Asymptotics of Young diagrams through Matrix Models

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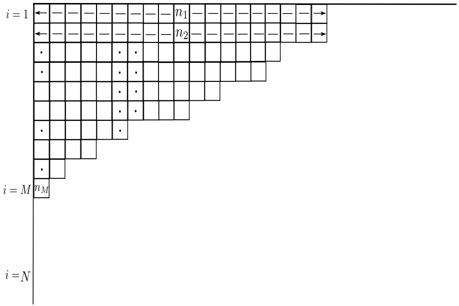
Young diagrams : a collection of boxes arranged in a particular manner, characterised by an integer k (total number of boxes) and its partitions $\{n_i\}$



A typical Young diagram in English notation

(left-justified rows with non-increasing length)

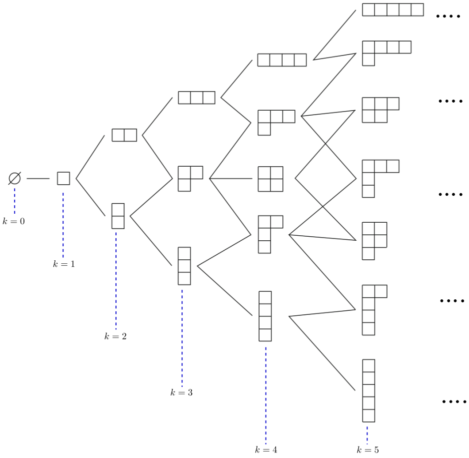
$k = \sum_i n_i$

Young diagrams provide a convenient diagrammatic way to describe the irreducible representations of symmetric groups, general linear groups, special unitary groups. The questions about asymptotic behaviour of representations can be translated to questions about asymptotics of Young diagrams.

The asymptotic theory of representations is helpful in the investigation of high rank groups and infinite dimensional groups, which are pertinent to physics.

Growth of Young diagrams

All diagrams at level $k+1$ can be obtained by adding one box to each diagram at level k in all possible allowed ways.



$\mathcal{Y}_k$: Set of all Young diagrams with k boxes
λ_k: Young diagram at level k

Plancherel Growth Process :

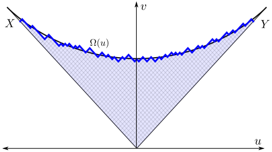
$$\mathcal{P}_{\text{transition}}(\lambda_k, \lambda_{k+1}) = \frac{1}{k+1} \frac{\text{dim}\lambda_{k+1}}{\text{dim}\lambda_k}$$

when λ_{k+1} is obtained from λ_k by addition of a single box, and zero, otherwise. This is a **Markovian** growth process.

The probability associated to a Young diagram at level k is given by the **Plancherel measure**

$$\mathcal{P}(\lambda_k) = \frac{(\text{dim}\lambda_k)^2}{k!}$$

Limit shape theorem : Young diagrams following Plancherel growth process converge to a universal diagram in the large k limit when scaled appropriately. The boundary becomes smooth under scaling, and takes a particular form, called the limit shape. [Vershik and Kerov, 1977] [Logan and Shepp, 1977]



$$\lim_{k\to\infty} \hat{v}_k(u) \equiv \Omega(u) = \begin{cases} \frac{2}{\pi}\left(u \sin^{-1}\frac{u}{2} + \sqrt{4-u^2}\right) & \text{if} \quad |u| \leq 2 \\ |u| & \text{if} \quad |u| > 2. \end{cases}$$

Kerov introduced a differential model to capture the growth of Young diagrams.

He associated a 'time' parameter with Young diagrams in the continuum limit and showed that the Young diagrams equipped with Plancherel measure follow a first order partial differential equation termed as the automodel equation

$$\partial_t \hat{v}(u,t) = \frac{1}{2t}\left(\hat{v}(u,t) - u\partial_u \hat{v}(u,t)\right), \quad \hat{v}(u,t) = \sqrt{t}\ \hat{v}(u/\sqrt{t}) \quad \text{for} \quad t > 0$$

The limit shape is a unique solution of the automodel equation in far future with $\emptyset$ as initial condition in far past.

