Scaling Black Holes and Modularity

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Summary

- N=2 string theory admits multi-centered black holes.
- Scaling black holes are multi-centered black holes, such that centers can come arbitrarily close.
  This near-coincident regime of scaling black holes is “similar” to single centered black holes.
- Black hole degeneracies can be packed inside certain generating functions.
  These generating functions have interesting modular properties.
- Scaling black holes give additional contribution to such generating functions.
- We show for scaling black holes with 3-centers, this gives a depth 2 mock modular form and further compute its modular completion.

Quick Recap of Black Holes in String Theory

black holes entropy puzzle

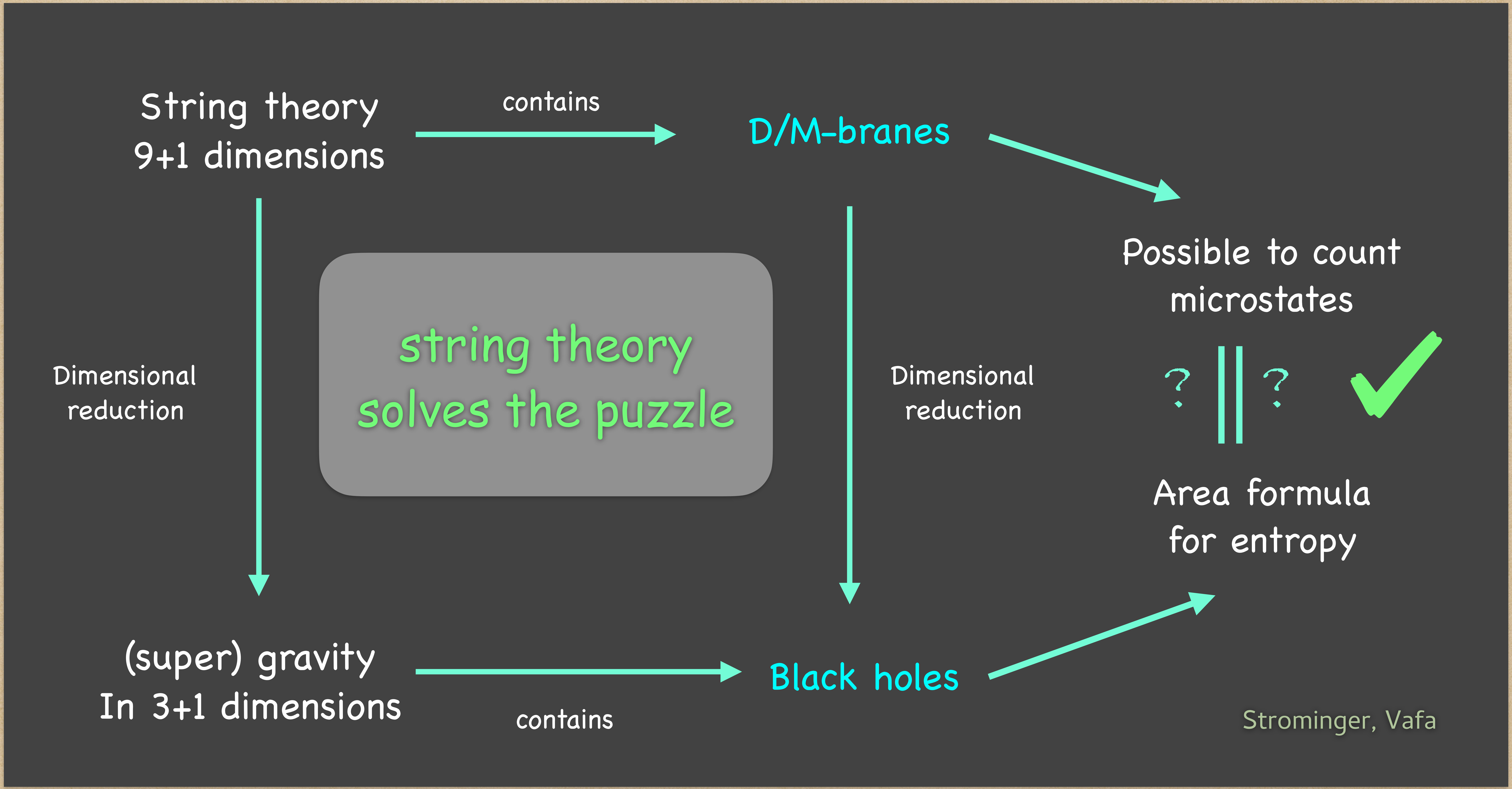
- Classically black holes have no microstates ⇒ no entropy.
  What happened to the entropy of the collapsed star? ⇒ 2^{nd} law is in danger !

- Area of event horizon always increases, just like entropy
  ⇒ $S_{BH} = \frac{Area}{4\, l_P^2}$ Bekenstein, Hawking

