Different approaches to gravity from Yang-Mills squared

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The double copy

- Double copy = the idea that gravity can be expressed as a "product" (to be defined later) of two Yang-Mills theories.
- Inspired by the Kawai-Lewellen-Tye (KLT) relations of string theory, has experienced a revival through the Bern-Carrasco-Johannson (BCJ) duality and corresponding double copy for amplitudes.
- Appeal comes from possible simplifications arising by ***translating problems in gravity to simpler counterparts in gauge theory***.
- Extended in many directions
  - scattering amplitudes
  - classical solutions
  - precision gravity
  - symmetries
  - Lagrangian constructions
  - supergravity
  - non-gravitational theories
  - effective theories
  - ...
- **A number of different formulations .... is there a unique double copy ?**

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Why symmetries ?

- Important in off-shell constructions in the double copy (e.g. Lagrangian constructions, building gravity solutions with control over gauge choices etc.)
- Double copy for symmetries well understood at linear order [Anastasiou, Borsten, Duff, Hughes, Jubb, Makwana, SN, Zoccali].
- One can go beyond linear level and construct Lagrangians perturbatively to higher orders using techniques from the amplitudes double copy [Bern, Dennen, Huang, Kiermaier, SN, Borsten, Jurco, Kim, Macrelli, Saemann, Wolf, Ferrero, Francia], but we need field redefinitions to map to the standard GR description because we lack a direct relation to symmetries.

Why asymptotic symmetries ?

- Unlike standard diffeomorphisms and gauge transformations, do not vanish at boundary of space-time (null infinity).
- Crucial in the study of soft theorems.
- Potentially linked to experiment via memory effect.
- Double copy at null infinity:
  - Classical solutions [Adamo, Kol, Godazgar, Monteiro, Peinador Veiga, Pope].
  - Celestial holography [Casali, Puhm, Pasterski, Donnay, Sharma, Kalyanapuram...]
  - Amplitudes in the soft limit

Self-dual sector

- Simplification: start with the self-dual sector, where we have a simple description of the "kinematic algebra" [Monteiro, O'Connell]
- Kinematic algebra = additional structure in Yang-Mills theory which facilitates double copy constructions.
- Self-duality conditions:
  - Yang-Mills

$$\tilde{F}_{\mu\nu} := \frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma} = iF_{\mu\nu}.$$

  - Gravity

$$\tilde{R}_{\mu\nu\rho}{}^{\sigma} := \frac{1}{2}\epsilon_{\mu\nu}{}^{\eta\lambda}R_{\eta\lambda\rho}{}^{\sigma} = iR_{\mu\nu\rho}{}^{\sigma}$$

Notation

- Light-cone: $U = \frac{X^0 - X^3}{\sqrt{2}}$, $V = \frac{X^0 + X^3}{\sqrt{2}}$, $Z = \frac{X^1 + iX^2}{\sqrt{2}}$, $\bar{Z} = \frac{X^1 - iX^2}{\sqrt{2}}$.
- In light-cone gauge, we can write the YM field as

$$A_U = 0, \quad A_V = \partial_{\bar{Z}}\Phi, \quad A_Z = \partial_U \Phi, \quad A_{\bar{Z}} = 0$$

  and similarly for the graviton, but it helps to take a more covariant approach. Define

$$x^i := (U, \bar{Z}), \quad y^\alpha := (V, Z).$$

  Metric:

$$ds^2 = 2\eta_{i\alpha} dx^i dy^\alpha = -2dUdV + 2dZd\bar{Z}.$$

- Introduce the tensors

$$\Omega_{ij} dx^i dx^j = dUd\bar{Z} - d\bar{Z}dU, \quad \Pi_{\alpha\beta} dy^\alpha dy^\beta = dVdZ - dZdV$$

  They are left/right inverses of each other.
- We can write the self-dual YM and gravity fields as

$$\mathcal{A}_\alpha = \Pi_\alpha{}^i \partial_i \Phi, \quad h_{\alpha\beta} = \Pi_\alpha{}^i \Pi_\beta{}^j \partial_i \partial_j \phi$$

Light-cone fields

- We can write the self-dual YM and gravity fields as

$$\mathcal{A}_\alpha = \Pi_\alpha^{\ i}\partial_i\Phi, \quad h_{\alpha\beta} = \Pi_\alpha^{\ i}\Pi_\beta^{\ j}\partial_i\partial_j\phi$$

  Note this is fully non-perturbative .