Unitary matrix models and Young diagrams

The classical solutions of UMMs can be described in terms of asymptotic (large number of boxes) Young diagrams.

$$\mathcal{Z} = \int dU\ e^{S(U)}, \qquad U : N \times N \text{ unitary matrix}$$

Single plaquette model:

$$S(U) = N \sum_{n=1}^{Q} \frac{\beta_n}{n} \left(\text{Tr}U^n + \text{Tr}U^{\dagger n}\right), \quad \text{or} \quad S(U) = N \sum_{n=1}^{Q} \frac{a_n}{n} \left(\text{Tr}U^n \text{Tr}U^{\dagger n}\right)$$

Using the Frobenius formula

$$\prod_n (\text{Tr}U^n)^{k_n} = \sum_R \chi_R(C(\vec{k}))\text{Tr}_R[U]$$

a UMM can be analysed in terms of representations of the unitary group. Further, the sum over representations of $U(N)$ can be written as a sum over different Young diagrams

$$\sum_R \to \sum_k \sum_{\{n_i\}} \delta\left(k - \sum_i n_i\right), \quad n_1 \geq \ldots\ldots \geq n_N$$

$\chi_{\mathbf{R}}(\mathbf{C}(\tilde{\mathbf{k}}))$: character of the conjugacy class $C(\vec{k})$ of the permutation group S_k, $k = \sum_n n k_n$

Large-N

We introduce new variables $h_i = n_i - i + N$ such that $h_1 > .... > h_N$.

In the large-N limit, we define continuous variables

$$h(x) = \frac{h_i}{N}, \quad k = N^2 k' \quad \text{where} \quad x = \frac{i}{N}, \quad x \in [0, 1]$$

The partition function takes the form of a path integral with an effective action

$$\mathcal{Z} = \int [Dh(x)] e^{-N^2 S_{eff}[h(x)]}$$

Regarding N^2 as $\hbar^{-1}$, $N \to \infty$ is equivalent to doing a semi-classical approximation ($\hbar \to 0$). In that limit, the partition function is dominated by saddle-points of the effective action.

To characterize different dominant diagrams, we introduce a Young diagram density function

$$u(h) = -\frac{\partial x}{\partial h}$$

Since $h(x)$ is monotonically decreasing function of x, $u(h) \leq 1$.

UMM for Plancherel growth

Define a Young lattice

$$\mathcal{Y} = \bigcup_{k=0}^{\infty} \mathcal{Y}_k$$

$\mathcal{Y}_k$ can be thought of as a canonical ensemble of Young diagrams with the same macroscopic variable k. One can then write a grand canonical partition function for the Young lattice

$$\mathcal{Q}_{\mathcal{Y}} = \sum_{k=0}^{\infty} z^k \mathcal{Z}_{\mathcal{Y}_k}, \;\; z > 0 \quad \text{with} \quad \mathcal{Z}_{\mathcal{Y}_k} = \sum_{\lambda_k} \mathcal{P}(\lambda_k)\delta(k - |\lambda_k|)$$

$$\boxed{\mathcal{Q}_{\mathcal{Y}} = \sum_{k=0}^{\infty} z^k \sum_{\lambda_k} \frac{(\mathrm{dim}\lambda_k)^2}{k!}\delta(k - |\lambda_k|) = \frac{1}{1-z}, \quad z > 0}$$

We regularise by introducing a large positive integer N and constraining the Young diagrams in lattice $\mathcal{Y}$ to not have more than N rows. The regularised partition function is exactly same as the partition function (a close cousin of Gross-Witten-Wadia model)

$$\mathcal{Z}_c = \int [dU] \; e^{a \mathrm{Tr} U \mathrm{Tr} U^\dagger}$$

with z identified with a. The large-N (large k) solutions of this model are well-known.

Symmetric solution of GWW: The large N solution of the saddle-point equation must be invariant under transposition, since any two Young diagrams related to each other by transposition have the same probability $\mathcal{P}(\lambda_k)$.

$$u(h) = \begin{cases} 1 & h \in [0,p) \\ \frac{1}{\pi}\cos^{-1}\left[\frac{h-1}{2\xi}\right] & h \in (p,q] \end{cases} \quad \text{with} \quad p = 1-2\xi, \quad q = 1+2\xi$$

