Constructing new solutions of the Yang-Baxter equation

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Based on arXiv:1911.01439, 2003.04332, 2010.11231 and 2109.00017 in collaboration with Marius de Leeuw, Chiara Paletta, Anton Pribytok and Paul Ryan

1. Introduction
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4. $\mathrm{AdS}_{2,3}$ deformations
5. Conclusions and Further developments

Introduction

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- They are known as **Integrable models**.

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- $AdS_2 \rightarrow$ XYZ-like model.

- When I say that these models can be solved what I mean is that Integrable models have many very effective techniques that were developed specifically to deal with them such as
  - Coordinate Bethe ansatz (CBA);
  - (Nested) Algebraic Bethe ansatz (ABA);
  - Thermodynamic Bethe ansatz (TBA);
  - Q-operators;
  - Quantum spectral Curve;
- With all these techniques we can in most of the cases solve these models.

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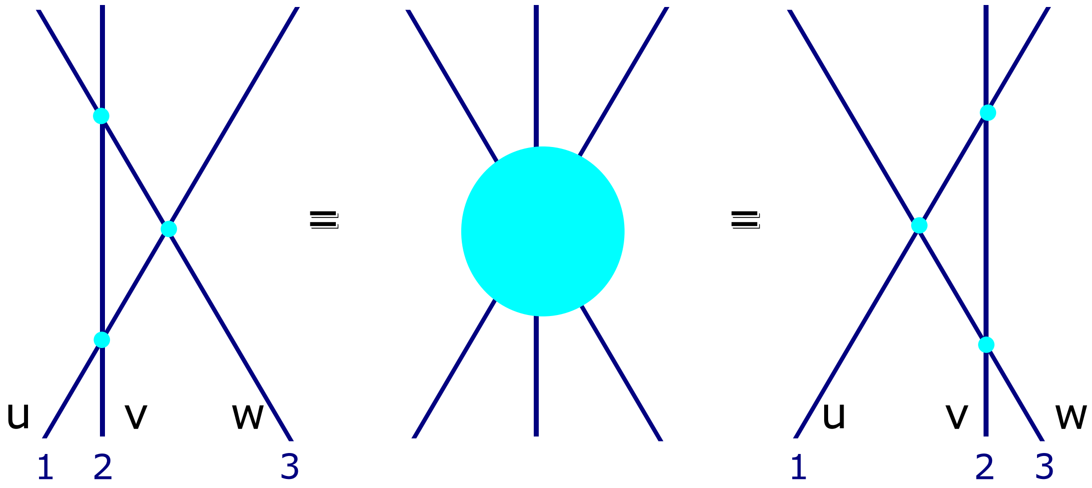
    - $3 \rightarrow 3$-particles scattering $\Rightarrow \{2 \rightarrow 2\}$-particles scattering



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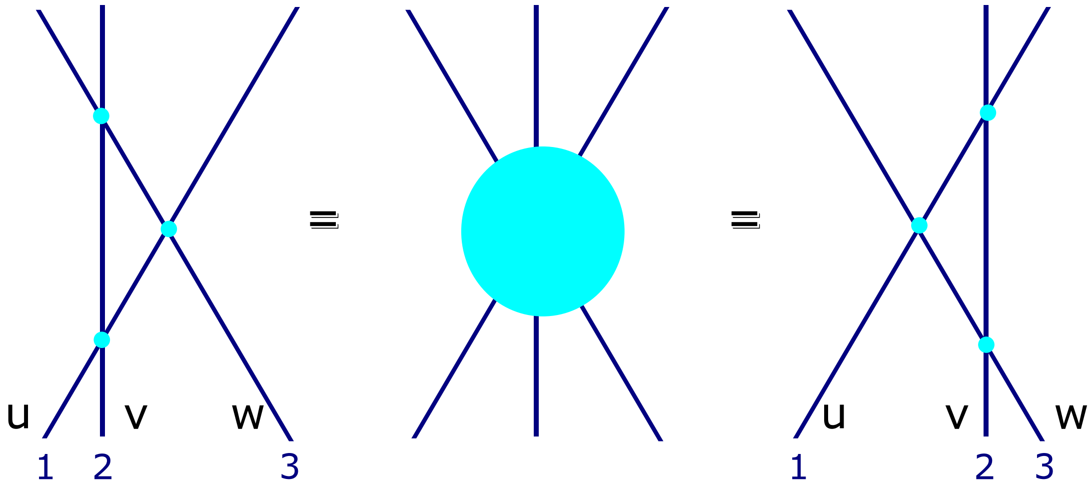