- The scalar fields satisfy the equations (following from the SD conditions):

$$\Box\Phi = -i\Pi^{ij}[\partial_i\Phi, \partial_j\Phi], \quad \Box\phi = \frac{1}{2}\Pi^{ij}\Pi^{kl}\partial_i\partial_k\phi\partial_j\partial_l\phi$$

  Introduce the Poisson bracket

$$\{f, g\} := \Pi^{ij}\partial_i f\partial_j g$$

  corresponding to area-preserving diffeomorphisms (*kinematic algebra*) of $x^i := (U, \bar{Z})$, and rewrite

$$\Box\Phi = -i\{[\Phi, \Phi]\}, \quad \Box\phi = \frac{1}{2}\{\{\phi, \phi\}\}$$

  Double copy prescription for the e.o.m. [Monteiro, O'Connell]

$$\boxed{\Phi \to \phi, \quad -i[\ ,\ ] \to \frac{1}{2}\{\ ,\ \}}$$

Symmetries [Campiglia ,SN '21]

- Remember $x^i := (U, \bar{Z})$, $\quad y^\alpha := (V, Z)$.
- We are seeking residual symmetries preserving

$$\mathcal{A}_\alpha = \Pi_\alpha^{\ i}\partial_i\Phi, \quad h_{\alpha\beta} = \Pi_\alpha^{\ i}\Pi_\beta^{\ j}\partial_i\partial_j\phi$$

  and the e.o.m. for Φ and ϕ.
- Yang-Mills transfomation:

$$\delta_\Lambda\mathcal{A}_\mu = \partial_\mu\Lambda + i[\Lambda, \mathcal{A}_\mu].$$

  We want to preserve $\mathcal{A}_i = 0$, so we get

$$\Lambda = \Lambda(y)$$

  and finally we can read off the transformation of the scalar Φ

$$\delta\Phi = x^i\Omega_i^{\ \alpha}\partial_\alpha\Lambda + i[\Lambda, \Phi], \qquad \Lambda = \Lambda(y)$$

Symmetries - 1st Family

- Gravity

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad h_{\mu\nu} = \Pi_\mu{}^\rho \Pi_\nu{}^\sigma \partial_\rho \partial_\sigma \phi \quad \text{(nonperturbative)}$$

Transformation

$$\delta_\xi h_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = \mathcal{L}_\xi \eta_{\mu\nu} + \mathcal{L}_\xi h_{\mu\nu}$$

We want to preserve $h_{i\mu} = 0 \Rightarrow$ two families of diffeomorphisms.

- 1st family has parameters

$$\xi_i = 0, \quad \xi_\alpha = b_\alpha(y)$$

and finally we can read off the transformation of the scalar Φ

$$\delta_\xi \phi = \Omega_i{}^\alpha \Omega_j{}^\beta x^i x^j \partial_\alpha b_\beta + \eta^{i\alpha} b_\alpha \partial_i \phi.$$

1st Family of Symmetries - Double copy

- The transformation rules for the scalars

$$\begin{cases} \delta\Phi & = & x^i \Omega_i{}^\alpha \partial_\alpha \Lambda + i[\Lambda, \Phi], & \text{YM} \\ \delta_\xi \phi & = & \Omega_i{}^\alpha \Omega_j{}^\beta x^i x^j \partial_\alpha b_\beta + \eta^{i\alpha} b_\alpha \partial_i \phi, & \text{gravity} \end{cases}$$

- To make the double copy manifest, we define

$$\lambda \equiv 2\Omega_i{}^\alpha x^i b_\alpha$$

which can be thought of as a "Hamiltonian" w.r.t. the Poisson bracket, and then

$$\begin{cases} \delta\Phi & = & x^i \Omega_i{}^\alpha \partial_\alpha \Lambda + i[\Lambda, \Phi], & \text{YM} \\ \delta_\xi \phi & = & \frac{1}{2} x^i \Omega_i{}^\alpha \partial_\alpha \lambda - \frac{1}{2}\{\lambda, \phi\}, & \text{gravity} \end{cases}$$

Then the double copy is

$$\boxed{\Phi \to \phi, \quad -i[\ ,\ ] \to \frac{1}{2}\{\ ,\ \}, \quad \Lambda \to \lambda}$$

together with a factor $\mathfrak{r} = \frac{1}{2}$ for the first term. ***This is the first non-perturbative double copy result for (a subset of) symmetries.***

1st Family of Symmetries - Double copy

- The factor $\mathfrak{r} = \frac{\deg(\Lambda)+1}{\deg(\lambda)+1}$, where $\deg(a)$ counts the order of x^i in a dissapears in the replacement rules for

$$\delta \mathcal{A}_\alpha = \Pi_\alpha{}^i \partial_i \delta\Phi \quad \rightarrow \quad \delta_\xi h_{\alpha\beta} = \Pi_\alpha{}^i \Pi_\beta{}^j \partial_i \partial_j \delta_\xi \phi,$$