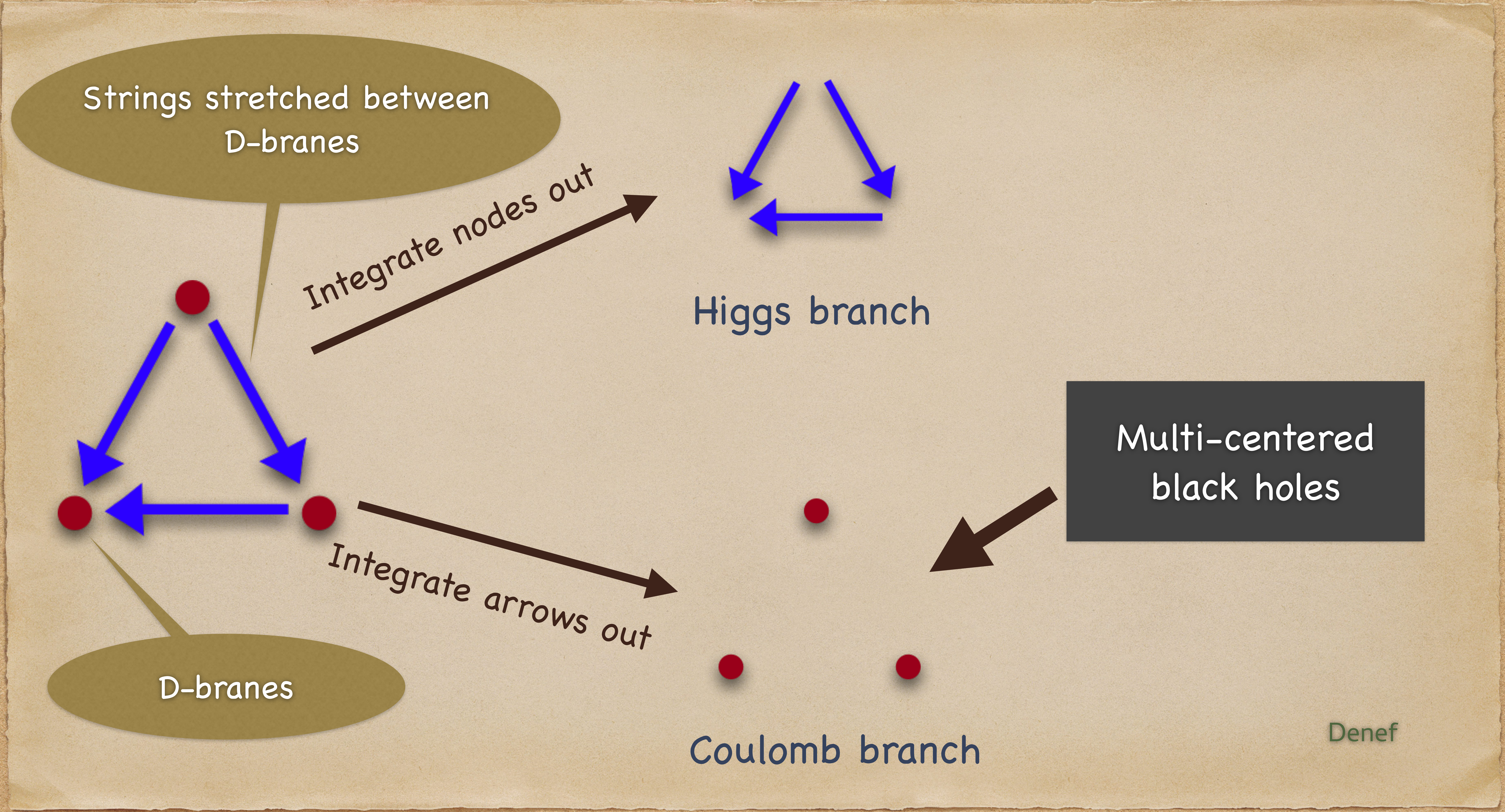
- Puzzle: what are the underlying microstates?

- Hint: $l_P \sim 1.6 \times 10^{-35} m$ is the scale of quantum gravity
  ⇒ Ask a theory of quantum gravity.

- A correct theory of Quantum gravity must explain black hole entropy.


String theory
9+1 dimensions
contains
D/M-branes
Possible to count
microstates
?
?
Dimensional
reduction
string theory
solves the puzzle
Dimensional
reduction
Area formula
for entropy
(super) gravity
In 3+1 dimensions
contains
Black holes
Strominger, Vafa

Scaling Black Holes


Strings stretched between D-branes
Integrate nodes out
Higgs branch
D-branes
Integrate arrows out
Multi-centered black holes
Coulomb branch
Denef

- Denef's eqns:

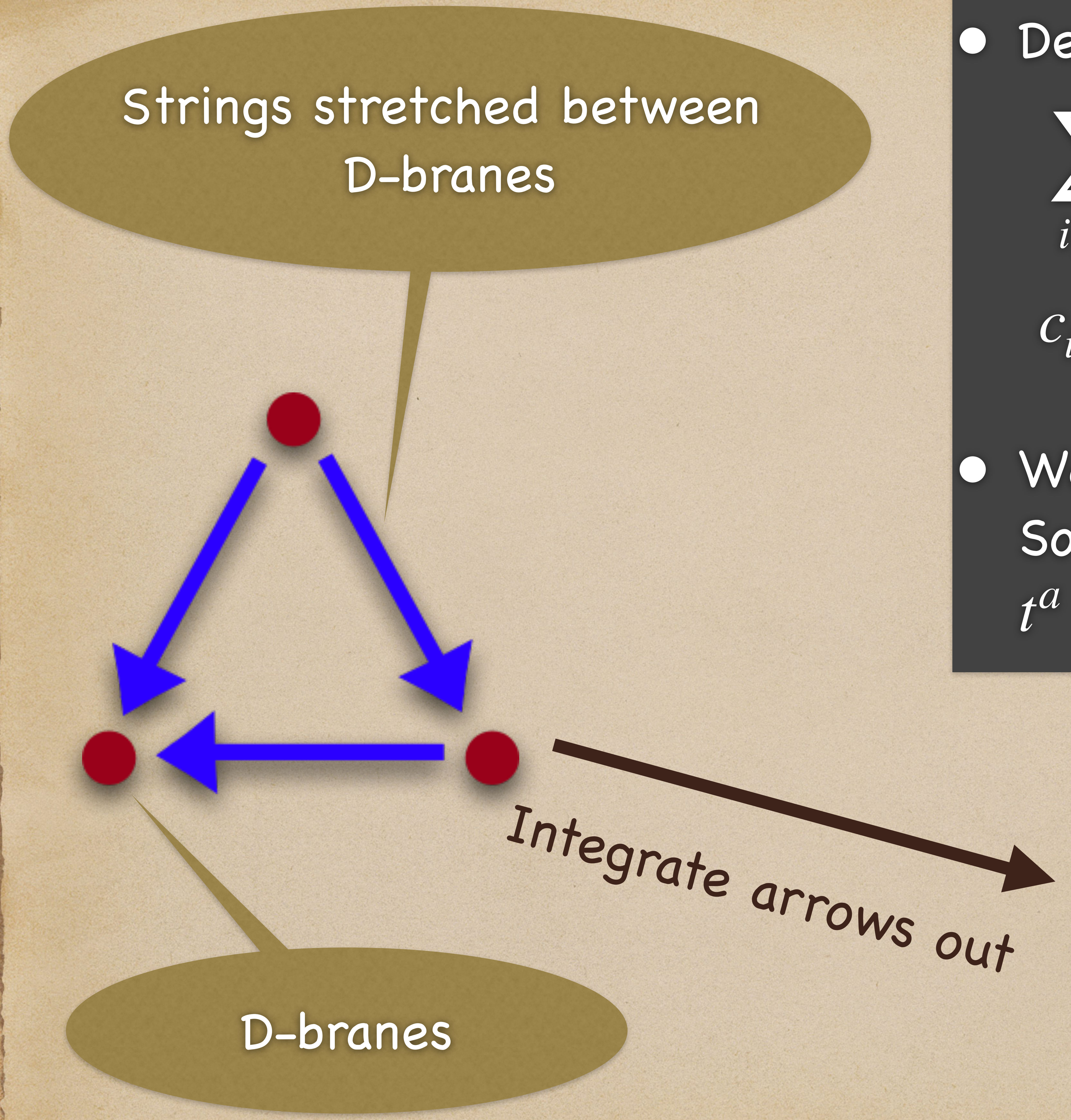
$$\sum_{i\neq j} \frac{\langle \gamma_i, \gamma_j \rangle}{r_{ij}} = c_i\,, \quad \langle \gamma_i, \gamma_j \rangle = P_i \,.\, Q_j - P_j \,.\, Q_i\,,$$