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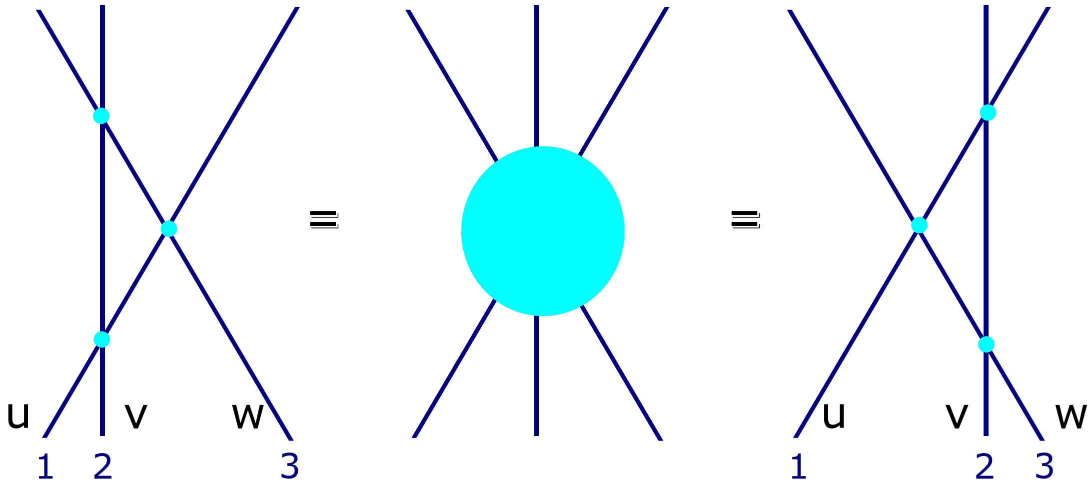
This is called **Yang-Baxter equation (YBE)**;



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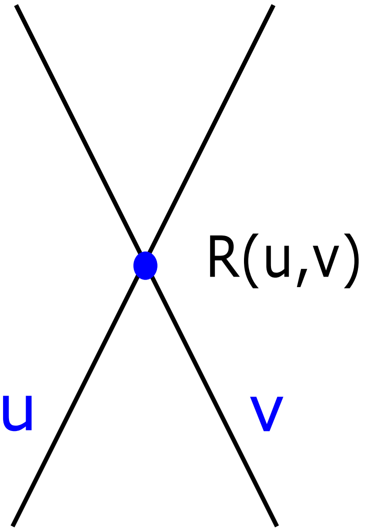
This is called **Yang-Baxter equation (YBE)**;

u, v and w can be interpreted as rapidities of the particles.

So, the main object to define a quantum integrable model is the R-matrix



where

$$R: \quad V \otimes V \to V \otimes V$$

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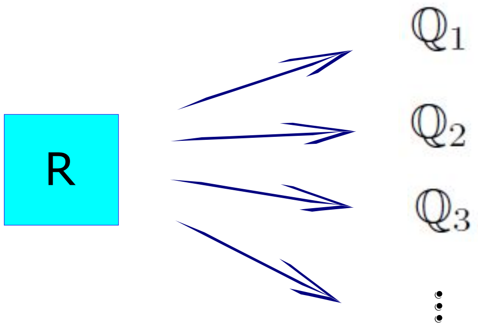

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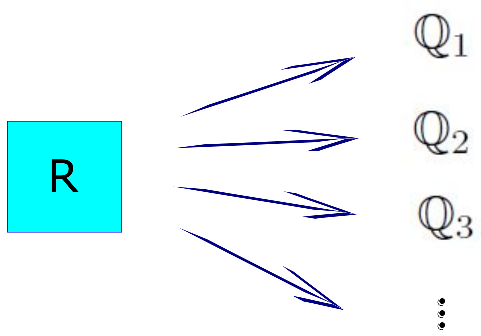

- We are interested in R-matrices with the **regularity** property:

$$R(u,u) = P, \quad \text{where} \quad P_{12}(v_1 \otimes v_2) = v_2 \otimes v_1$$

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difference form R-matrix:

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non-difference form R-matrix:

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Example: Hubbard-model

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Solving YBE "directly" (Vieira, Lima-Santos, ...)

- you derivate YBE with respect to one of the variables and solve the differential equations;

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For example:

$$H = \begin{pmatrix} h_1(u) & 0 & 0 & h_8(u) \\ 0 & h_5(u) & h_3(u) & 0 \\ 0 & h_2(u) & h_6(u) & 0 \\ h_7(u) & 0 & 0 & h_4(u) \end{pmatrix}.$$

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- For that we use the so called Boost operator (see Tetelman, 1982, Loebbert, 2016, Grabowski and Mathieu, 1994):

$$B\left[\mathbb{Q}_2\right] = \partial_\theta + \sum_{n=-\infty}^{\infty} n\,\mathcal{H}_{n,n+1}(\theta);$$

- The advantage of this object is that we can use it to construct higher charges in a recursive way:

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- So, Q_3 is given by

$$\mathbb{Q}_3(\theta) = [B[\mathbb{Q}_2], \mathbb{Q}_2]$$

$$\mathbb{Q}_3(\theta) = \sum_{i=1}^{L} [\mathcal{H}_{i-1,i}, \mathcal{H}_{i,i+1}] + \frac{d\,\mathbb{H}}{d\theta}$$