- If we want $\delta_\xi \phi$ to preserve the e.o.m. for ϕ, we have to restrict

$$\xi_\alpha = b_\alpha(y) = \partial_\alpha b(y) \quad \Rightarrow \quad \lambda = 2x^i \Omega_i{}^\alpha \partial_\alpha b$$

- It is convenient to define the operator $\boxed{S \equiv x^i \Omega_i{}^\alpha \partial_\alpha}$, and then

$$\begin{aligned} \delta_\Lambda \Phi &= S(\Lambda) + i[\Lambda, \Phi] \\ \delta_\lambda \phi &= \tfrac{1}{2} S(\lambda) - \tfrac{1}{2}\{\lambda, \phi\}, \quad \text{with} \quad \lambda = 2S(b) \end{aligned}$$

and we can write the double copy rules as

$$\boxed{\Phi \rightarrow \phi, \quad -i[\ ,\ ] \rightarrow \frac{1}{2}\{\ ,\ \}, \quad \Lambda \rightarrow \lambda, \quad S(\Lambda) \rightarrow \mathfrak{r} S(\lambda)}$$

1st Family - Asymptotic Symmetries

- Work in Bondi-flat coordinates, which are related to light-cone coordinates via

$$U = rz\bar{z} + u, \quad V = r, \quad Z = rz, \quad \bar{Z} = r\bar{z},$$

in which the Minkowski line element takes the form

$$ds^2 = -2dudr + 2r^2 dzd\bar{z}.$$

- At null infinity, the YM and metric fields are captured by

$$\begin{aligned} \mathcal{A}_z(r,u,z,\bar{z}) &\overset{r\to\infty}{=} A_z(u,z,\bar{z}) + \cdots \\ h_{zz}(r,u,z,\bar{z}) &\overset{r\to\infty}{=} rC_{zz}(u,z,\bar{z}) + \cdots \end{aligned}$$

- Assume a standard fall-off for scalars

$$\phi(r,u,z,\bar{z}) \overset{r\to\infty}{=} \frac{\phi_{\mathcal{I}}(u,z,\bar{z})}{r} + \cdots, \quad \Phi(r,u,z,\bar{z}) \overset{r\to\infty}{=} \frac{\Phi_{\mathcal{I}}(u,z,\bar{z})}{r} + \cdots$$

- Then, in the self-dual sector

$$\begin{aligned} A_z &= \partial_u \Phi_{\mathcal{I}}, \quad A_{\bar{z}} = 0 \qquad \text{YM} \\ C_{zz} &= \partial_u^2 \phi_{\mathcal{I}}, \quad C_{\bar{z}\bar{z}} = 0 \qquad \text{gravity} \end{aligned}$$

1st Family - Asymptotic Symmetries

- For YM:

$$A_z = \partial_u \Phi_{\mathcal{I}}, \quad A_{\bar{z}} = 0$$

- A subset of our gauge transformations will give the asymptotic symmetries preserving the YM self-dual sector at null infinity:

$$\Lambda(y) = \Lambda(V, Z) \;\rightarrow\; \Lambda(V/Z) = \Lambda(z) \equiv \Lambda_0(z)$$

where the subscript denotes the order in r. Then

$$\delta A_z = \partial_z \Lambda_0 + i[\Lambda_0, A_z], \quad \delta A_{\bar{z}} = 0, \quad \text{as needed}$$

- The operator $S \equiv x^i \Omega_i{}^{\alpha} \partial_\alpha$ acts on a function of the form $r^k F_k(u, z, \bar{z})$ as

$$S(r^n F_n) = r^n S_0(F_n) + r^{n-1} S_{-1}(F_n)$$

where

$$S_0 = -\bar{z} u \partial_u + n\bar{z} - \bar{z}^2 \partial_{\bar{z}}, \quad S_{-1} = u\partial_z.$$

so finally

$$\delta_\Lambda \Phi_{\mathcal{I}} = S_{-1}(\Lambda_0) + i[\Lambda_0, \Phi_{\mathcal{I}}].$$

1st Family - Asymptotic Double Copy

- On the gravity side, the ("Hamiltonian" of the) symmetry parameter is

$$\lambda = r\lambda_1 + \lambda_0, \quad \text{with} \quad \lambda_1 = -2\bar{z} f(z), \quad \lambda_0 = -2u\partial_z f(z).$$

The Poisson bracket in Bondi coordinates is

$$\{a, b\} = r^{-1}\{a, b\}_{-1}, \quad \text{with} \quad \{a, b\}_{-1} = \partial_{\bar{z}} a \partial_u b - \partial_u a \partial_{\bar{z}} b.$$

