Universal Statistics of Vortices in a Holographic Superconductor: Beyond the Kibble-Zurek Mechanism

Hai-Qing Zhang
Beihang University

Collaborators: Adolfo del Campo, Fernando Javier Gómez-Ruiz, Zhi-Hong Li, Chuan-Yin Xia, Hua-Bi Zeng. JHEP06(2021)061

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Kibble-Zurek mechanism (KZM) predicts the universal relation between the mean value of the number density of topological defects n and the quench time τ_Q:

$$n \sim (1/\tau_Q)^{\frac{d\nu}{1+\nu z}}$$

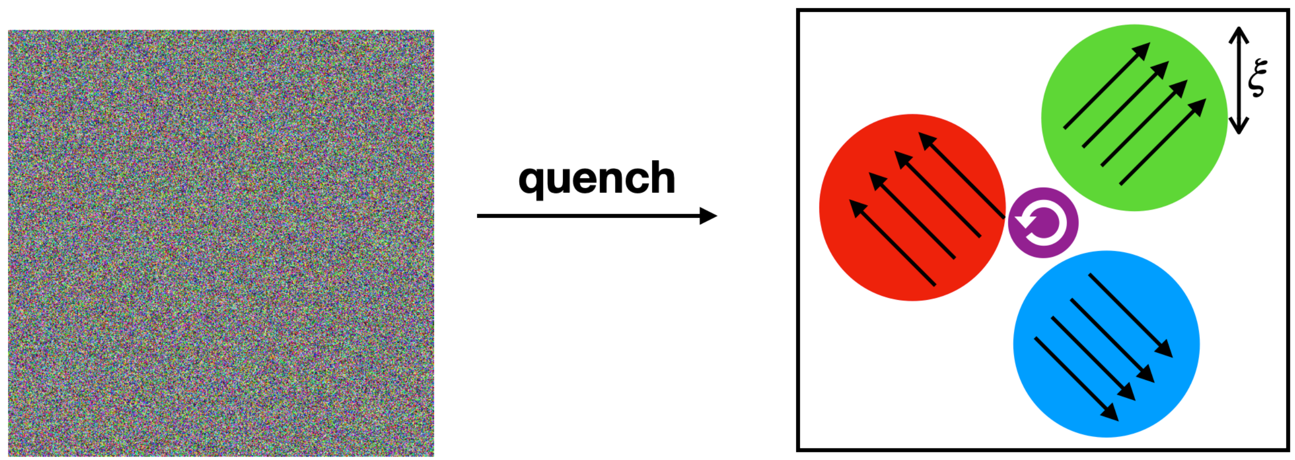
Our work investigates the statistical properties of vortices (point defects in 2-dim) beyond KZM:

1. The effect of periodic boundary conditions (PBC);
2. Probability density function (PDF) and the large fluctuations away from the mean of vortices number.

KZM: In a phase transition induced by a finite quench, due to the causality, the system can only be correlated in a finite size ξ.

From the viewpoint of symmetry, the previous higher symmetry is broken to a lower symmetry.

Example: U(1) symmetry breaking in superconductor system during a quench, vortices will turn out.