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$$[\mathbb{Q}_2(\theta), \mathbb{Q}_3(\theta)] = 0 = [\mathbb{Q}_3(\theta), \mathbb{Q}_4(\theta)] = ...$$

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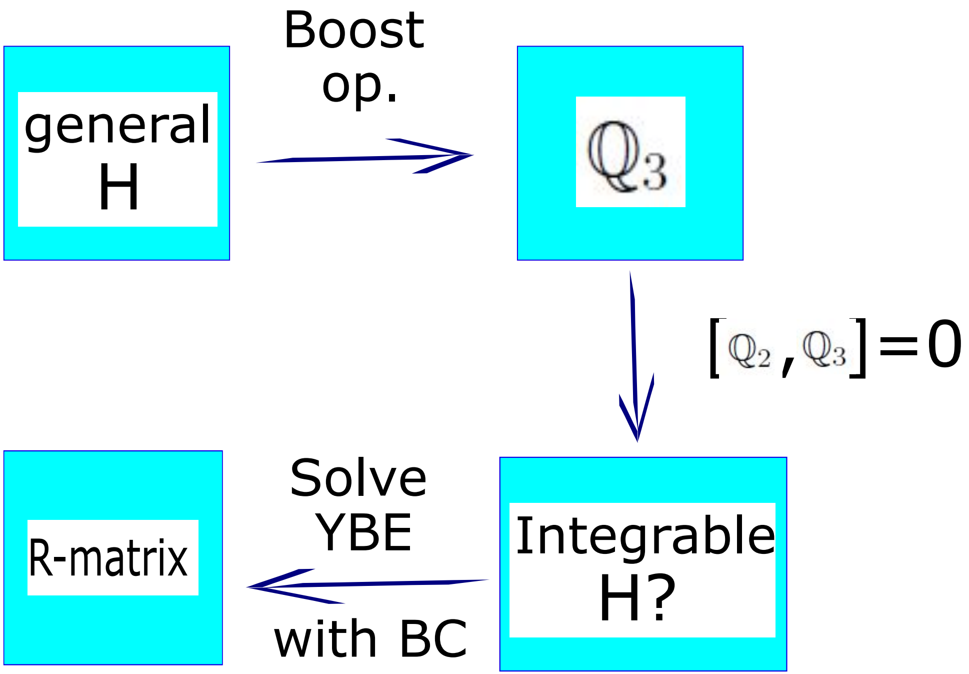
- The last step is to check that $R(u,v)$ satisfies YBE.

Summarizing...



with boundary conditions:

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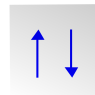
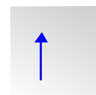
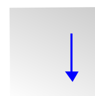
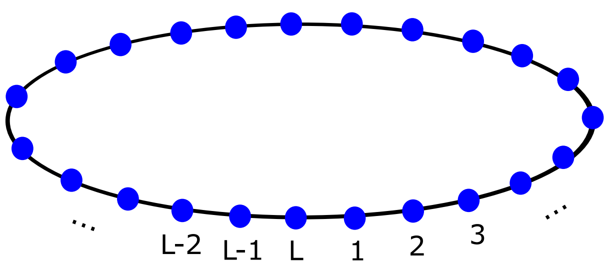
- It has 256 components, so solving $[\mathbb{Q}_2, \mathbb{Q}_3] = 0$ for so many coefficients is not feasible at the moment;
- So, we assumed $su(2) \oplus su(2)$ symmetry;

Two sets of vectors: $\{|\phi_1\rangle, |\phi_2\rangle\}$ and $\{|\psi_1\rangle, |\psi_2\rangle\}$

- $$|\phi_1\rangle = |0\rangle$$
- $$|\phi_2\rangle = c^\dagger_\uparrow c^\dagger_\downarrow |0\rangle$$ 
- $$|\psi_1\rangle = c^\dagger_\uparrow |0\rangle$$ 
- $$|\psi_2\rangle = c^\dagger_\downarrow |0\rangle$$ 

where $\left\{c^\dagger_i, c_j\right\} = \delta_{ij}$



- With this symmetry our two-sites Hamiltonian has the form

$$
\begin{aligned}
\mathcal{H}|\phi_a\phi_b\rangle &= A|\phi_a\phi_b\rangle + B|\phi_b\phi_a\rangle + C\epsilon_{ab}\epsilon^{\alpha\beta}|\psi_\alpha\psi_\beta\rangle \\
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- Using this $\mathcal{H}$ and applying the method we found :
- 12 independent solutions

The 12 solutions are:

Model	**A**	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**K**	**L**
1	0	0	0	0	0	0	a	b	c	d
2	0	0	0	$a+c$	0	0	a	b	c	d
3	0	0	0	a	0	0	b	0	c	0
4	ρ	$-\rho$	0	0	0	0	a	$\rho e^{-\phi}$	$2\rho-a$	ρe^{ϕ}
5	ρ	$-\rho$	0	ρ	$-\rho$	0	a	$\rho e^{-\phi}$	$2\rho-a$	ρe^{ϕ}
6	0	0	0	ρ	ρ	0	a	$\rho e^{-\phi}$	$2\rho-a$	ρe^{ϕ}
7	ρ	$-\rho$	0	ρ	ρ	0	a	$\rho e^{-\phi}$	$2\rho-a$	ρe^{ϕ}
8	ρ	$-\rho$	$\rho e^{-\phi}$	$-\rho$	ρ	$-\rho e^{\phi}$	0	0	0	0
9	ρ	$-\rho$	$\rho e^{-\phi}$	ρ	$-\rho$	ρe^{ϕ}	0	0	0	0
10	$\frac{7}{4}\rho$	$-\rho$	$\frac{1}{2}\rho e^{-\phi}$	$\frac{7}{4}\rho$	$-\rho$	$\frac{1}{2}\rho e^{\phi}$	0	0	0	0
11	ρ	$-\rho$	$\frac{1}{2}\rho e^{-\phi}$	ρ	$-\rho$	$\frac{1}{2}\rho e^{\phi}$	$\frac{3}{2}\rho$	$-\frac{3}{2}\rho$	$\frac{3}{2}\rho$	$-\frac{3}{2}\rho$
12	0	0	$-\rho e^{-\phi}$	0	0	ρe^{ϕ}	0	ρ	0	$-\rho$

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- They have $G = H = K = L = 0$;

Why is relevant that $G = H = K = L = 0$?

Remember the form of the Hamiltonian

$$\begin{aligned}
\mathcal{H}|\phi_a \phi_b\rangle &= A|\phi_a \phi_b\rangle + B|\phi_b \phi_a\rangle + C\epsilon_{ab}\epsilon^{\alpha\beta}|\psi_\alpha \psi_\beta\rangle \\
\mathcal{H}|\phi_a \psi_\beta\rangle &= G|\phi_a \psi_\beta\rangle + H|\psi_\beta \phi_a\rangle \\
\mathcal{H}|\psi_\alpha \phi_b\rangle &= K|\psi_\alpha \phi_b\rangle + L|\phi_b \psi_\alpha\rangle \\
\mathcal{H}|\psi_\alpha \psi_\beta\rangle &= D|\psi_\alpha \psi_\beta\rangle + E|\psi_\beta \psi_\alpha\rangle + F\epsilon^{ab}\epsilon_{\alpha\beta}|\phi_a \phi_b\rangle
\end{aligned}$$

$G = H = K = L = 0$ means that electrons can not move in the spin chain by themselves, they only move when in pairs.

$$\mathcal{H}|\phi_a\phi_b\rangle = A|\phi_a\phi_b\rangle + B|\phi_b\phi_a\rangle + C\epsilon_{ab}\epsilon^{\alpha\beta}|\psi_\alpha\psi_\beta\rangle$$
$$\mathcal{H}|\phi_a\psi_\beta\rangle = 0$$
$$\mathcal{H}|\psi_\alpha\phi_b\rangle = 0$$
$$\mathcal{H}|\psi_\alpha\psi_\beta\rangle = D|\psi_\alpha\psi_\beta\rangle + E|\psi_\beta\psi_\alpha\rangle + F\epsilon^{ab}\epsilon_{\alpha\beta}|\phi_a\phi_b\rangle$$

$$
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\end{aligned}
$$

Let us think in L=5 (number of sites):

$$
\mathbb{H} = \mathcal{H}_{12} + \mathcal{H}_{23} + \mathcal{H}_{34} + \mathcal{H}_{45} + \mathcal{H}_{51}
$$

$$\mathcal{H}|\phi_a\phi_b\rangle = A|\phi_a\phi_b\rangle + B|\phi_b\phi_a\rangle + C\epsilon_{ab}\epsilon^{\alpha\beta}|\psi_\alpha\psi_\beta\rangle$$

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Let us look the state

$$|\phi_1\,\psi_1\,\phi_1\,\psi_2\,\phi_1\rangle$$

$$\mathbb{H}|\phi_1\,\psi_1\,\phi_1\,\psi_2\,\phi_1\rangle = (A+B)|\phi_1\,\psi_1\,\phi_1\,\psi_2\,\phi_1\rangle$$

i.e. Electrons did not move!

$$\begin{aligned}
\mathcal{H}|\phi_a\phi_b\rangle &= A|\phi_a\phi_b\rangle + B|\phi_b\phi_a\rangle + C\epsilon_{ab}\epsilon^{\alpha\beta}|\psi_\alpha\psi_\beta\rangle \\
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$$|\phi_1\,\phi_1\,\psi_1\,\psi_2\,\phi_1\rangle$$

$$\mathbb{H}|\phi_1\,\phi_1\,\psi_1\,\psi_2\,\phi_1\rangle = ?$$

Now they move!