The double copy copy structure appears again

$$\begin{cases} \delta_\Lambda \Phi_{\mathcal{I}} & = & S_{-1}(\Lambda_0) + i[\Lambda_0, \Phi_{\mathcal{I}}] & \text{YM} \\ \delta \phi_{\mathcal{I}} & = & \frac{1}{2} S_{-1}(\lambda_0) - \frac{1}{2}\{\lambda_1, \phi_{\mathcal{I}}\}_{-1} & \text{gravity} \end{cases}$$

The transformation of the gravity scalar can be written as

$$\delta \phi_{\mathcal{I}} = -u^2 \partial_z^2 f(z) + f(z) \partial_u \phi_{\mathcal{I}}$$

and remembering that $C_{zz} = \partial_u^2 \phi_{\mathcal{I}}$:

$$\delta C_{zz} = -2\partial_z^2 f + f \partial_u C_{zz} \quad \text{supertranslation!}$$

- ***Holomorphic large gauge transformations in YM double copy to holomorphic supertranslations in gravity.***

Symmetries - 2nd Family

- Start on the gravity side (in the bulk). The 2nd family of symmetries preserving $h_{\mu\nu} = \Pi_\mu{}^\rho \Pi_\nu{}^\sigma \partial_\rho \partial_\sigma \phi$ and the e.o.m. for ϕ has diffeomorphism parameters

$$\xi^i = \eta^{i\alpha} \Omega_j{}^\beta x^j \partial_\alpha \partial_\beta c, \quad \xi^\alpha = -\Omega^{\alpha\beta} \partial_\beta c, \quad c = c(y)$$

- The scalar field transforms as

$$\delta_c \phi = \frac{1}{3} \Omega_i{}^\alpha \Omega_j{}^\beta \Omega_k{}^\gamma x^i x^j x^k \partial_\alpha \partial_\beta \partial_\gamma c + \xi^i \partial_i \phi + \xi^\alpha \partial_\alpha \phi$$

- In analogy with the 1st Family, we define a "Hamiltonian":

$$\tilde{\lambda} = \Omega_i{}^\alpha \Omega_j{}^\beta x^i x^j \partial_\alpha \partial_\beta c$$

so that:

$$\delta\phi = \frac{1}{3} \Omega_i{}^\alpha x^i \partial_\alpha \tilde{\lambda} - \tfrac{1}{2} \{\tilde{\lambda}, \phi\} - \Omega^{\alpha\beta} \partial_\alpha \phi \partial_\beta c.$$

- What YM transformation double copies to this under

$$\boxed{\Phi \to \phi, \quad -i[\ ,\ ] \to \frac{1}{2}\{\ ,\ \}, \quad \Lambda \to \lambda, \quad S(\Lambda) \to \mathfrak{r}S(\lambda)} \quad ?$$

2nd Family- perturbation theory

- Write the gravity scalar as

$$\phi = \phi^{(0)} + \phi^{(1)} + \cdots ,$$

and similarly for YM.

$$\delta\phi^{(0)} = \frac{1}{3}\Omega_i{}^{\alpha} x^i \partial_\alpha \tilde{\lambda}$$

- Under the double copy rules $\Phi \to \phi,\ \Lambda \to \lambda,\ S(\Lambda) \to \mathfrak{r}S(\lambda)$, this comes from the linearised YM transformation:

$$\delta^{(0)}\Phi = \frac{1}{2}\Omega_i{}^{\alpha} x^i \partial_\alpha \tilde{\Lambda}, \quad \text{with} \quad \tilde{\Lambda} = \Omega_i{}^{\alpha} x^i \partial_\alpha B(y)$$

- To go to higher orders, we can make use of the perturbative expansion of the e.o.m. to write

$$\Box\delta\Phi^{(1)} = -2i\Pi^{ij}[\partial_i\Phi^{(0)}, \partial_j\delta\Phi^{(0)}]$$

to get

$$\delta\Phi^{(1)} = -i[\Phi^{(0)}, \tilde{\Lambda}] + 2i\frac{1}{\Box}\eta^{i\alpha}[\partial_\alpha\Phi^{(0)}, \partial_i\tilde{\Lambda}].$$

- This is perturbative and non-local, but the gravity transformation was non-perturbative and local !