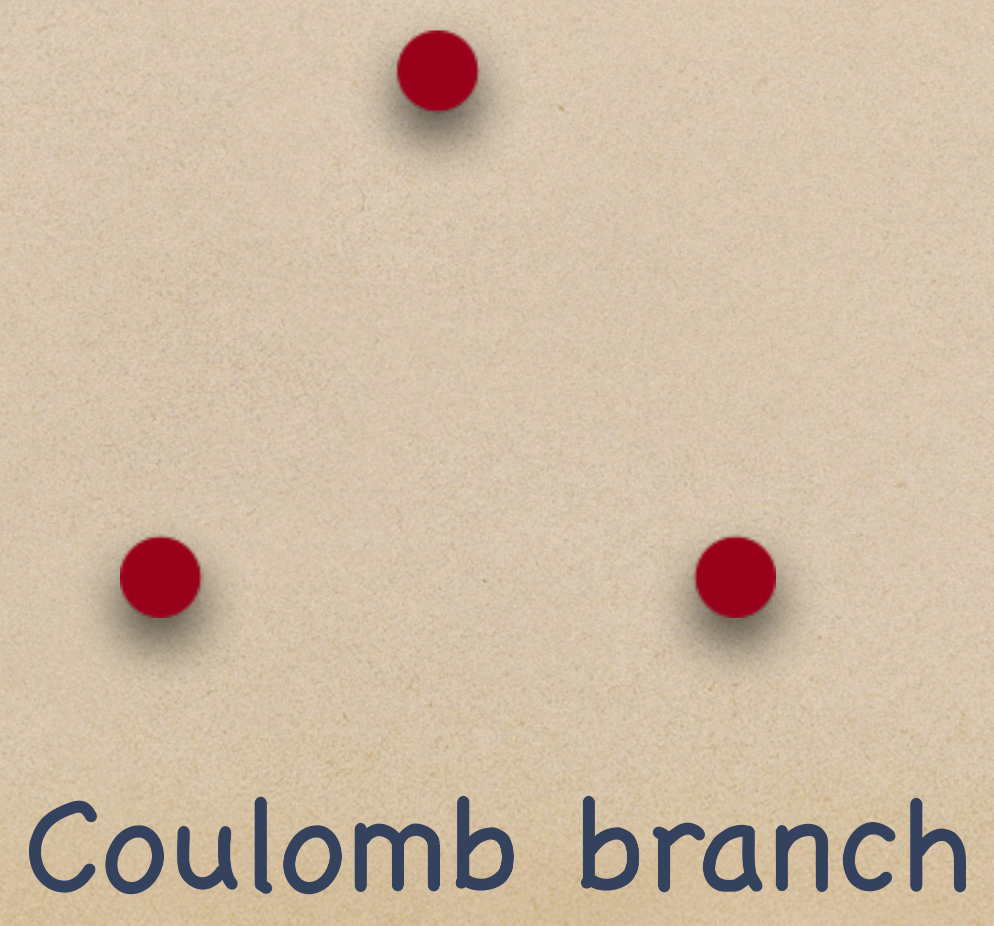
$$c_i = 2\,\mathrm{Im}\,\left(e^{-i\alpha} Z(\gamma_i, t)\right)|_{r=\infty}$$

- Wall crossing:

  Solutions may or may not exist depending on moduli $t^a = B^a + iJ^a$.

  Denef, Moore





D4-D2-D0 Black Holes

- Charge vector $\gamma = (0, P, Q, Q_0)$
  D6-brane charge 0, D4-brane charge P, D2-brane charge Q, D0-brane charge Q_0.

- Let D_{abc}, $a, b, c = 1, \ldots, b_2(X)$ be the triple intersection numbers of the Calabi-Yau 3-fold X.

- Magnetic charge P lives on a lattice Λ, electric charge Q lives on dual lattice Λ^*.