Spectrum:

- For 4 sites for example:

Model 8:	$\{1,1,1,1,14,14,224\}$;
Model 9:	$\{1,15,16,30,194\}$;
Model 10:	$\{1,1,1,1,1,1,6,6,8,8,14,16,16,32,44,100\}$

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- So the three models despite their similarities have a very different spectrum;

Spectrum:

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- So the three models despite their similarities have a very different spectrum;
- And also probably have some extra symmetries we still do not understand;

$$R(u) =$$

$$\begin{pmatrix}
r_{1,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & r_1 & 0 & 0 & r_2 & 0 & 0 & 0 & 0 & 0 & 0 & -r_8 & 0 & 0 & r_8 & 0 \\
0 & 0 & r_4 & 0 & 0 & 0 & 0 & 0 & r_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & r_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_{10} & 0 & 0 & 0 \\
0 & r_2 & 0 & 0 & r_1 & 0 & 0 & 0 & 0 & 0 & 0 & r_8 & 0 & 0 & -r_8 & 0 \\
0 & 0 & 0 & 0 & 0 & r_{1,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & r_4 & 0 & 0 & 0 & 0 & 0 & r_{10} & 0 & 0 \\
0 & 0 & r_7 & 0 & 0 & 0 & 0 & 0 & r_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & r_7 & 0 & 0 & r_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_{5,6} & 0 & 0 & 0 & 0 & 0 \\
0 & -r_9 & 0 & 0 & r_9 & 0 & 0 & 0 & 0 & 0 & 0 & r_5 & 0 & 0 & r_6 & 0 \\
0 & 0 & 0 & r_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_3 & 0 & 0 & 0 \\
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0 & r_9 & 0 & 0 & -r_9 & 0 & 0 & 0 & 0 & 0 & 0 & r_6 & 0 & 0 & r_5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_{5,6}
\end{pmatrix}$$

where $r_{i,j} = r_i + r_j$.

Model 8

$$\begin{array}{l} r_1 = -r_5 = -\tan(u\ \rho) \\ r_2 = 1 - r_1 \\ r_6 = 1 + r_1 \end{array} \quad , \quad \begin{array}{l} r_7 = r_{10} = 1 \\ r_8 = e^{\phi}\ r_1 \\ r_9 = -e^{-\phi}\ r_1 \end{array}$$

Model 9

$$\begin{array}{l} r_1 = r_5 \\ r_2 = r_6 = 1 - r_1 \\ r_7 = r_{10} = 1 \end{array} \ , \quad \begin{array}{l} r_8 = -e^{\phi}\ r_1 \\ r_9 = -e^{-\phi}\ r_1 \end{array}$$

$$r_1 = 2 + \sqrt{3}\coth\left(\sqrt{3}\rho u + \log\left(2 - \sqrt{3}\right)\right)$$

Model 10

$$\begin{array}{l} r_1 = r_5 = \frac{2(e^{\frac{3\rho u}{2}} - 1)}{e^{\frac{3\rho u}{2}} - 4} \\ r_7 = r_{10} = e^{-\frac{1}{4}(3\rho u)} \end{array} \quad , \quad \begin{array}{l} r_2 = r_6 = -\frac{e^{\frac{3\rho u}{2}} + 2}{e^{\frac{3\rho u}{2}} - 4} \\ e^{-2\phi} r_9 = r_8 = -\frac{1}{2} e^{\frac{3\rho u}{4} + \phi}\ r_1 \end{array}$$

Generalized Hubbard model

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- So we decided to see which terms we could add and still keep integrability;
- but we would like to study only models we could interpret as electrons moving on a one-dimensional lattice;
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- K_{pair}: moves one pair of electrons from one site to the next;

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- K_{flip}: flips spins in neighbor sites;

$$\begin{aligned} K_{flip} = {} & A_3 \mathbf{c}^{\dagger}_{\uparrow,1}\mathbf{c}^{\dagger}_{\downarrow,2}\mathbf{c}_{\downarrow,1}\mathbf{c}_{\uparrow,2} + A_4 \mathbf{c}^{\dagger}_{\downarrow,1}\mathbf{c}^{\dagger}_{\uparrow,2}\mathbf{c}_{\uparrow,1}\mathbf{c}_{\downarrow,2} \\ & + A_5 \mathbf{c}^{\dagger}_{\uparrow,1}\mathbf{c}^{\dagger}_{\uparrow,2}\mathbf{c}_{\downarrow,1}\mathbf{c}_{\downarrow,2} + A_6 \mathbf{c}^{\dagger}_{\downarrow,1}\mathbf{c}^{\dagger}_{\downarrow,2}\mathbf{c}_{\uparrow,1}\mathbf{c}_{\uparrow,2}. \end{aligned}$$