2nd Family- perturbation theory

- When we rewrite the gravity transformation in terms of $\tilde{\lambda}$, it becomes perturbative and non-local

$$\delta\phi^{(1)} = \frac{1}{2}\{\phi^{(0)}, \tilde{\lambda}\} - \Box^{-1}\eta^{i\alpha}\{\partial_\alpha\phi^{(0)}, \partial_i\tilde{\lambda}\}$$

and then we see that

$$\begin{cases} \delta\Phi^{(1)} &= -i[\Phi^{(0)}, \tilde{\Lambda}] + 2i\frac{1}{\Box}\eta^{i\alpha}[\partial_\alpha\Phi^{(0)}, \partial_i\tilde{\Lambda}], \qquad \text{YM} \\ \delta\phi^{(1)} &= \frac{1}{2}\{\phi^{(0)}, \tilde{\lambda}\} - \frac{1}{\Box}\eta^{i\alpha}\{\partial_\alpha\phi^{(0)}, \partial_i\tilde{\lambda}\}, \quad \text{gravity} \end{cases}$$

are related by ***the same*** double copy rules as the first family

$$\boxed{\Phi \to \phi, \quad -i[\ ,\ ] \to \frac{1}{2}\{\ ,\ \}, \quad \tilde{\Lambda} \to \tilde{\lambda}, \quad S(\tilde{\Lambda}) \to \mathfrak{r}S(\tilde{\lambda})}$$

- One can see recursively that these rules work at all orders in perturbation theory.
- We related a perturbative, non-local transformation on the YM side, to a non-perturbative, local transformation on the gravity side.
- The gravity transformation is a subset of the usual diffeomorphisms, but the YM transformation is **not** a subset of gauge transformations - it is a symmetry that appears exclusively in the self-dual sector.

2nd Family - Asymptotics

Taking the limit to null infinity, we have two types of transformations arising from the 2nd Family

- Fist :

$$\delta^{(0)}\Phi_{\mathcal{I}} = \frac{1}{2}S_0(\tilde{\Lambda}_{-1}) = -u\bar{z}\partial_z\Lambda_0(z) - i\bar{z}[\Lambda_0, \Phi_{\mathcal{I}}],$$

  double copies to a **supertranslation** with parameter $f(z,\bar{z}) = \bar{z}g(z)$

$$\delta C_{zz} = -2\partial_z^2(\bar{z}g(z)) + \bar{z}g(z)\partial_u C_{zz}$$

- Second:

$$\partial_u\delta^{(1)}\Phi_{\mathcal{I}} = -i\bar{z}\partial_u[\Lambda_0, \Phi_{\mathcal{I}}] + i\partial_z[\Lambda_1, \Phi_{\mathcal{I}}] - i[\partial_u\Phi_{\mathcal{I}}, S_{-1}(\Lambda_1)]$$

  double copies to a **holomorphic superrotation**

$$\delta\phi_{\mathcal{I}} = -\frac{u^3}{6}\partial_z^3Y(z) + (Y(z)\partial_z + \frac{u}{2}\partial_zY(z)\partial_u + \frac{1}{2}\partial_zY(z))\phi_{\mathcal{I}}$$

An infinite tower of Double Copies

- The self-dual sectors of gravity and YM are integrable, so they possess an infinite tower of symmetries.
- Remember the self-dual equations

$$
\begin{aligned}
\Box\Phi &= -i\Pi^{ij}[\partial_i\Phi, \partial_j\Phi] \\
\Box\phi &= \frac{1}{2}\Pi^{ij}\Pi^{kl}\partial_i\partial_k\phi\partial_j\partial_l\phi
\end{aligned}
$$

- Let $\delta\Phi$ and $\delta\phi$ be symmetries of the SDYM and SDE equations respectively:

$$
\begin{aligned}
\Box\delta\Phi &= -2i\Pi^{ij}[\partial_i\Phi, \partial_j\delta\Phi] \\
\Box\delta\phi &= \Pi^{ij}\Pi^{kl}\partial_i\partial_k\phi\partial_j\partial_l\delta\phi
\end{aligned}
$$

- One can then obtain new symmetries $\tilde{\delta}\Phi$ and $\tilde{\delta}\phi$, defined implicitly by the condition

$$
\begin{aligned}
\partial_i\tilde{\delta}\Phi &= \Omega_i{}^{\alpha}\partial_\alpha\delta\Phi - i[\partial_i\Phi, \delta\Phi] \\
\partial_i\tilde{\delta}\phi &= \Omega_i{}^{\alpha}\partial_\alpha\delta\phi + \frac{1}{2}\{\partial_i\phi, \delta\phi\}
\end{aligned}
$$

An infinite tower of Double Copies

- The self-dual sectors of gravity and YM are integrable, so they possess an infinite tower of symmetries.

$$\begin{array}{ccc} \delta_1\Phi & \xrightarrow{\text{DC}} & \delta_1\phi \\ \downarrow & & \downarrow \\ \delta_2\Phi & \xrightarrow{\text{DC}} & \delta_2\phi \\ \downarrow & & \downarrow \\ \vdots & & \vdots \\ \delta_n\Phi & \xrightarrow{\text{DC}} & \delta_n\phi \\ \vdots & & \vdots \end{array}$$

- At any level n, we have the double copy relations

$$\Phi \to \phi, \quad -i[\ ,\ ] \to \tfrac{1}{2}\{\ ,\ \}, \quad \Lambda_n \to \lambda_n, \quad S \to \mathfrak{r}S, \quad \mathfrak{r} = \frac{\deg(\Lambda_n)+1}{\deg(\lambda_n)+1}.$$

- Proof by recursion.