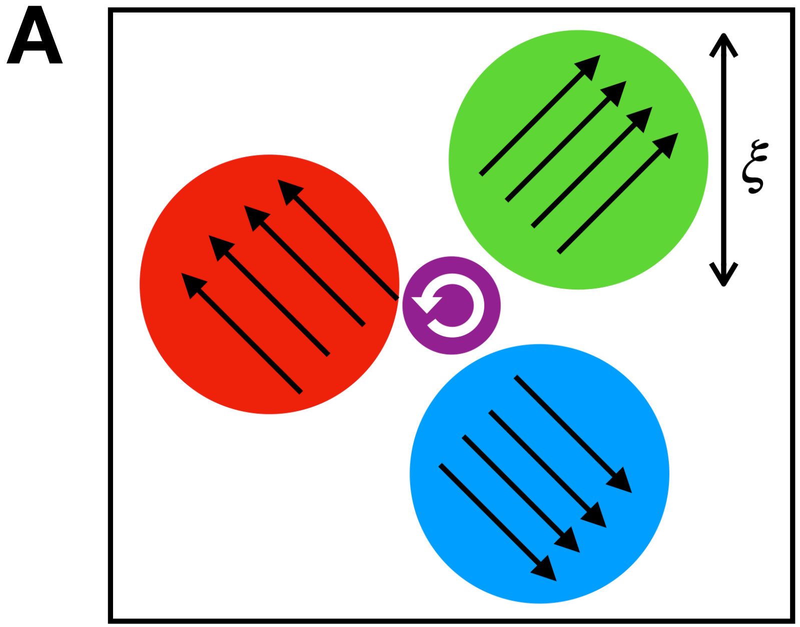


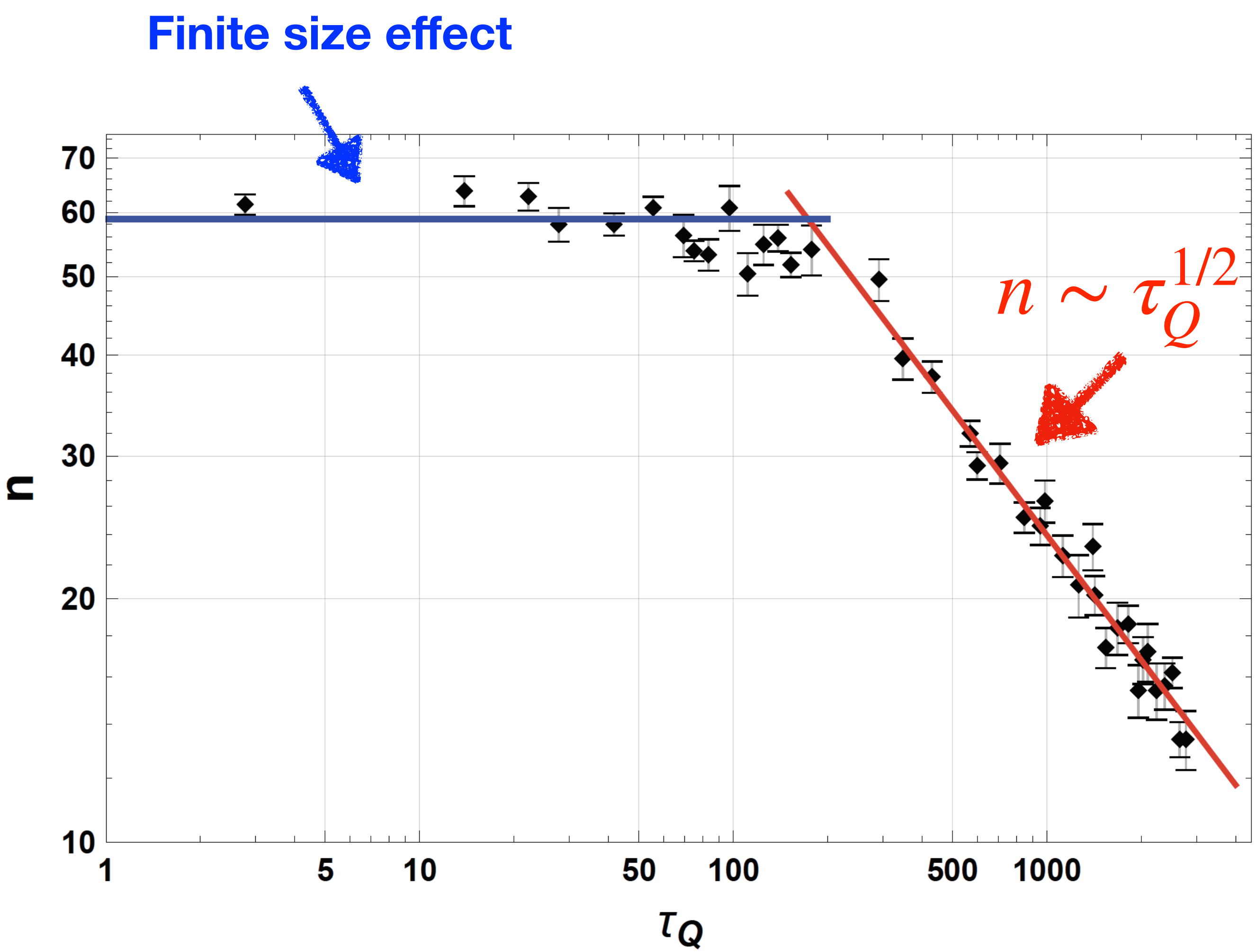
The symmetry broken domains are randomly distributed.

System size: A;

Symmetry broken domain size: ξ^d ;

Number of defects is $n \approx A/\xi^d$





This relation holds in a relatively slow quench (τ_Q is bigger)

For fast quench (τ_Q is smaller), due to the finite size effect of the vortices, the number is almost constant.

KZM in condensed matter physics

- **Liquid crystals**: Chuang, et.al., Science 251 (1991) 1336; Bowick, et.al.,Science 263 (1994) 943;Digal, et.al., PRL 83 (1999) 5030
- **He3 superfluids**: Baeuerle,et.al.,Nature 382 (1996) 332; Ruutu et al. ,Nature 382 (1996) 334
- **Thin-film superconductors**: Maniv,et.al., PRL 91 (2003) 197001; PRL 104, 247002 (2010).
- **Quantum optics**: Xu, et.al., PRL,112, 035701(2014)

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Reviews: T. Kibble, Phys. Today 60N9 (2007) 47;
A. del Campo and W. H. Zurek, Int. J. Mod. Phys. A 29 (2014) no.8, 1430018

Holographic KZM

- **Holographic 1d system:**

Sonner, del Campo, Zurek, Nature Communications 6, 7406 (2015);
Li, HQZ, arXiv:2111.05568
Li, Shi, HQZ,arXiv: 2111.15230