- Λ has quadratic form $D_{ab} = D_{abc}P^c$, with signature $(1, b_2 - 1)$.

attractor point

- Irrespective of their values at infinity t_∞ , moduli fields flows to the "attractor values" t_γ (determined by charge γ of the black hole) at the event horizon of the black hole.

$$t_\gamma^a = B^a + iJ^a = Q^a + iP^a$$

$$c_j^* = |Z(\gamma, t_\gamma)| \langle \gamma, \gamma_i \rangle = -M \sum_{j \neq i} \gamma_{ij}$$

- Denef's eqns at the attractor point read $\sum_{j \neq i} \frac{\gamma_{ij}}{|r_{ij}|} = -M \sum_{j \neq i} \gamma_{ij}$
- If for any given i, if all the γ_{ij}-s have same sign, there is no solution.

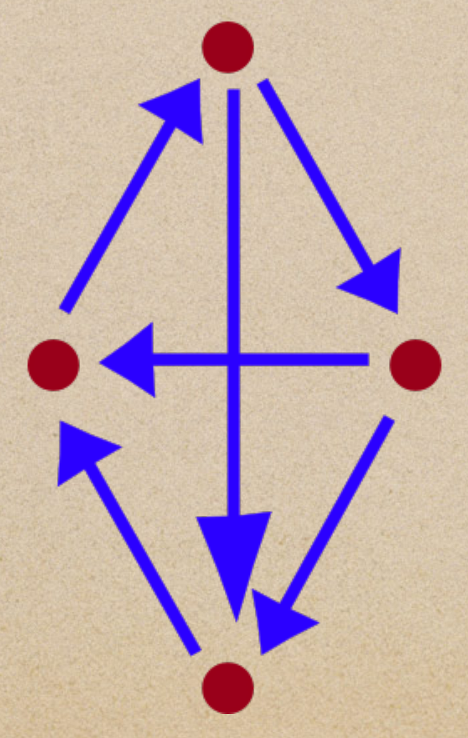
$\Rightarrow$ many multi-centered black holes do not survive at the attractor point !

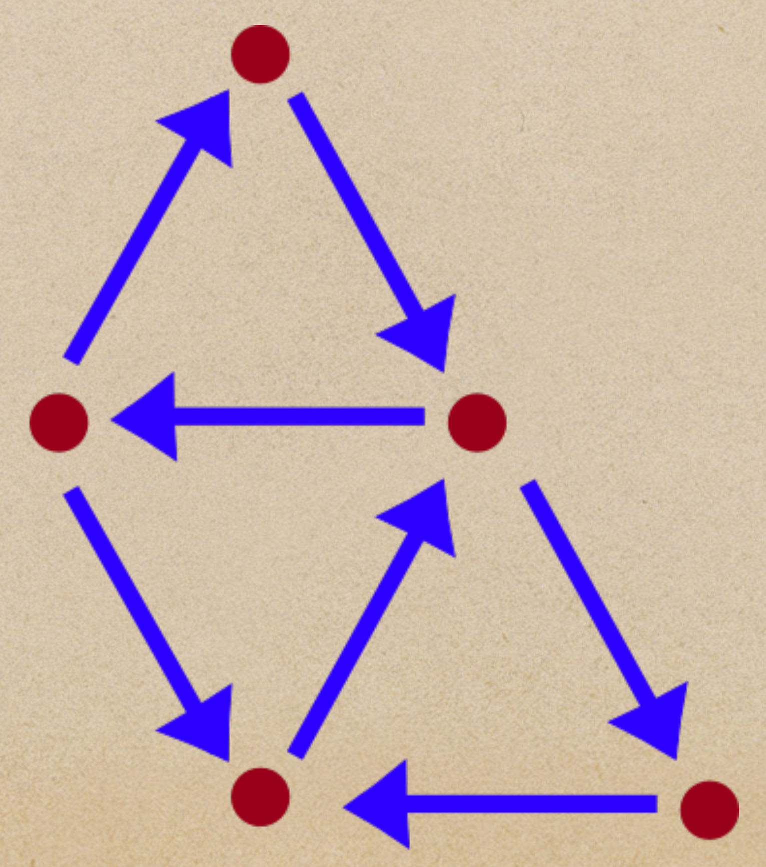
attractor survivors

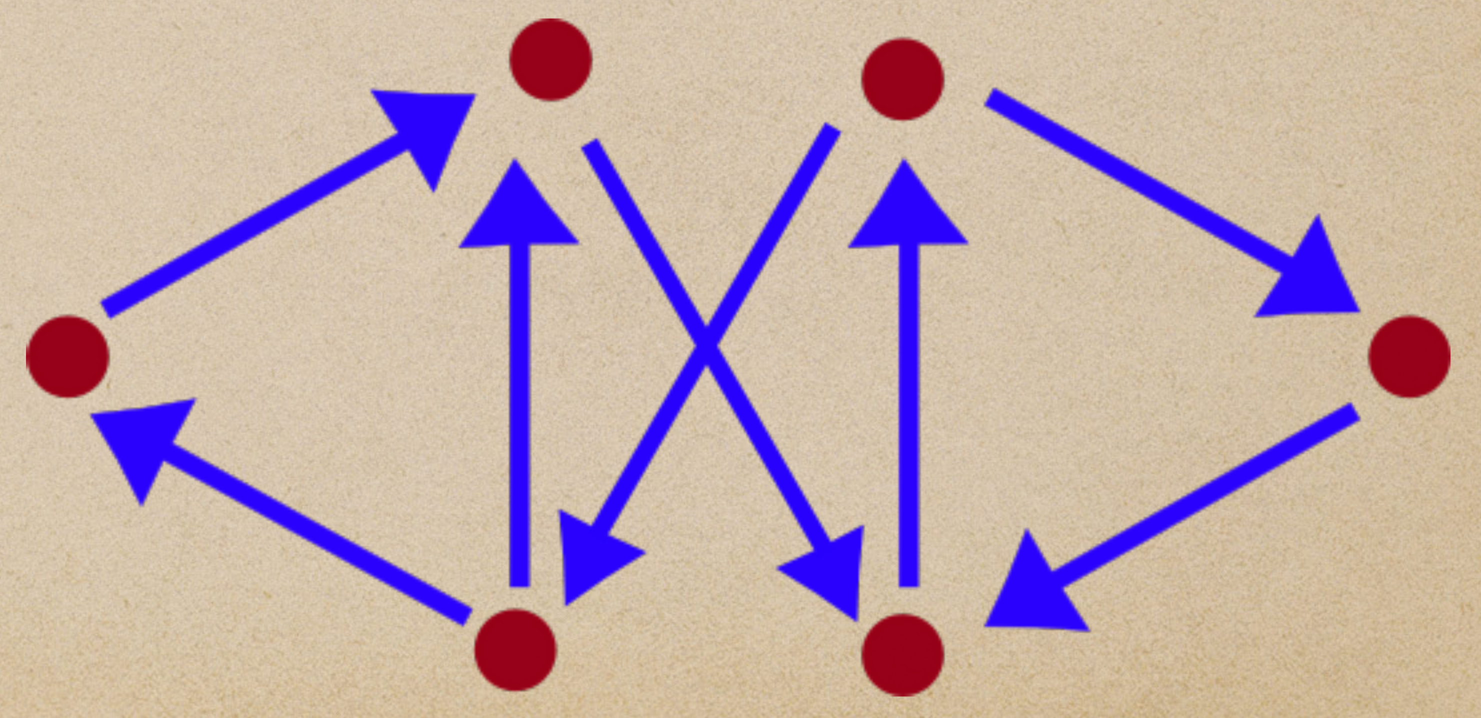
- Note, for a given i,
  all the $\gamma_{ij} > 0$ $\Rightarrow$ only outgoing arrows $\Rightarrow$ the node is a source.
  all the $\gamma_{ij} < 0$ $\Rightarrow$ only incoming arrows $\Rightarrow$ the node is a sink.