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The spinorial formalism

- GR typically uses the language of tensors and four-vectors.
- Alternative formulation in terms of two-component spinors $\pi^A \equiv (\pi^0, \pi^1)$, and their higher-rank generalisations. Spinor indices raised and lowered with

$$\pi_A = \epsilon_{AB}\pi^B, \quad \pi^B = \pi_A \epsilon^{AB}.$$

- Any multi-rank spinor can be decomposed into a sum of terms, each of which involves symmetric spinors, multiplying Levi-Civita symbols.
- Any symmetric spinor factorises into a symmetrised product of spinors e.g.

$$S_{AB\ldots C} = S_{(AB\ldots C)} \quad \Rightarrow \quad S_{AB\ldots C} = \alpha_{(A}\beta_B \ldots \gamma_{C)}.$$

  with $\alpha_A, \ldots$ called *pricipal spinors*.
- Any tensorial quantity can be translated into the spinorial language using

$$\sigma^0_{AA'} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1_{AA'} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\sigma^2_{AA'} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3_{AA'} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The spinorial formalism

- For a 4-vector this gives

$$V_\alpha \sigma^\alpha_{AA'} = \frac{1}{\sqrt{2}} \begin{pmatrix} V_0 + V_3 & V_1 - iV_2 \\ V_1 + iV_2 & V_0 - V_3 \end{pmatrix},$$

where the determinant of the matrix on the right-hand side is

$$\det\left(V_\alpha \sigma^\alpha_{AA'}\right) = \frac{1}{2}\left((V_0)^2 - (V_1)^2 - (V_2)^2 - (V_3)^2\right).$$

This is proportional to the norm of the 4-vector, such that the determinant vanishes if V_α is null.

- Then the matrix must factorise i.e.

$$V_\alpha V^\alpha = 0 \quad \Rightarrow \quad V_\alpha \sigma^\alpha_{AA'} = \pi_A \pi_{A'}, \qquad (1)$$

where $\pi_{A'} = (\pi_A)^*$ given that the matrix in eq. (26) is clearly Hermitian.

- Conversely, given any spinor π_A, we may construct a matrix $M_{AA'} = \pi_A \pi_{A'}$, which in turn corresponds to a null 4-vector in spacetime. In particular, each of the so-called *principal spinors* appearing in the decomposition of a general symmetric tensor can be associated with a *principal null direction* in spacetime.

The spinorial formalism - gravity

- The Riemann tensor $R_{\alpha\beta\gamma\delta}$ can translate this into the spinor language as

$$
\begin{aligned}
R_{\alpha\beta\gamma\delta} \to R_{AA'BB'CC'DD'} &= \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \bar{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} \\
&+ \Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} + \bar{\Phi}_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'} \\
&+ 2\Lambda(\epsilon_{AC}\epsilon_{BD}\epsilon_{A'B'}\epsilon_{C'D'} + \epsilon_{AB}\epsilon_{CD}\epsilon_{A'D'}\epsilon_{B'C'}),
\end{aligned}
$$

- For *vacuum spacetimes*, we are left with the *Weyl tensor*: $C_{\alpha\beta\gamma\delta}$. We have the spinorial identification

$$C_{\alpha\beta\gamma\delta} \to \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \bar{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD}.$$

  where, Ψ_{ABCD} and $\bar{\Psi}_{A'B'C'D'}$ are the *anti-self-dual* and *self-dual* parts of the Weyl tensor respectively.
- The dynamics of the Weyl tensor is constrained by the Bianchi identity for the Riemann tensor, which leads to:

$$\nabla^{AA'}\Psi_{ABCD} = 0, \quad \nabla^{AA'}\bar{\Psi}_{A'B'C'D'} = 0.$$

- Ψ_{ABCD} is usually referred to as the *Weyl spinor*.

The spinorial formalism - various spins

- Electromagnetism:

$$F_{\alpha\beta} \to F_{AA'BB'} = \phi_{AB}\epsilon_{A'B'} + \bar{\phi}_{A'B'}\epsilon_{AB},$$

  where the symmetric spinors ϕ_{AB} and $\bar{\phi}_{A'B'}$ are the anti-self-dual and self-dual parts.

- The Maxwell equations then imply

$$\nabla^{AA'}\phi_{AB} = 0, \quad \nabla^{AA'}\bar{\phi}_{A'B'} = 0.$$

- General spinorial equations:

$$\nabla^{AA'}\phi_{AB\ldots C} = 0, \quad \nabla^{AA'}\bar{\phi}_{A'B'\ldots C'} = 0 \tag{2}$$

  where $\phi_{AB\ldots C}$ is assumed symmetric, with n indices. These are known as the *massless free field equations*.