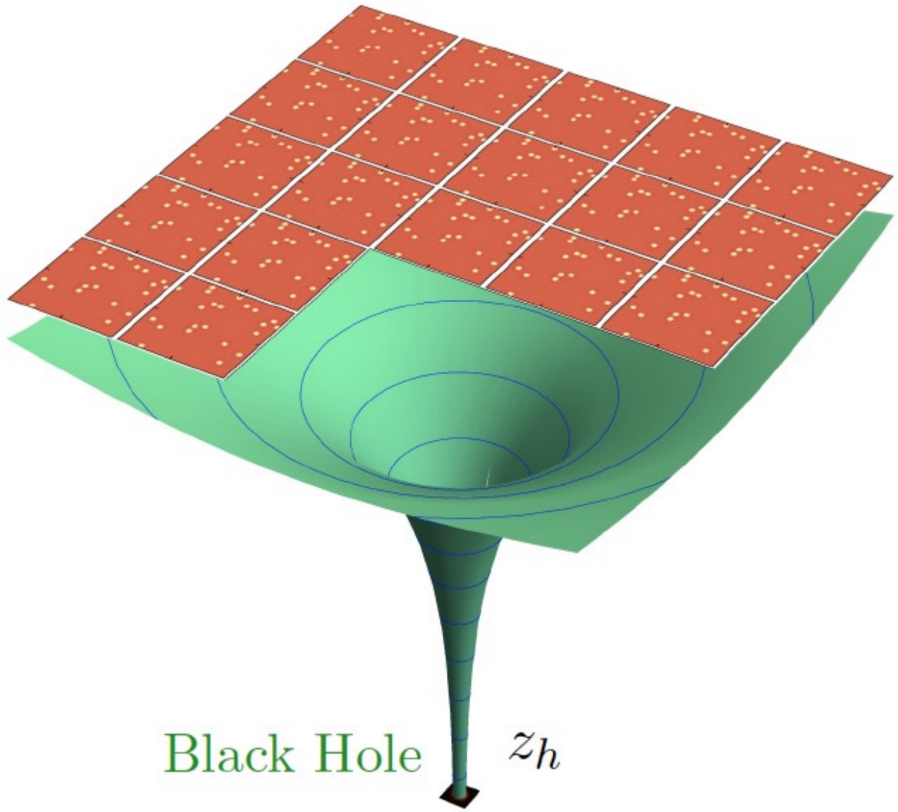
- **Holographic superfluid in 2d system:**

Chesler, Garcia-Garcia, Liu, PRX 5 (2015) 2, 021015

- **Holographic superconductor in 2d system:**

Zeng, Xia, HQZ, *JHEP* 03 (2021) 136
Li, Xia, Zeng, HQZ, *JHEP* 04 (2020) 147
Li, Zeng, HQZ, *JHEP* 04 (2021) 295
Li, Xia, Zeng, HQZ, *JHEP* 10 (2021) 124

Holographic Principle



- **Lagrangian:**

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - |\partial\Psi - iA\Psi|^2 - m^2|\Psi|^2 .$$

- **Eddington-Finkelstein coordinates:**

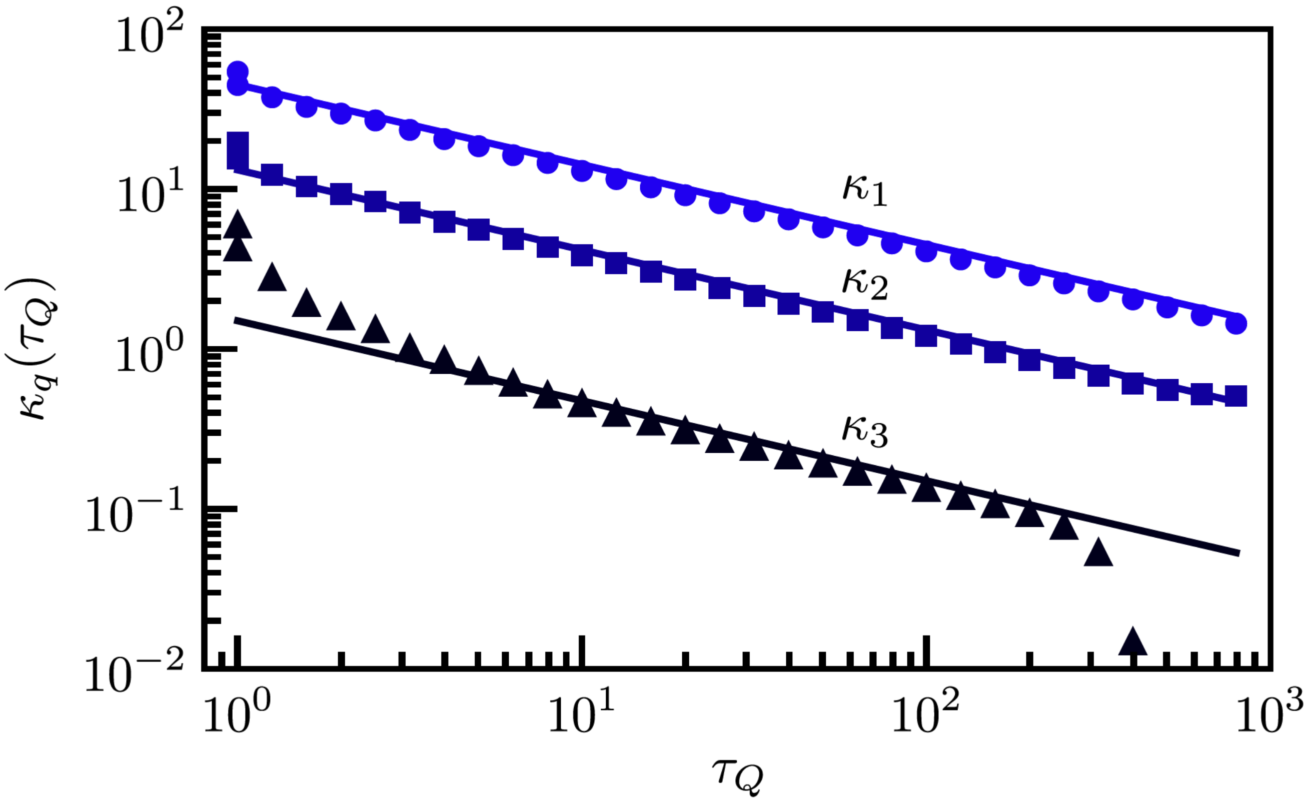
$$ds^2 = \frac{L^2}{z^2}(-f(z)dt^2 - 2dtdz + dx^2 + dy^2)$$

where $f(z) = 1 - z^3$.