- V: potential term

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- V: potential term

The density Hamiltonian whose integrability we investigate is

$$\mathcal{H} = K_{Hub} + K_{pair} + K_{flip} + V,$$

It has 22 free parameters

If we consider a non-trivial K_{flip} and also a potential part we found that

$$\begin{aligned}\mathcal{H} = \mathcal{K}_{Hub} + a\Big(&\mathbf{c}^{\dagger}_{\uparrow,1}\mathbf{c}^{\dagger}_{\downarrow,2}\mathbf{c}_{\downarrow,1}\mathbf{c}_{\uparrow,2} + \mathbf{c}^{\dagger}_{\downarrow,1}\mathbf{c}^{\dagger}_{\uparrow,2}\mathbf{c}_{\uparrow,1}\mathbf{c}_{\downarrow,2}\\
&+\mathbf{c}^{\dagger}_{\uparrow,1}\mathbf{c}^{\dagger}_{\uparrow,2}\mathbf{c}_{\downarrow,1}\mathbf{c}_{\downarrow,2} + \mathbf{c}^{\dagger}_{\downarrow,1}\mathbf{c}^{\dagger}_{\downarrow,2}\mathbf{c}_{\uparrow,1}\mathbf{c}_{\uparrow,2}\Big)\\
&+ (2a-b)(\mathbf{n}_{\uparrow,1} + \mathbf{n}_{\downarrow,1}) + b(\mathbf{n}_{\uparrow,2} + \mathbf{n}_{\downarrow,2})\\
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$$+\mathbf{c}^{\dagger}_{\uparrow,1}\mathbf{c}^{\dagger}_{\uparrow,2}\mathbf{c}_{\downarrow,1}\mathbf{c}_{\downarrow,2} + \mathbf{c}^{\dagger}_{\downarrow,1}\mathbf{c}^{\dagger}_{\downarrow,2}\mathbf{c}_{\uparrow,1}\mathbf{c}_{\uparrow,2}\Big)$$
$$+ (2a-b)(\mathbf{n}_{\uparrow,1} + \mathbf{n}_{\downarrow,1}) + b(\mathbf{n}_{\uparrow,2} + \mathbf{n}_{\downarrow,2})$$
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- It does not conserve spin orientation, so it is XYZ deformation of the Hubbard model;
- Bethe ansatz does not work;
- It has two free parameters, so it may have a phase diagram;

$AdS_{2,3}$ deformations

- It is known that in addition to $AdS_5 \times S^5$, lower dimensional versions of AdS like:

$$AdS_3 \times S^3 \times T^4 \text{ (Borsato, Ohlsson Sax, Sfondrini, B. Stefanski, 2014)}$$

$$AdS_3 \times S^3 \times S^3 \times S^1 \text{ (Borsato, Ohlsson Sax, Sfondrini, B. Stefanski, 2015)}$$

$$AdS_2 \times S^2 \times T^6 \text{ (Hoare, Pittelli, Torrielli, 2014).}$$

are also integrable.

- The R-matrix for $AdS_3 \times S^3 \times T^4$, for example, was obtained by assuming that

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  both remain at quantum level;

- This was enough to fix the S-matrix up to the dressing factor;

AdS$_{2,3}$ deformations

- Focusing on the $\mathrm{su}(1|1)^2_{\mathrm{ce}}$ sector, one can write the S-matrix as

$$\check{\mathbb{S}} = \begin{pmatrix} \check{S}^{\mathrm{LL}} & \check{S}^{\mathrm{RL}} \\ \check{S}^{\mathrm{LR}} & \check{S}^{\mathrm{RR}} \end{pmatrix}$$

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- it satisfies the Yang-Baxter equation;
- each of these blocks are an embedding of a 4×4 R-matrix;
- the blocks with same chirality come from regular R-matrices while the opposite-chirality ones come from non-regular R-matrices;

$\mathrm{AdS}_{2,3}$ deformations

- For $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{M}^4$ the diagonal blocks are regular 6-vertex regular R-matrices, i.e.

$$R(u,v) = \begin{pmatrix} r_1(u,v) & 0 & 0 & 0 \\ 0 & r_2(u,v) & r_6(u,v) & 0 \\ 0 & r_5(u,v) & r_3(u,v) & 0 \\ 0 & 0 & 0 & r_4(u,v) \end{pmatrix}, \qquad R(u,u) = P,$$

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which means that only scatterings like

$$\begin{aligned} \phi\,\phi &\to \phi\,\phi \\ \psi\,\psi &\to \psi\,\psi \\ \phi\,\psi &\to \phi\,\psi + \psi\,\phi \\ \psi\,\phi &\to \psi\,\phi + \phi\,\psi \end{aligned}$$

are allowed. Spin is conserved.