- Quivers with no source/sink = quivers with loops.

Examples of quivers with loops







large volume attractor point

- Attractor point lives deep "inside" the moduli space,
  whereas certain simplifications occur in large volume regime of Kähler moduli space.

- Best of both worlds: "analog" of attractor point in large volume regime ?

- Indeed, such a point exists and called large volume attractor point, given by

$$(t_\gamma^\lambda)^a = Q^a + i\lambda P^a, \lambda \to \infty$$

$$c_j^\lambda = 2\lambda\langle\gamma, \gamma_j\rangle \propto c_j^*,$$

equivalent to attractor point, as far as survival is concerned.

Denef's eqns
$$\frac{a}{r_{12}} - \frac{c}{r_{13}} = c_1, \qquad \frac{b}{r_{23}} - \frac{a}{r_{12}} = c_2$$
are invariant under
$$\frac{1}{r_{12}} \to \frac{1}{r_{12}} + \frac{1}{\varepsilon a}, \qquad \frac{1}{r_{23}} \to \frac{1}{r_{23}} + \frac{1}{\varepsilon b}, \qquad \frac{1}{r_{31}} \to \frac{1}{r_{31}} + \frac{1}{\varepsilon c}$$

- For $\varepsilon \to 0$, centers are arbitrarily close, provided (a,b,c) obey triangle inequalities.

  These are called scaling solutions.

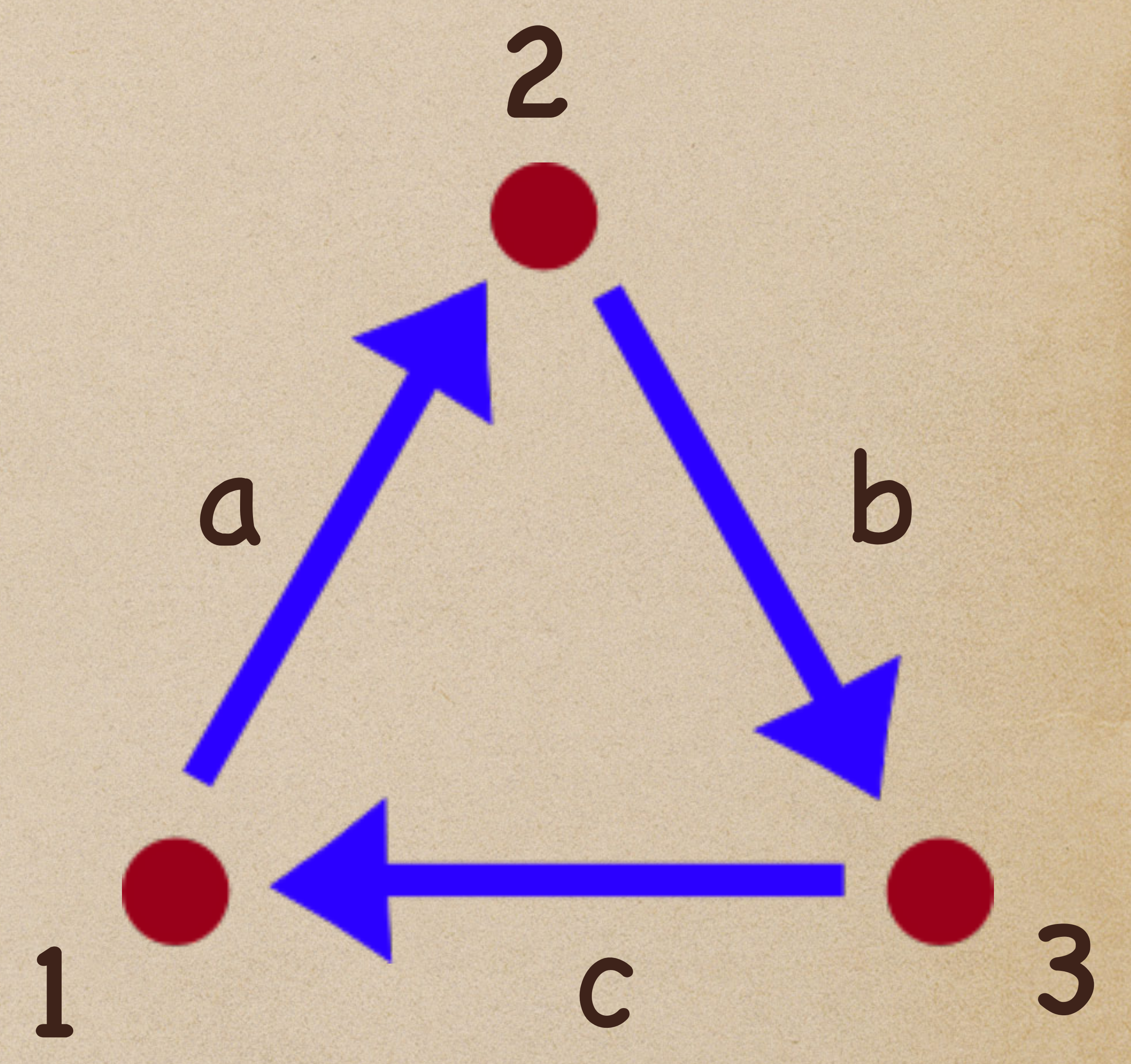
  Bena, Berkooz, de Boer, El-Showk, den Bleeken, Messamah, Wang, Warner, Denef, Moore