Go Beyond KZM

del Campo (PRL 122, 014103 (2019)) studied the kink formation in a ‘transverse field quantum Ising model’.

Kinks statistics satisfy the Poisson binomial distribution, and all cumulants exhibiting a universal power-law scaling with the quench rate.



It describes the number of successes in a sequence of n independent yes/no experiment with success probabilities p1, p2, …, pn.

The ordinary binomial distribution is a special case of the Poisson binomial distribution, when all the success probabilities are the same, that is p1=p2=…=pn.

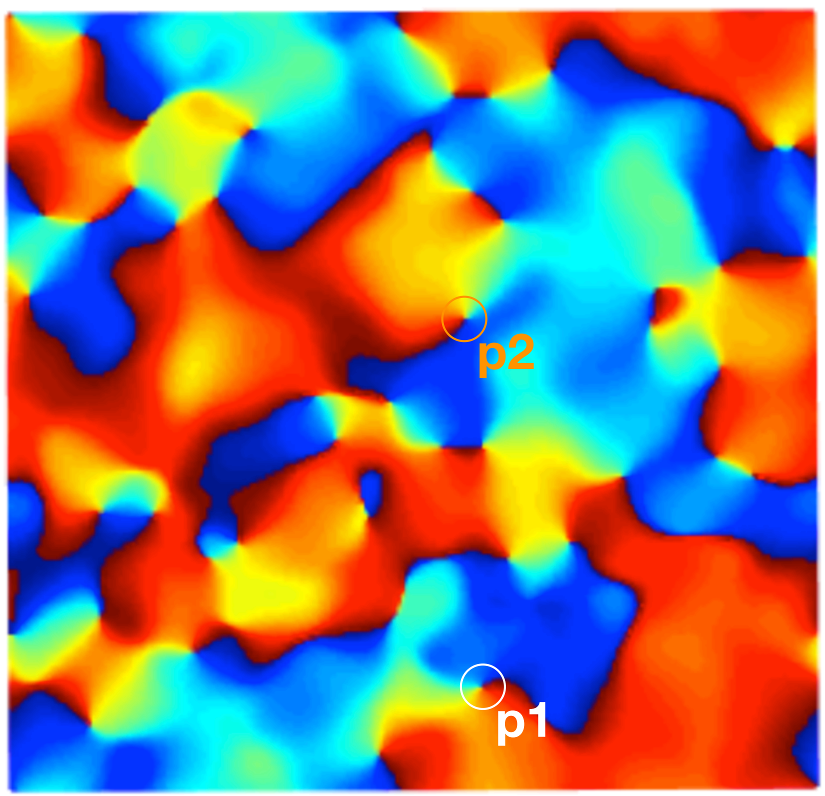
A binomial distribution with N independent `trial' and 'success probability' p: $P(n) \sim B(n, N, p) = \binom{N}{n} p^n (1-p)^{N-n}$

N: number of symmetry breaking domain $N \sim A/\xi^d$;
p: probability succeeds to form one defect;
1-p: probability fails to form one defect.

Therefore, in KZM, the mean value of defects number is:

$$n \sim pA/\xi^d$$

However, for a Poisson binomial distribution, things get complicated since each probability p_i may not be identical



The PDF of Poisson binomial distribution is

$$P(n) = \sum_{A \in F_n} \prod_{i \in A} p_i \prod_{j \in A^c} (1 - p_j)$$

Where, F_n is all subsets of n integers chosen from N. Therefore, F_n contains C_n^N elements.

A^c is the complement of A, i.e., $A^c = \{1,2,\cdots,N\} \backslash A$

Complicated! One can transform the PDF into its ‘momentum’ space by $P(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \tilde{P}(\theta) e^{-i\theta n}$.

$\tilde{P}(\theta)$ is the ‘characteristic function’ of Poisson binomial function.

Finally, $\tilde{P}(\theta) = \prod_k \left((1-p_k) + p_k e^{i\theta}\right)$, k is the k-th level of the eigenenergy.

From probability theory, expansions of $\log \tilde{P}(\theta)$ can generate the cumulants $\{\kappa_q\}$ of the distribution:

$$\log \tilde{P}(\theta) = \sum_{q=1}^{\infty} \frac{(i\theta)^q}{q!} \kappa_q$$

Finally, one gets $\kappa_1 \propto \kappa_2 \propto \kappa_3 \propto \sqrt{1/\tau_Q}$. Indeed, all the cumulants are proportional to $\sqrt{1/\tau_Q}$.

The distribution is not 'normal distribution' as previously thought, since higher cumulants $(q \geq 3)$ of 'normal distribution' is vanishing.