- The spin of the field is given by the number of spinor indices divided by two,

The spinorial formalism - classifying solutions

- An immediate use of the spinorial language is that it allows us to classify different types of solutions in electromagnetism and gravity in terms of the degeneracy of the spinors.
- Electromagnetism

$$\phi_{AB} = \alpha_{(A}\beta_{B)},$$

and there are then two different "types" of field strength spinor:

  - (i) those with distinct null directions ($\alpha_A \not\propto \beta_A$);
  - (ii) those with a degenerate null direction, so that $\alpha_A \propto \beta_A$.

The spinorial formalism - classifying solutions

- For the Weyl tensor there are more possibilities. In general we have

$$\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}$$

then we can classify solutions as

Weyl type	Petrov label
$\{1,1,1,1\}$	I
$\{2,1,1\}$	II
$\{3,1\}$	III
$\{4\}$	N
$\{2,2\}$	D
$\{-\}$	O

Weyl Double Copy

- Given an electromagnetic field strength spinor ϕ_{AB}, one may construct a Weyl spinor according to the rule [Luna, Monteiro, Nicholson, O'Connell]

$$\Psi_{ABCD} = \frac{1}{S}\phi_{(AB}\phi_{CD)}$$

where S is a scalar function.

- This procedure was shown to hold for arbitrary type D and N vacuum spacetimes.
- Can we generalise away from these ?

Twistors

- Define twistor space as the set of solutions of the *twistor equation*

$$\nabla_{A'}^{(A}\Omega^{B)} = 0$$

whose general solution in Minkowksi space is

$$\Omega^A = \omega^A - ix^{AA'}\pi_{A'} .$$

- Twistors:

$$Z^\alpha = \left(\omega^A, \pi_{A'}\right) = \left(\omega^0, \omega^1, \pi_{0'}, \pi_{1'}\right) .$$

- The "location" of a twistor in Minkowski space is defined to be the region in which its associated spinor field Ω^A vanishes. This implies the *incidence relation*

$$\omega^A = ix^{AA'}\pi_{A'}$$

invariant under simultaneous rescalings

$$\omega^A \to \lambda\omega^A, \quad \pi_{A'} \to \lambda\pi_{A'}, \quad \lambda \in \mathbb{C},$$

so twistor space is projective. A point $x^{AA'}$ in position space defines a complex projective line in twistor space, which can be thought of as a Riemann sphere.

Twistors - the Penrose transform

- Correspondence between solutions of the massless free field equations and twistor space:

$$\phi_{A'B'\ldots C'}(x) = \frac{1}{2\pi i}\oint_{\Gamma} \pi_{E'}\, d\pi^{E'} \pi_{A'}\pi_{B'}\ldots\pi_{C'}[\rho_x f(Z^\alpha)],$$

where the symbol ρ_x denotes that we must restrict to the line in projective twistor space corresponding to the spacetime point $x^{AA'}$. The contour Γ for this integral is defined on the related Riemann sphere.

- The integrand (including the measure) must be homogeneous of degree zero under rescalings $\pi_{A'} \to \lambda\pi_{A'}$ (or $Z^\alpha \to \lambda Z^\alpha$). This in turn implies that the function $f(Z^\alpha)$ must have degree $(-n-2)$, where n is the number of indices appearing on the left-hand side.
- We can deal with exact solutions which linearise the e.o.m., or with general but linearised solutions:

$$\partial^{A'}_{D}\phi_{A'B'\ldots C'} = 0 \qquad (3)$$

- Works in arbitrary conformally flat spacetime.

Twistors - the Penrose transform

- There are some tricks for formulating representative twistor functions for spacetime fields possessing certain properties.
- Note that the factorisation property of symmetric spinors means that if a given n-index spinor has a k-fold principal spinor $\xi_{A'}$, it will vanish if contracted with $(n-k+1)$ factors of $\xi_{A'}$, but not if only $(n-k)$ factors are contracted. See

$$\phi_{A'B'\dots F'} = \underbrace{\xi_{(A'}\xi_{B'}\dots\xi_{C'}}_{k\ \text{factors}}\underbrace{\alpha_{D'}\beta_{E'}\dots\gamma_{F')}}_{(n-k)\ \text{factors}}$$

- Contracting the Penrose Transform with m factors of $\eta^{A'}$ gives

$$\underbrace{\eta^{A'}\eta^{B'}\dots\eta^{C'}}_{m\ \text{factors}}\underbrace{\phi_{A'B'\dots C'D'\dots F'}}_{n\ \text{indices}}(x) = \frac{1}{2\pi i}\oint_\Gamma \pi_{E'}d\pi^{E'}[\pi\eta]^m\pi_{C'}\dots\pi_{F'}[\rho_x f(Z^\alpha)]$$

we see that the field $\phi_{A'B'\dots F'}$ has at least a $(n-m+1)$-fold principal spinor $\eta_{A'}$, if the twistor function $f(Z^\alpha)$ has a single m^{th}-order pole as $\pi_{A'}\to\eta_{A'}$, enclosed by Γ.