Since $p \geq 0$, this solution is valid for $0 \leq \xi \leq 1/2$. The limiting value, $\xi = 1/2$ corresponds to the limit shape,

$$\boxed{\mathcal{P}_{\lambda_k} = 1 + \mathcal{O}\left(\frac{1}{L}\right)}$$

The Young diagram density at the limiting value $\xi = \frac{1}{2}$ satisfies the automodel equation with $k^{'}$ playing the role of t:

$$\boxed{\partial_{k'} u(h,k') + \frac{h-1}{2k'}\partial_h u(h,k') = 0}$$

This gives an alternate proof of limit shape theorem of Vershik-Kerov and Logan-Shepp.

q-deformed Plancherel growth

$0 < q < 1$ is a deformation parameter

(I) [S. Kerov, '93, E. Strahov, '07]

$$\mathcal{P}_q(\lambda_k) = (1-q)^k q^{b(\lambda_k)} \dim \lambda_k \frac{\prod_{1\le i<j\le N}[h_i - h_j]}{\prod_{i=1}^N [h_i]!}$$

$$\text{where} \quad b(\lambda) = \sum_{i=1}^{l(\lambda)} (i-1)\lambda_i,$$

The square bracket represents the q-analogue of a positive integer defined as

$$[x] = 1 - q^x$$

Follows from representation theory of Iwahori-Hecke algebras.

(II) [B. Eynard, '08]

$$\mathcal{P}_q(\lambda_k) = \left(\frac{\mathrm{dim}_q \lambda_k}{k!}\right)^2 = k!(1-q)^{2k} q^{2b(\lambda_k)} \frac{\prod_{1\le i<j\le N}[h_i - h_j]^2}{\prod_{i=1}^N ([h_i]!)^2}$$

Appears in topological string theory partition function on certain Calabi-Yau threefolds and Gromov-Witten invariants of $\mathbb{P}^1$.

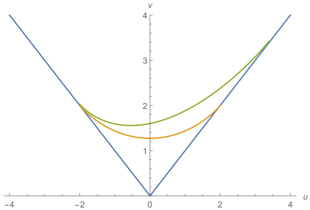
Grand canonical partition function :

$$\mathcal{Z}_q = \sum_{k=0}^{\infty} \sum_{\lambda_k} t^k \mathcal{P}_q(\lambda_k) \delta(k - |\lambda_k|), \quad t > 0$$

In order to go to the continuum limit, we first re-define the parameter $q = e^{-g_s}$, and then take the double scaling limit $N \to \infty$, $g_s \to 0$ such that $\lambda = N g_s$ is finite.

q-automodel diagrams :

$$\tilde{u}(h) = \frac{1}{\pi} \cos^{-1} \left[\frac{1 + \lambda^2 \xi^2 - e^{-\lambda(h-1)}}{2\lambda\xi} \right], \quad \text{where} \quad \xi^2 = tk'$$



supported between $a = 1 - \frac{2}{\lambda} \log(1 + \lambda\xi)$ and $b = 1 - \frac{2}{\lambda} \log(1 - \lambda\xi)$. This solution is valid for $\xi \leq \frac{1}{\lambda}(e^{\lambda/2} - 1)$. The limiting value of ξ gives the q-limit shape.

q-automodel equation :

$$\partial_\xi \tilde{u}(h, \xi, \lambda) + \frac{\left[(1 - \lambda^2\xi^2)e^{\lambda(h-1)} - 1\right]}{\lambda\xi} \partial_h \tilde{u}(h, \xi, \lambda) = 0$$

UMM for q-deformed Plancherel growth

One can diagonalise the matrices, $U \to \{e^{i\theta_1}, ..., e^{i\theta_i}, ...e^{i\theta_N}\}$, and describe a UMM in terms of eigenvalues. The large-N solutions are given in terms of the eigenvalue density

$$\rho(\theta) = \frac{1}{N}\sum_i \delta(\theta - \theta_i)$$

The eigenvalue density and Young diagram density are related to each other [Dutta and Gopakumar, 2008], [Dutta and Dutta, 2016, 2017, Chattopadhyay, Dutta and Dutta, 2017]

$$\begin{aligned} \pi u(h) &= \theta \\ h^2 - 2S(\theta)h + S^2(\theta) - \pi^2\rho^2(\theta) &= 0, \quad \text{where} \quad S(\theta) = \frac{1}{2} + \sum_n \beta_n \cos n\theta \end{aligned}$$

From the $u(h)$ and above relations, one can find that for q-deformed growth process

$$S(\theta) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{\xi^n \lambda^{n-1}}{n} \cos n\theta$$