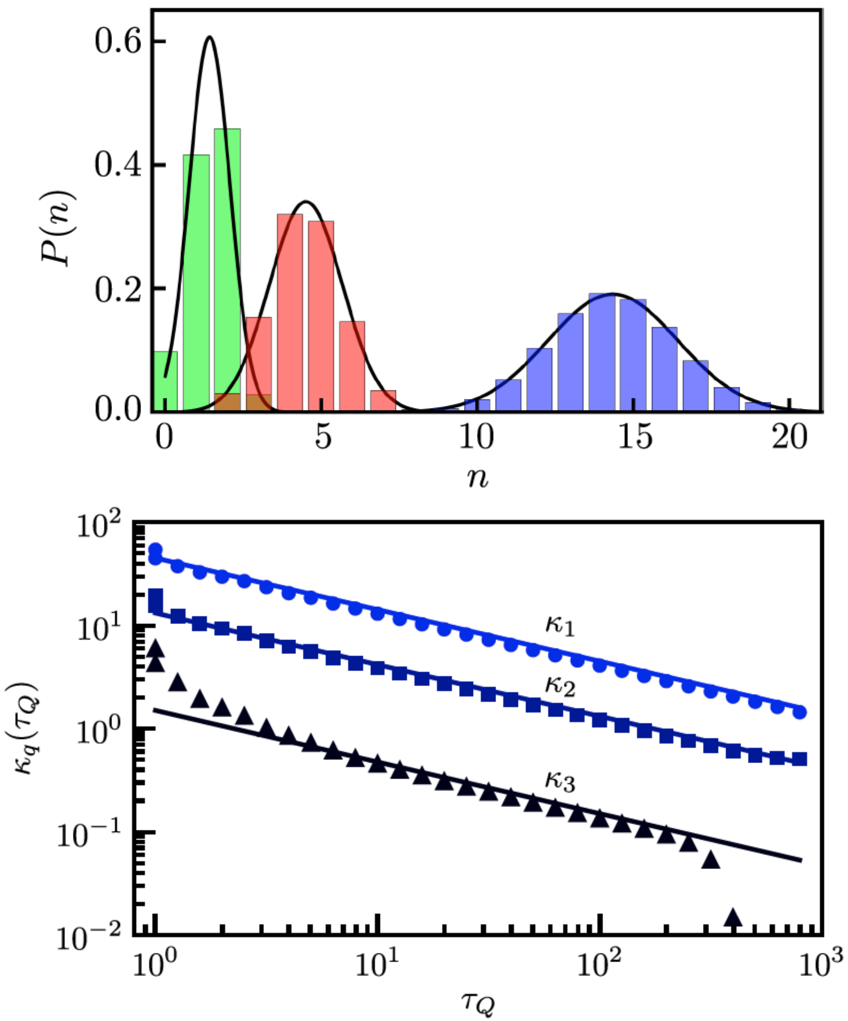
Physically,

$\kappa_1 = \langle n \rangle$ is the mean;

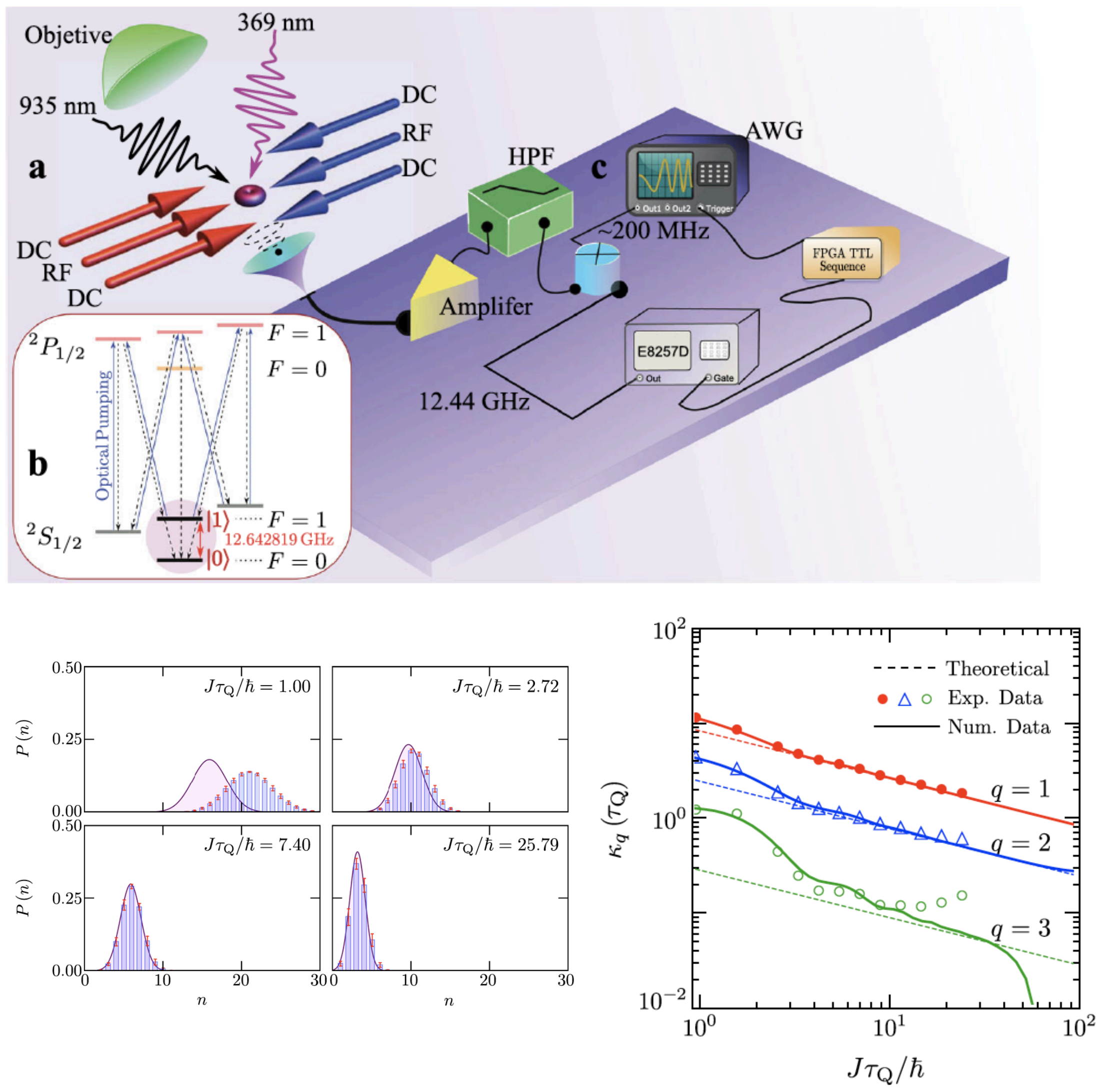
$\kappa_2 = \langle n^2 \rangle - \langle n \rangle^2$ is the variance;

