coassociativity $\Rightarrow$ the seven-term identity
  - In this category, both d and h are a part of the ambient structure

Homotopy algebra of CK duality

The homotopy algebra of CK duality

- Homotopy algebra: $BV^{\square}_{\infty}/_{\mathbf{Hdg}}$-algebra
- Corresponds to integrating out auxiliary fields
- Homotopy relations of $BV^{\square}_{\infty}/_{\mathbf{Hdg}}$-algebra $\leftrightarrow$ kinematic Jacobi relations

Homotopy algebra of CK duality

The homotopy algebra of CK duality

- Homotopy algebra: $BV^{\square}_{\infty/\mathbf{Hdg}}$-algebra
- Corresponds to integrating out auxiliary fields
- Homotopy relations of $BV^{\square}_{\infty/\mathbf{Hdg}}$-algebra $\leftrightarrow$ kinematic Jacobi relations
- Computational efficiency:
  - Purely tree-level calculations
  - One identity at any order (the rest follow axiomatically)

$$\sum_{p+q=n+2} n\text{-point tree with two internal } (p\text{-ary and } q\text{-ary}) \text{ vertices}$$
$$= n\text{-point tree with one internal } (n\text{-ary}) \text{ vertex}$$

  - But, work with Feynman diagrams - marry with on-shell methods?

Future work

- AdS background [Zhou '21; Diwakar-Herderschee-Roiban-Teng '21 . . .] $\to$ Hopf algebra of universal enveloping algebra of AdS isometries

- Bagger-Lambert-Gustavsson CK duality [Bargheer-He-McLoughlin '12; Huang-Johansson '12] $\to$ m-ary $BV^{\square}$ operads

- Matter coupling [Johansson-Ochirov '14] $\to$ many-sorted $BV^{\square}$ operads

- String theory (modular envelope over) $BV_{\infty}^{L_0}$

$$\{d, h\} = \square \quad \longrightarrow \quad \{Q, b_0\} = L_0$$

  Cf. BV_{∞} structure on TVOA [Galvez-Carrillo–Tonks–Vallette '09] lifting the BV-algebra structure on the BRST (co)homology [Lian-Zuckerman '93]

- Counterterms?

Thanks for listening