$\mathrm{AdS}_{2,3}$ deformations

- While for massive $\mathrm{AdS}_2 \times \mathrm{S}^2 \times \mathrm{T}^6$ the RR and LL blocks are described by an 4×4 8-vertex regular R-matrix:

$$R(u,v) = \begin{pmatrix} r_1(u,v) & 0 & 0 & r_8(u,v) \\ 0 & r_2(u,v) & r_6(u,v) & 0 \\ 0 & r_5(u,v) & r_3(u,v) & 0 \\ r_7(u,v) & 0 & 0 & r_4(u,v) \end{pmatrix}, \quad R(u,u) = P.$$

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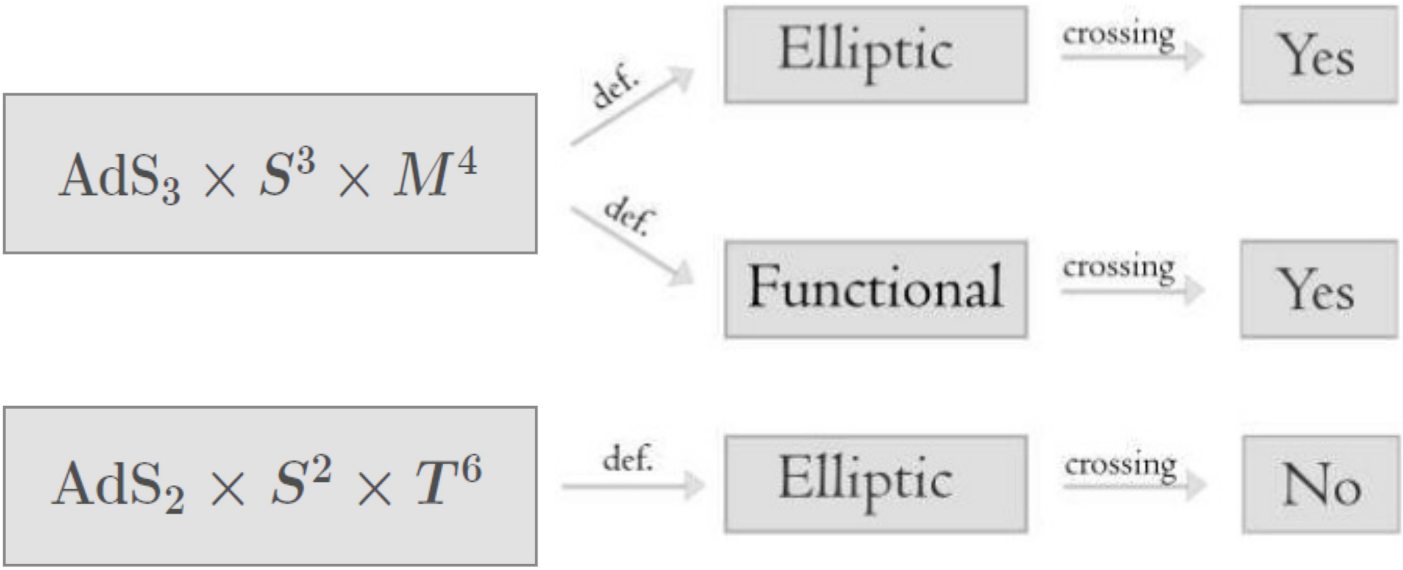
$$\begin{aligned} \phi\,\phi &\to \phi\,\phi + \psi\,\psi, \\ \psi\,\psi &\to \psi\,\psi + \phi\,\phi, \\ \phi\,\psi &\to \phi\,\psi + \psi\,\phi, \\ \psi\,\phi &\to \psi\,\phi + \phi\,\psi \end{aligned}$$

are allowed.

$AdS_{2,3}$ deformations

Goal: Find the most general integrable deformations of AdS_3 and AdS_2 R-matrices.

Spoiler:



$AdS_3 \times S^3 \times M^4$
def.
Elliptic
crossing
Yes
def.
Functional
crossing
Yes
$AdS_2 \times S^2 \times T^6$
def.
Elliptic
crossing
No

We found 4 independent solutions:

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Two of 6 vertex form;

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Two of 6 vertex form;

and

Two of 8 vertex form

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But in only one of the 6-vertex and one of the 8-vertex, $AdS_{2,3}$ known R-matrices could be embedded.

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But in only one of the 6-vertex and one of the 8-vertex, $AdS_{2,3}$ known R-matrices could be embedded.

We called them **6-vertex B** and **8-vertex B**

6-vertex B

$$
\begin{aligned}
r_1 &= \frac{h_2(q) - h_1(p)}{h_2(p) - h_1(p)}, \\
r_2 &= (h_2(p) - h_2(q))X(p)Y(p), \\
r_3 &= \frac{h_1(p) - h_1(q)}{(h_2(p) - h_1(p))(h_2(q) - h_1(q))} \frac{1}{X(q)Y(q)}, \\
r_4 &= \frac{h_2(p) - h_1(q)}{h_2(q) - h_1(q)} \frac{X(p)Y(p)}{X(q)Y(q)}, \\
r_5 &= \frac{Y(p)}{Y(q)}, \\
r_6 &= \frac{X(p)}{X(q)}.
\end{aligned}
$$

6-vertex B

$$
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r_2 &= (h_2(p) - h_2(q))X(p)Y(p), \\
r_3 &= \frac{h_1(p) - h_1(q)}{(h_2(p) - h_1(p))(h_2(q) - h_1(q))} \frac{1}{X(q)Y(q)}, \\
r_4 &= \frac{h_2(p) - h_1(q)}{h_2(q) - h_1(q)} \frac{X(p)Y(p)}{X(q)Y(q)}, \\
r_5 &= \frac{Y(p)}{Y(q)}, \\
r_6 &= \frac{X(p)}{X(q)}.
\end{aligned}
$$

We will assume $R^{RR}(u,v)$ and $R^{LL}(u,v)$ as two independent copies of 6-vertex B.