Therefore, the matrix model that describes q-deformed Plancherel growth process is given by

$$\mathcal{Z} = \int dU \; e^{\sum_{n=1}^{\infty} \frac{\beta_n}{n}\left(\mathrm{Tr}U^n + \mathrm{Tr}U^{\dagger n}\right)}, \quad \text{with} \quad \beta_n = \frac{\xi^n \lambda^{n-1}}{n}$$

The phase space

The solutions $h_\pm(\theta)$ of

$$h^2 - 2S(\theta)h + S^2(\theta) - \pi^2\rho^2(\theta) = 0$$

describe the boundary of a two-dimensional phase space spanned by (h, θ).
Define a **phase space distribution function**

$$\omega(h,\theta) = \Theta\left(\frac{(h - h_-(\theta))(h_+(\theta) - h)}{2}\right),$$

such that $\omega(h,\theta) = 1$ for $h_-(\theta) < h < h_+(\theta)$ and zero otherwise.

$$u(h) = \frac{1}{2\pi}\int d\theta\ \omega(h,\theta), \quad \rho(\theta) = \frac{1}{2\pi}\int dh\ \omega(h,\theta)$$

The distribution function $\omega(h,\theta)$ is similar to Thomas-Fermi distribution at zero temperature

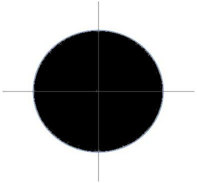
$$\Delta(p,q) = \Theta(\mu - \mathfrak{h}(p,q))$$

Comparing the two, one can find the single particle Hamiltonian density:

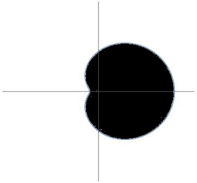
$$\mathfrak{h}(h,\theta) = \frac{h^2}{2} - S(\theta)h + \frac{g(\theta)}{2} + \mu, \quad \text{where} \quad g(\theta) = h_+(\theta)h_-(\theta)$$

Phase space for automodel class:

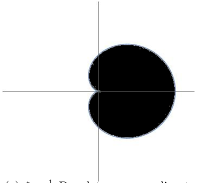
$$h_+(\theta) = 1 + 2\xi \cos\theta, \quad h_-(\theta) = 0$$



(a) $\xi = 0$ Droplet corresponding to YD with no box $\emptyset$



(b) $0 < \xi < \frac{1}{2}$ Droplet corresponding to generic YD in automodel class



(c) $\xi = \frac{1}{2}$ Droplet corresponding to *limit shape*

Hilbert space description : fluctuations of the limit shape

In terms of variables $\bar{h}(\theta) = h(\theta) - S(\theta)$, the boundary evolution equations following from $\mathfrak{h}(h, \theta)$ are given by (double-copy of dispersionless KdV equation)

$$\dot{\bar{h}}_{\pm}(t, \theta) = \bar{h}_{\pm}(t, \theta)\bar{h}'_{\pm}(t, \theta)$$

We introduce Poisson brackets

$$\{\bar{h}_{\pm}(t, \theta), \bar{h}_{\pm}(t, \theta')\} = \pm\pi\hbar\delta'(\theta - \theta') \quad \text{and} \quad \{\bar{h}_{+}(t, \theta), \bar{h}_{-}(t, \theta')\} = 0$$

such that the evolution equation follows from

$$\dot{\bar{h}}_{\pm}(t, \theta) = \{\bar{h}_{\pm}(t, \theta), H_h\} \quad \text{where} \quad H_h = \frac{1}{2\pi\hbar}\int dh \int d\theta\, \omega(h, \theta)\, \mathfrak{h}(h, \theta)$$

We quantize the system by promoting Poisson brackets to commutation relations and study fluctuations about the classical solutions. The modes of fluctuations satisfy Kac-Moody algebra.

$$[a_m, a_n] = \frac{1}{2}m\delta_{m+n}, \quad [b_m, b_n] = \frac{1}{2}m\delta_{m+n}, \quad \text{and} \quad [a_m, b_n] = 0.$$

We can construct a Hilbert space over a one parameter family of vacua, which are eigenstates of the zero modes of fluctuations.

$$a_n|s\rangle = 0, \quad b_n|s\rangle = 0 \quad \text{for } n > 0$$
$$a_0|s\rangle = -b_0|s\rangle = \pi_0|s\rangle = s|s\rangle$$