- Scaling solutions exist for any value of $\{c_i\}$, including attractor point.

- $\varepsilon \to 0$ region, often dubbed deep scaling regime, is tricky.

  Beaujard, SM, Pioline

Simplest quiver with loop



On moduli space of Scaling solutions

- Denef's equations can be exactly solved
$$\frac{1}{r_{12}} = \frac{1}{a\varepsilon} - M\,, \quad \frac{1}{r_{23}} = \frac{1}{b\varepsilon} - M\,, \quad \frac{1}{r_{31}} = \frac{1}{c\varepsilon} - M\,.$$
the scaling parameter ε is free.

- Since distances are positive, there is an upper limit on ε, where 3 centers align and one has $r_{12} + r_{23} = r_{13}$ or some permutation thereof.

- There is no lower cutoff on ε.
$\varepsilon \to 0$ region makes the solution space non-compact.

Roughly, scaling solution space = $\underbrace{\text{regular part}}_{\text{easy}} \cup \underbrace{\text{deep scaling part}}_{\text{tricky}}$ Manschot, Pioline, Sen; Beaujard, SM, Pioline

Scaling ($\varepsilon \to 0$ region) vs single centered black holes

- An asymptotic observer can't tell them apart.
- Both exist for any moduli
- Gives zero angular momentum states (pure Higgs states), a trait of single single-centered black hole microstates.

Chowdhury, Garavuso, S.M , Sen

- Both develop a "near horizon AdS_2" region.

Mirfendereski, Raeymaekers, Van den Bleeken

- Starting with 5-dimensional black holes, one can take a "decoupling limit", upon which only objects of length scale ℓ_5^3 or less, survive and go over to $AdS_3 \times S^2$.

de Boer, Denef, El-Showk, Messamah, Bleeken

For vanishing D6 charge, this leaves only scaling and single centered black holes.

This AdS_3 is expected to be dual of the microscopic CFT.
So CFT can't differentiate between single centered and scaling black holes either.

Modularity

2 avatars of modularity

- Modular group is $SL(2,\mathbb{Z})$ and it appears in 2 avatars:

1. In type IIB supergravity, it appears as S-dualtiy:

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad where \quad \tau = C_0 + \frac{i}{g_B}$$

$$\begin{pmatrix} C \\ B \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C \\ B \end{pmatrix},$$

$$J \to |c\tau + d| J,$$

   type IIA supergravity is related by T-duality along time circle: $g_A = \frac{\beta_A}{l_s} g_B$

2. In MSW CFT, is defined on $T^2 = S^1_{time} \times S^1_M$. Modularity emerges as modular group of torus.

Maldacena, Strominger, Witten

- Note same τ appears in both cases. E.g.

$$(\tau_2)_{CFT} = \frac{\text{circumference of time circle}}{\text{circumference of M theory circle}} = \frac{\beta_A}{g_A l_s} = (\tau_2)_{SUGRA}$$

$\Rightarrow$ We can analyze modularity either in SUGRA or CFT.

Partition functions

Supergravity partition function

In large volume regime,

$$\mathcal{Z}_{SG}(\tau, C, t) = \sum_{Q_0, Q_a} \bar{\Omega}(\gamma, t)\, (-1)^{P.Q}\, e^{-2\pi\tau_2 |Z(\gamma,t)| + 2\pi i \tau_1 \left(Q_0 - Q\,.\,B + B^2/2\right) + 2\pi i C.(Q - B/2)}$$

$$Z(\gamma, t) = \frac{1}{2} P\,.\,(J^2 - B^2) + Q\,.\,B - Q_0 + i(Q - BP)\,.\,J$$

$$|Z(\gamma, t)| = \frac{1}{2} P\,.\,(J^2 - B^2) + Q\,.\,B - Q_0 + \underbrace{\frac{\left((Q - BP)\,.\,J\right)^2}{P\,.\,J^2}}_{:=\hat{Q}_+^2} + \mathcal{O}(1/J)$$

Spectral flow symmetry

- In large volume limit, Supergravity has a $Sp(2b_2+2,\mathbb{Z})$ symmetry:

$$Q_a \to Q_a + d_{abc}k^b P^c\,,$$
$$Q_0 \to Q_0 + k^a Q_a + \frac{1}{2}d_{abc}k^a k^b P^c\,,$$
$$t^a \to t^a + k^a\,.$$

$\mathcal{Z}_{SG}$ changes by a phase under spectral flow transformation.

- This symmetry also shows up in MSW CFT, as symmetry of the algebra.

- Some spectral flow invariant combinations: $\hat{Q}_{\bar{0}} := -Q_0 + \frac{1}{2}Q^2\,,\, \hat{Q} := Q - B\,,$ and conjugacy class μ.