Twistorial Double Copy[White'20, Chackon, SN,White'21]

- Remember the general (mixed) type D Weyl double copy may be written as

$$\phi_{A'B'C'D'} = \frac{1}{\phi}\phi^{(1)}_{(A'B'}\phi^{(2)}_{C'D')}.$$

- Consider two twistor functions $f^{(1,2)}_{\mathrm{EM}}(Z^\alpha)$ of homogeneity -4, and a further twistor function $f(Z^\alpha)$ of homogeneity -2.
- These will necessarily correspond to electromagnetic spinors $\phi^{(1,2)}_{A'B'}$ and a scalar field ϕ in spacetime. One may then form a product

$$f_{\mathrm{grav.}}(Z^\alpha) = \frac{f^{(1)}_{\mathrm{EM}}(Z^\alpha)\, f^{(2)}_{\mathrm{EM}}(Z^\alpha)}{f(Z^\alpha)},$$

such that the function on the left-hand side necessarily has homogeneity -6, and thus potentially corresponds to a spacetime field solving the spin-2 massless free field equation i.e. to a self-dual gravity solution.
- For a suitable choice of twistor functions, this spacetime relationship is precisely the type D Weyl double copy.

Twistorial Double Copy

- Define a family of twistor functions

$$f_m(Z^\alpha) = \frac{1}{m!}\left[Q_{\alpha\beta} Z^\alpha Z^\beta\right]^{-m},$$

  for some constant $Q_{\alpha\beta}$. This will produce a type D Weyl tensor (for $m = 3$), that is related to an electromagnetic field strength ($m = 2$) and scalar field ($m = 1$).
- One can show that this is indeed true by carrying out the Penrose transform in each case.
- Opens the door to a formulation on curver backgrounds.
- Allows us to go beyond Type D/N.

Twistorial Double Copy- More general solutions

- Consider the homogeneity -4 functions related to two different electromagnetic spinors

$$f^{(0,2)}_{\mathrm{EM}} = \frac{1}{(A_\alpha Z^\alpha)(B_\beta Z^\beta)^3} = \frac{1}{[\pi\mathcal{A}][\pi\mathcal{B}]^3}, \quad \mathcal{A}^{A'} = ix^{AA'}A_A + A^{A'}$$

$$f^{(1,1)}_{\mathrm{EM}} = \frac{1}{(A_\alpha Z^\alpha)^2(B_\beta Z^\beta)^2} = \frac{1}{[\pi\mathcal{A}]^2[\pi\mathcal{B}]^2},$$

as well as the homogeneity -2 function

$$f^{(0,0)} = \frac{1}{(A_\alpha Z^\alpha)(B_\beta Z^\beta)} = \frac{1}{[\pi\mathcal{A}][\pi\mathcal{B}]}.$$

Then we can construct the twistor representative for Type II solutions

$$f^{(\mathrm{II})}_{\mathrm{grav.}} = \frac{1}{f^{(0,0)}} f^{(1,1)}_{\mathrm{EM}} \left(-\frac{[\mathcal{C}\mathcal{B}]}{[\mathcal{A}\mathcal{B}]} f^{(0,2)}_{\mathrm{EM}} + \frac{[\mathcal{C}\mathcal{A}]}{[\mathcal{A}\mathcal{B}]} f^{(1,1)}_{\mathrm{EM}} \right).$$

which in space-time becomes

$$\Psi^{(\mathrm{II})}_{A'B'C'D'} = \frac{1}{\phi}\left[3\frac{[\mathcal{C}\mathcal{A}]}{[\mathcal{A}\mathcal{B}]}\phi^{(0,2)}_{(A'B'}\phi^{(1,1)}_{C'D')} - 4\frac{[\mathcal{C}\mathcal{B}]}{[\mathcal{A}\mathcal{B}]}\phi^{(1,1)}_{(A'B'}\phi^{(1,1)}_{C'D')} \right],$$

Twistorial Double Copy- More general solutions

- One can also construct Type I solutions.
- We have generalised the Weyl Double copy beyond Type D/N - it is now a sum of products.
- The twistor language reduces the problem to finding combinations with the correct pole structure - this are then guaranteeed to satisfy the e.o.m. when we go to spacetime by performing the Penrose transform!

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Thank You !