$\kappa_3 = \langle \left(n - \langle n \rangle \right)^3 \rangle$ represents the skewness through the identity $\kappa_3 = \mathrm{Skew}(n) \kappa_2^{3/2}$

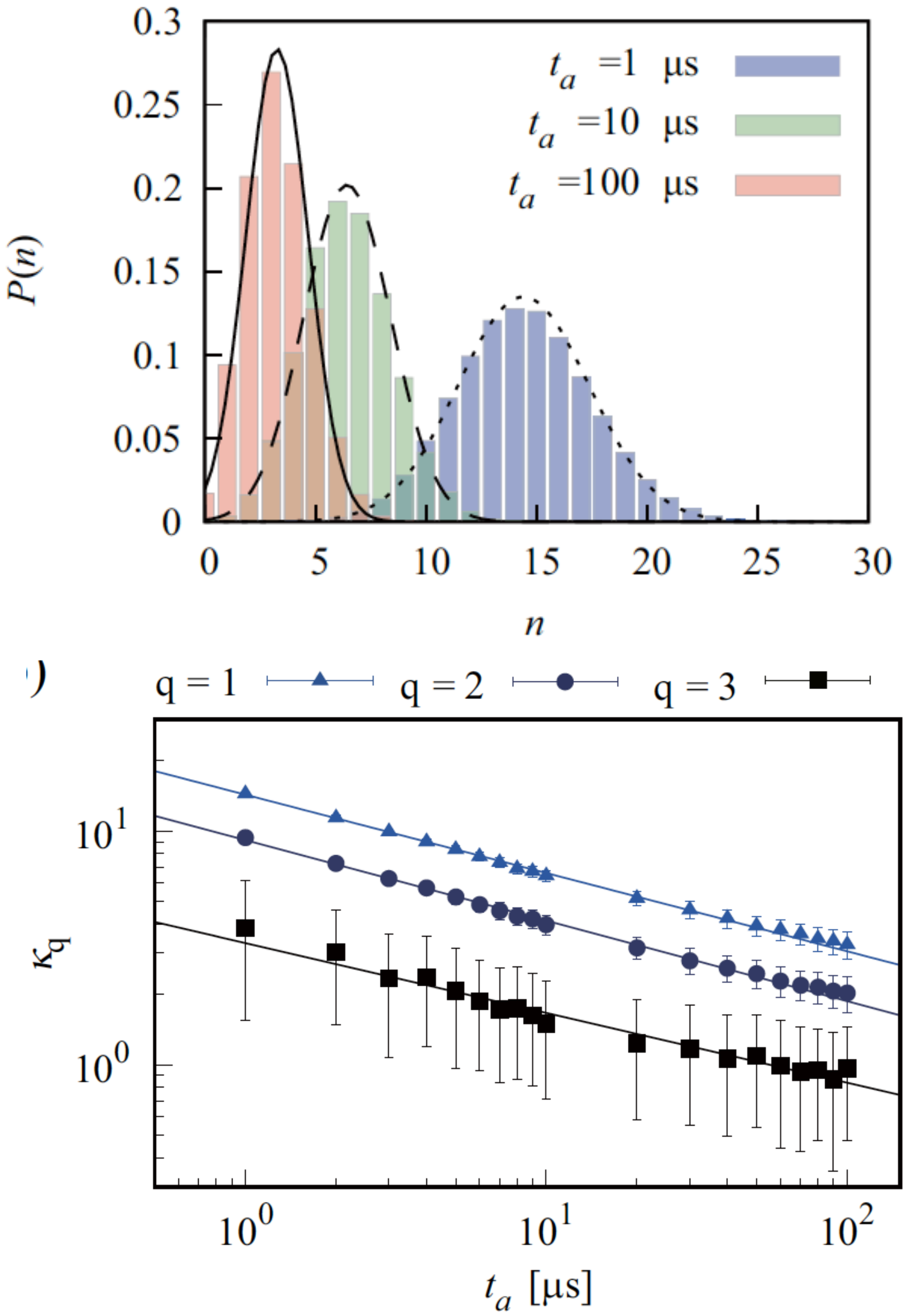
This theory was tested in various ways.