- Degeneracies are spectral flow invariant.

classical Black Hole entropy $\sim \sqrt{P^3\hat{Q}_0}$

attractor partition function

- We work with C=0.

$$\mathcal{Z}_{SG}(\tau, C, t) = e^{-\pi\tau_2 PJ^2} \sum_{Q_0,Q} \bar{\Omega}(P,Q,Q_0;t)\,(-1)^{P.Q}\,\bar{q}^{\hat{Q}_{\bar{0}} - \hat{Q}_-^2/2}\,q^{\hat{Q}_+^2/2}, \quad q = e^{2\pi i\tau}$$

- Attractor partition function: $Z(t_\infty)$ but $\Omega(t_\gamma)$

$$\mathcal{Z}_P^\lambda(\tau, C, t) = \sum_{Q_0,Q} \bar{\Omega}(\gamma, t_\gamma^\lambda)\,(-1)^{P.Q}\,\bar{q}^{\hat{Q}_{\bar{0}} - \hat{Q}_-^2/2}\,q^{\hat{Q}_+^2/2} = \sum_{\mu\in\Lambda^*/\Lambda} \overline{h_{P,\mu}(\tau)}\,\Theta_\mu(\tau,\bar{\tau},C=0,B),$$

where

$$\Theta_\mu(\tau,\bar{\tau},C=0,B) = \sum_{Q\in\Lambda_\mu^*} (-1)^{P.Q}\,q^{\hat{Q}_+^2/2}\,\bar{q}^{-\hat{Q}_-^2/2},$$

$$h_{P,\mu}(\tau) := \sum_{Q_0} \overline{\Omega}(\gamma; t_\gamma^\lambda)\,q^{\hat{Q}_{\bar{0}}},$$

Note, both Q_+^2 and $-Q_-^2$ are +ve, thus the sum converges

multi-center black hole

- Degeneracies have the structure $\bar{\Omega}(\{P_i, Q_i, Q_{0,i}\}) = f(\{P_i, Q_i\}) \prod_i \bar{\Omega}_i(\hat{Q}_{\bar{0},i})$

  We try to get a $\prod_i h_{P_i,\mu_i}$ in the partition function.

- To this end, we note $\bar{q}^{\hat{Q}_{\bar{0}}} = \bar{q}^{\frac{1}{2}Q^2} \; \bar{q}^{\sum_i (\hat{Q}_{\bar{0},i})} \; \bar{q}^{-\frac{1}{2}\sum_i Q_i^2}$

  upon substituting this, yellow terms give $\prod_i h_{P_i,\mu_i}$,

  whereas green terms lead to $\sum f(\{P_i, Q_i\}) \, \bar{q}^{-\frac{1}{2}\sum_i Q_i^2}$

- Problem: for fixed total charge, the exponent $-\frac{1}{2}\sum_i Q_i^2$ has signature $(n-1,(n-1)(b_2-1))$.

  It seems the Green sum may not converge!

  intuition: If f happens to vanish for -ve exponent, then the sum may converge.

specialize to n=3

- Since we are at the (large volume) attractor point, only scaling black holes survive.
- Scaling black holes exist only if $(\gamma_{12}, \gamma_{23}, \gamma_{31}) =: (a, b, c)$ form a triangle.
- Fact of life: unless (a,b,c) form a triangle, the following combination vanishes:

$$F_{total}(a,b,c) = \frac{1}{4}\Big[1 + sgn(a+b-c)\, sgn(a+c-b) + sgn(a+c-b)\, sgn(b+c-a) + sgn(b+c-a)\, sgn(a+b-c)\Big]$$

Manschot, Pioline, Sen;

- So the degeneracy must include a factor of F_{total}.

settle for a simpler problem: replace degeneracies by F_{total}.

- We note

$$a = P_1 . Q_2 - P_2 . Q_1 = C_a . \vec{Q}$$
$$b = P_2 . Q_3 - P_3 . Q_2 = C_b . \vec{Q}$$
$$c = P_3 . Q_1 - P_1 . Q_3 = C_c . \vec{Q}$$

where

$$C_a = (-P_2, P_1, 0)\ ,$$
$$C_b = (0, -P_3, P_2)\ ,$$
$$C_c = (P_3, 0, -P_1)\ ,$$
$$\vec{Q} = (Q_1, Q_2, Q_3)\ .$$

Condition for convergence

$$\Theta_\mu[\mathscr{K}](\tau; L) = \sum_{x \in L+\mu} \mathscr{K}(x)\, q^{-B(x)/2}$$

The sum

$$\mathscr{K}(x, \{V_i\}) = \frac{1}{4}\left(\underbrace{w(\{V_i\})}_{1\ FAPP} + \sum_{j=1}^{N} sgn(B(x, V_j)) sgn(B(x, V_{j+1})) \right)$$

$$w(\{V_i\}) = -\sum_{j=1}^{N} sgn(B(v, V_j)) sgn(B(v, V_{j+1}))\quad v^2 > 0\,,$$

converges, if the following Funke-Kudle conditions are satisfied

$$B(V_j, V_j) > 0,$$
$$B(V_j, V_j)\, B(V_{j+1}, V_{j+1}) - B(V_j, V_{j+1})^2 > 0,$$
$$B(V_j, V_j)\, B(V_{j-1}, V_{j+1}) - B(V_j, V_{j-1})\, B(V_j, V_{j+1}) < 0.$$

Funke Kudla

It is checked that for $V_1 = C_a + C_b - C_c, V_2 = C_a + C_c - C_b, V_3 = C_c + C_b - C_a$, these conditions are satisfied.

modular completion

Replace product of signs with error functions in the sum.