We define a mapping between states and phase space geometries,

$$\boxed{|\psi\rangle \to \langle\psi|\bar{h}_+(\theta)|\psi\rangle}$$

- The ground state corresponds to a constant shift over the classical value : $\langle s|\bar{h}_+|s\rangle = \frac{1}{2} + s\hbar$ with zero dispersion.
- An excited state $|\vec{k}\rangle = \prod_{n=1}^{\infty}(a_n^\dagger)^{k_n}|s\rangle$ corresponds to $O(\hbar)$ ripples (quantum excitations) over the classical surface : The expectation value of $\bar{h}_+$ is same that in the ground state with a non-zero dispersion.
- Coherent states correspond to classical deformations

$$|\tau_+\rangle = \text{exp}\left(\sum_{n=1}^{\infty}\frac{2\tau_n^+ a_n^\dagger}{n\hbar}\right)|s\rangle$$

$$\omega_{\tau_+}(\theta) = \frac{\langle\tau_+|\bar{h}_+|\tau_+\rangle}{\langle\tau_+|\tau_+\rangle} = \frac{1}{2} + s\hbar + 2\sum_{n>0}\tau_n^+\cos n\theta$$

The q-automodel diagrams correspond to coherent states

$$|\tau_q^+\rangle = \exp\left(\sum_{n=1}^{\infty} \frac{\tau_n^+ a_n^\dagger}{n\hbar}\right)|s\rangle, \quad \text{where} \quad \tau_n^+ = \frac{\xi^n \lambda^{n-1}}{n}$$

$\lambda \to 0$ corresponds to the automodel class.

Excitations over coherent states correspond to $O(\hbar)$ fluctuations of the limit shape. In particular,

$$\exp\left(\frac{a_1^\dagger \xi}{\hbar} + \sum_{n>1} \frac{2\alpha_n a_n^\dagger}{\sqrt{n}}\right)|0\rangle \quad \text{corresponds to} \quad \frac{1}{2} + \xi\cos\theta + 2\hbar\sum_{n>1}\sqrt{n}\alpha_n \cos n\theta$$

which is essentially the result of [Ivanov and Olshanki, 2003] about fluctuations of the limit shape corresponding to the Plancherel growth process.

$$\lim_{k\to\infty} \hat{\nu}_k(u) \sim \Omega(u) + \frac{2}{\sqrt{k}}\Delta(u), \quad \Delta(u) = \Delta(2\cos\theta) = \frac{1}{\pi}\sum_{n=2}^{\infty}\frac{\alpha_n}{\sqrt{n}}\sin(n\theta)$$

where α_n are independent Gaussian random variables with mean 0 and variance 1.

Summary

The growth processes of Young diagrams can be studied through UMMs. The Plancherel growth process can be described as the evolution of symmetric (no-gap) phase of GWW model while a q-deformed Plancherel growth process can be described by a single plaquette model with parameters of matrix model depending on the deformation parameter q.

- One can try to explore the holographic dual of the UMM describing the q-deformed Plancherel growth.
- The dispersionless KdV equation describes the boundary evolution of phase space geometries of UMMs. How does the dispersive term effect the evolution?

Thank You !

Saddle-point equations

GWW :

$$\fint \frac{u(h')dh'}{h-h'} = \ln\left(\frac{h}{\xi}\right), \quad \text{where} \quad \xi^2 = z\frac{k}{N^2} = zk'$$

q-deformation I :

$$\fint_{h_L}^{h_U} dh' u(h') \left[\coth\frac{\lambda(h-h')}{2} + \frac{2}{\lambda(h-h')}\right] = 1 + \frac{2}{\lambda}\ln\frac{h(1-e^{-\lambda h})}{t\lambda k'}$$

q-deformation II :

$$\fint_{h_L}^{h_U} dh' u(h') \frac{e^{-\lambda h}}{e^{-\lambda h} - e^{-\lambda h'}} = -\frac{1}{\lambda}\log\left[\frac{1-e^{-\lambda h}}{\lambda\xi}\right]$$

$$w = \frac{1-e^{-\lambda h}}{\lambda}$$

$$\fint_{w_L}^{w_U} dw' \frac{u(w')}{w-w'} = \frac{1}{(1-\lambda w)}\log\frac{w}{\xi}$$