6-vertex B - AdS_3

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- It actually corresponds to a deformation of the q-deformation introduced by Ben Hoare in 2015;
- It is what we are calling a functional deformation, because instead of $x^{\pm}_{R,L}(u)$ we have general functions $h^{R,L}_{1,2}(u)$

6-vertex B - AdS$_3$

- By making the following identifications

$$h_1^{\mathrm{R}}(p) = -\frac{x_R^-(p)}{\beta}, \qquad h_1^{\mathrm{L}}(p) = \beta\, x_L^-(p),$$

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where β is an arbitrary constant and

$$X^{\mathrm{L}}(p) = \frac{\rho}{\gamma_L(p)}, \qquad Y^{\mathrm{L}}(p) = \frac{1}{\beta\,\rho}\,\frac{\gamma_L(p)}{U_L(p)V_L(p)W_L(p)}\,\frac{1}{x_L^-(p) - x_L^+(p)},$$
$$Y^{\mathrm{R}}(p) = \frac{1}{\beta\,\rho}\,\frac{x_R^+(p)}{\gamma_R(p)}, \qquad X^{\mathrm{R}}(p) = -\frac{\rho\,\gamma_R(p)}{U_R(p)V_R(p)W_R(p)}\,\frac{x_R^+(p)}{x_R^-(p) - x_R^+(p)}$$

we recover the two parametric q-deformation;

6-vertex B - AdS_3

- But let us keep these functions $h_{1,2}^{R,L}(u)$ general;

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- But let us keep these functions $h_{1,2}^{R,L}(u)$ general;
- In such case we can interpret as the mass now depends on u;
- It has crossing symmetry;

8-vertex B model

$$r_1 = \frac{1}{\sqrt{\sin\eta(u)}\sqrt{\sin\eta(v)}}\left[\sin\eta_+\frac{\mathrm{cn}}{\mathrm{dn}} - \cos\eta_+\mathrm{sn}\right],$$
$$r_2 = \frac{1}{\sqrt{\sin\eta(u)}\sqrt{\sin\eta(v)}}\left[\cos\eta_-\mathrm{sn} + \sin\eta_-\frac{\mathrm{cn}}{\mathrm{dn}}\right],$$
$$r_3 = \frac{1}{\sqrt{\sin\eta(u)}\sqrt{\sin\eta(v)}}\left[\cos\eta_-\mathrm{sn} - \sin\eta_-\frac{\mathrm{cn}}{\mathrm{dn}}\right],$$
$$r_4 = \frac{1}{\sqrt{\sin\eta(u)}\sqrt{\sin\eta(v)}}\left[\sin\eta_+\frac{\mathrm{cn}}{\mathrm{dn}} + \cos\eta_+\mathrm{sn}\right],$$
$$r_5 = r_6 = 1,$$
$$r_7 = r_8 = k\,\mathrm{sn}\frac{\mathrm{cn}}{\mathrm{dn}},$$

with

$$\mathrm{sn} = \mathrm{sn}(G(u) - G(v), k^2), \quad \mathrm{cn} = \mathrm{cn}(G(u) - G(v), k^2), \quad \text{etc}$$

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- It is a deformation of:

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and

$$\text{AdS}_3 \text{ when } k \to 0$$

8-vertex B - AdS_3 deformation

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An 8-vertex deformation of AdS_3!

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- We constructed the full R-matrix for this model, and again we found that the LL and RR blocks can be deformed separately here;

8-vertex B - AdS$_3$ deformation

- This was the biggest surprise when we compared the models with the undeformed ones:

> An 8-vertex deformation of AdS$_3$!

- We constructed the full R-matrix for this model, and again we found that the LL and RR blocks can be deformed separately here;
- So, we have again a deformation of $AdS_3 \times S^3 \times M^4$;
- This is not however a deformation of the q-deformed model found by Hoare in 2014;

Conclusions

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- we presented a new method to construct R-matrices satisfying YBE;
- Some models with potential interesting physical properties were found;
- And three new integrable deformations of lower dimensional AdS were found,

Further developments

- Consider models with less symmetry and maybe try a full classification;
- Compute the spectrum of the new models where electrons can move only when in pairs ;
    - Maybe nested algebraic Bethe ansatz will work;
- Study physical properties of the deformed Hubbard-like model;

- Investigate if there are field theories whose S-matrix would correspond to the new R-matrices we found;
- Prove that $[\mathbb{Q}_2, \mathbb{Q}_3] = 0$ is always enough or find a counterexample;
- Construct the K-matrices;
- Study better the deformations of AdS_2 and AdS_3 we found, including its symmetries and solve the crossing equations.

Thank you!