$$sgn(V_1, x)sgn(V_2, x) \to E_2(\alpha, \sqrt{2\tau_2}\,\boldsymbol{u}),$$

$$E_2(\alpha; \boldsymbol{u}) = \int_{\mathbb{R}^2} e^{-\pi(u_1-u_1')^2-\pi(u_2-u_2')^2}\, sgn(u_2')\, sgn(u_1' + \alpha\, u_2')\, du_1'\, du_2',$$

$$\alpha = \alpha(V_1, V_2) = \frac{(V_1, V_2)}{\sqrt{V_1^2\, V_2^2 - (V_1, V_2)^2}},$$

$$\boldsymbol{u} = \boldsymbol{u}(V_1, V_2; x) = (u_1(V_1, V_2; x), u_2(V_1, V_2; x)),$$

where.

$$u_1(V_1, V_2; x) = \frac{(V_{1\perp 2}, x)}{\sqrt{(V_{1\perp 2}, V_{1\perp 2})}},$$

$$u_2(V_1, V_2; x) = \frac{(V_2, x)}{\sqrt{(V_2, V_2)}}$$

another representation of E_2

We find another representation useful.

$$E_2(\alpha;\boldsymbol{u}) = sgn(u_2)sgn(u_1+\alpha u_2)+sgn(u_1)M_1(u_2)$$
$$+sgn(u_2-\alpha u_1)\,M_1\left(\frac{u_1+\alpha u_2}{\sqrt{1+\alpha^2}}\right)+M_2(\alpha;u_1,u_2)\,.$$

where, $M_1(u), M_2(u)$ are nasty expressions involving iterated integrals.

Summary

- We have computed 3-centered scaling contributions to $h_{P,\mu}$, but with replacing bound state degeneracies (pure Higgs degeneracies) by 1.
- We have also made some progress in computing the contribution of "regular part" of the scaling solution space.
- Modularity can still be preserved, albeit at the cost of holomorphicity.

Future directions

- Check same line of thought works for quivers with more nodes and more loops.
- Compute the actual 3-centered scaling contributions to $h_{P,\mu}$, i.e. by reinstating pure Higgs degeneracies.

Þakka þér fyrir

Спасибо

ευχαριστώ

bedankt

Kiitos

Merci

நன்றி

Diolch

Tapadh leat

Danke

Thank You !

धन्यवाद

Grazie

ধন্যবাদ

ಧನ್ಯವಾದಗಳು

Go raibh maith agat !

Gracias

ધન્યવાદ

eskerrik asko

$$S:\ \Theta_\mu(-1/\tau, -1/\bar{\tau}, -B, C) = \frac{1}{\sqrt{|\Lambda^*/\Lambda|}}(-i\tau)^{b_2^+/2}(i\bar{\tau})^{b_2^-/2}e^{-i\pi P^2/2}$$

$$\times \sum_{\nu\in\Lambda^*/\Lambda} e^{-2\pi i\mu.\nu}\Theta_\nu(\tau, \bar{\tau}, C, B)\,,$$

$$T:\ \Theta_\mu(\tau+1, \bar{\tau}+1, C+B, B) = e^{i\pi(\mu+P/2)^2}\Theta_\mu(\tau, \bar{\tau}, B, C)\,.$$

$$S:\ \widehat{h}_{P,\mu}(-1/\tau, -1/\bar{\tau}) = -\frac{1}{\sqrt{|\Lambda^*/\Lambda|}}(-i\tau)^{-b_2/2-1}\varepsilon(S)^* e^{-i\pi P^2/2}\sum_{\delta\in\Lambda^*/\Lambda} e^{-2\pi i\delta.\mu}\widehat{h}_{P,\delta}(\tau, \bar{\tau})\,,$$

$$T:\ \widehat{h}_{P,\mu}(\tau+1, \bar{\tau}+1) = \varepsilon(T)^* e^{i\pi(\mu+P/2)^2}\,\widehat{h}_{P,\mu}(\tau, \bar{\tau}),$$

An alternative representation of E_2

$$E_2(\alpha;\boldsymbol{u}) = \operatorname{sgn}(u_2)\operatorname{sgn}(u_1+\alpha u_2) + \operatorname{sgn}(u_1)M_1(u_2) + \operatorname{sgn}(u_2-\alpha u_1)\, M_1\!\left(\frac{u_1+\alpha u_2}{\sqrt{1+\alpha^2}}\right) + M_2(\alpha;u_1,u_2)\,.$$

$$M_1(u) = \begin{cases} \frac{iu}{\sqrt{2\tau_2}} q^{\frac{u^2}{4\tau_2}} \int_{-\bar\tau}^{i\infty} \frac{e^{\frac{i\pi u^2 w}{2\tau_2}}}{\sqrt{-i(w+\tau)}} dw, & u\neq 0, \\ 0, & u=0. \end{cases}$$

$$m_2(u_1,u_2) = \begin{cases} \frac{u_1 u_2}{2\tau_2} q^{\frac{u_1^2}{4\tau_2}+\frac{u_2^2}{4\tau_2}} \int_{-\bar\tau}^{i\infty} dw_2 \int_{w_2}^{i\infty} dw_1\, \frac{e^{\frac{\pi i u_1^2 w_1}{2\tau_2}+\frac{\pi i u_2^2 w_2}{2\tau_2}}}{\sqrt{-(w_1+\tau)(w_2+\tau)}}, & u_1\neq 0 \\ 0, & u_1 = 0. \end{cases}$$

$$M_2(\alpha;u_1,u_2) = \begin{cases} -m_2(u_1,u_2) - m_2\left(\frac{u_2-\alpha u_1}{\sqrt{1+\alpha^2}}, \frac{u_1+\alpha u_2}{\sqrt{1+\alpha^2}}\right) & u_1\neq 0, u_2-\alpha u_1\neq 0\,, \\ -m_2\left(\frac{u_2-\alpha u_1}{\sqrt{1+\alpha^2}}, \frac{u_1+\alpha u_2}{\sqrt{1+\alpha^2}}\right) & u_1=0, u_2\neq 0\,, \\ -m_2(u_1,u_2) & u_1\neq 0, u_2-\alpha u_1 = 0\,, \\ \frac{2}{\pi}\arctan\alpha & u_1=u_2=0. \end{cases}$$