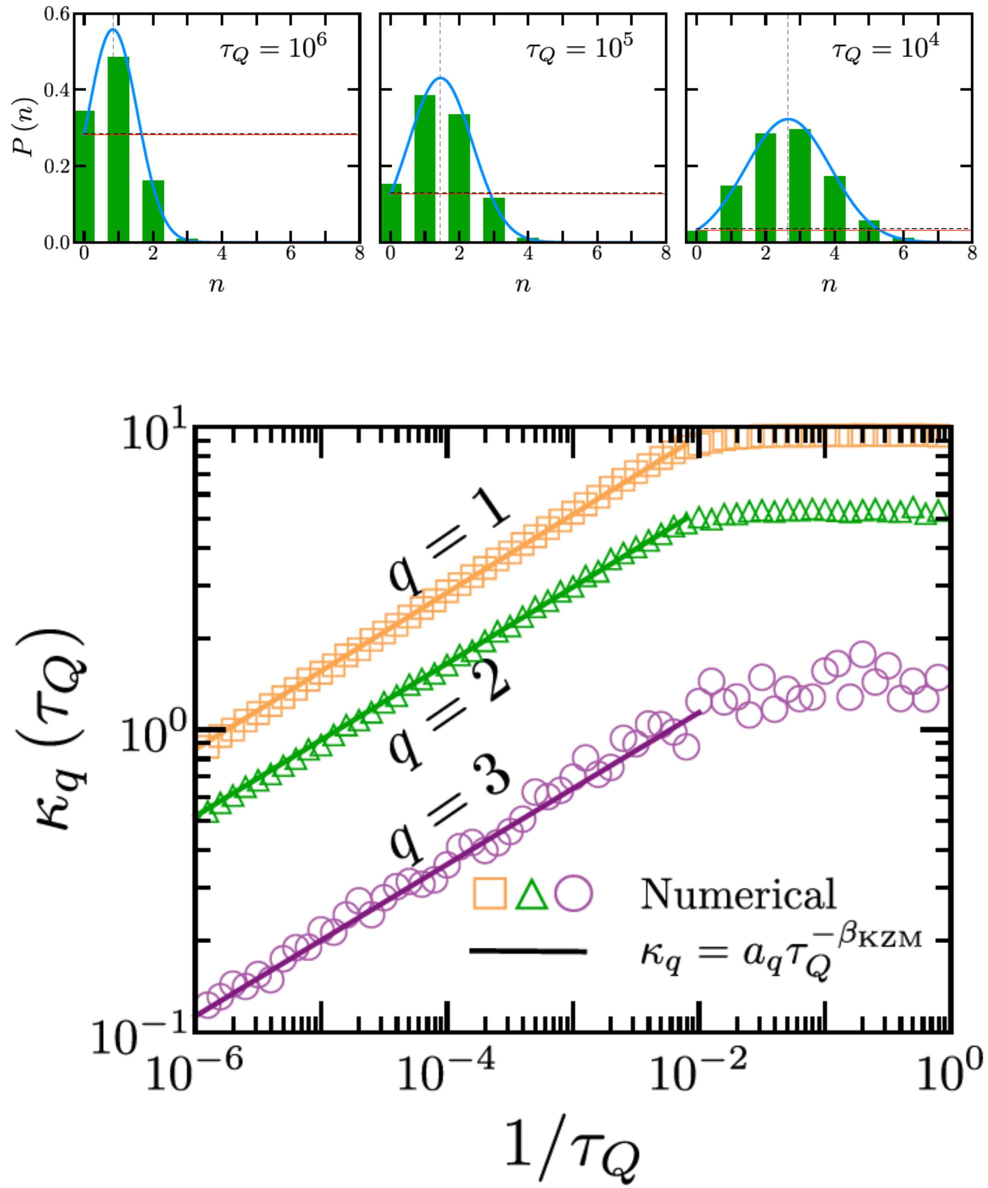
del Campo (PRL 122, 014103 (2019))



Jing-Min Cui, et.al.
Communications Physics, (2020) 3:44



Y. Bando, et al. PR Research 2, 033369 (2020)



F. J. Gomez-Ruiz, et al. PRL 124 240602 (2020)

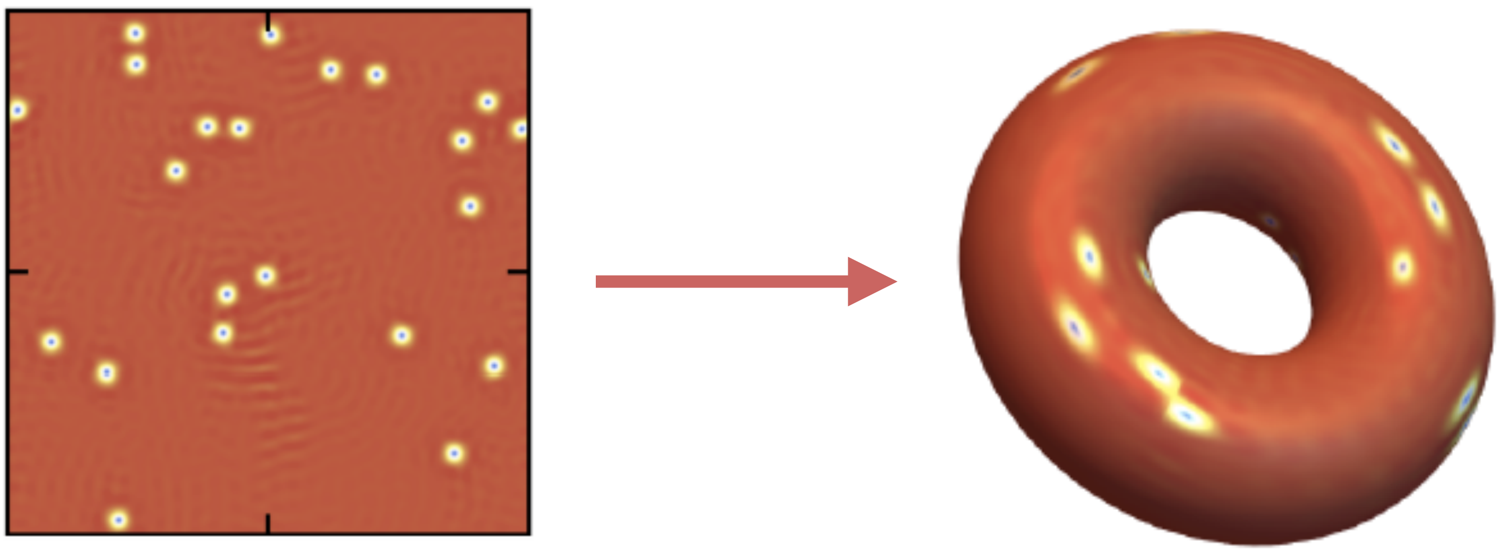
Extensions:

1). From 1-dim to 2-dim;

2). They indeed study the number of pairs of the kinks, rather than the individual number of kinks;

3). With strong couplings.

Holographic vortices in 2-dim boundary, with periodic boundary conditions (2-torus). JHEP06(2021)061



A 2-torus T^2 has zeros Euler characteristic, i.e., $\chi(T^2) = 2 - 2g = 0$, g is the number of genus. As a result of Poincare-Hopf theorem, the total vorticity of superconductor equals χ. (Laver, Forgan, Nat. Commun. 1 (2010) 45.)

Thus, the net vorticity on the 2-torus should be vanishing. Besides, in real simulations we never found any vortices with vorticity $|V| > 1$. Thus, the positive and negative vortices have the same number, i.e, there are always even number of vortices.

Averagely, we can assume the successful probability for a vortex as p, while failure to form a vortex as 1-p. (Please note that here we have simplified the complicated Poisson binomial distribution to binomial distribution with reasonable assumptions).

The distribution of the vortices is binomial distributions restricted to even outcomes. We call it as ‘even-binomial’ (EB) distributions.

Assume $p + q = 1, \quad (p, q \geq 0)$

$$\begin{aligned}(-p+q)^N &= \sum_{k=0}^{N} \binom{N}{k} (-p)^k q^{N-k} = \sum_{k=0}^{N} \binom{N}{k} (-1)^k p^k q^{N-k} \\ &= \underbrace{\sum_{\substack{k=0 \\ k \in \text{even number}}}^{N} \binom{N}{k} p^k q^{N-k}}_{A} - \underbrace{\sum_{\substack{k=1 \\ k \in \text{odd number}}}^{N} \binom{N}{k} p^k q^{N-k}}_{B}\end{aligned}$$

We already have $A + B = (p+q)^N = 1$. Therefore, it is easy to get

$$\begin{aligned}A &= \frac{1 + (-p+q)^N}{2} = \frac{1 + (1-2p)^N}{2}, \\ B &= 1 - A = \frac{1 - (1-2p)^N}{2}.\end{aligned}$$

So the distribution of the even number of vortices is

$$P_{\mathrm{eB}} = \frac{1}{A}\binom{N}{k} p^k (1-p)^{N-k}, \qquad k \in \text{ non-negative even integer}$$

The first three cumulants are:

$$\begin{aligned}
\kappa_1 &= Np\,\frac{1-(1-2p)^{N-1}}{1+(1-2p)^N}, \\
\kappa_2 &= \frac{Np(1-p)}{(1+(1-2p)^N)^2}\left(1-(1-2p)^{2N-2}+4(N-1)(p-p^2)(1-2p)^{N-2}\right), \\
\kappa_3 &= \frac{Np(1-p)}{(1+(1-2p)^N)^3}\Big(1-2p-(1-2p)^{3N-3} \\
&\quad +(1-4(1-p)p(1-(N-1)(3-2(N+4)(1-p)p)))\,(1-2p)^{N-3} \\
&\quad -(1-4(1-p)p(1+(N-1)(3+2(N-2)(1-p)p)))(1-2p)^{2N-3}\Big).
\end{aligned}$$

They satisfy the recursion relation which is typical for binomial distributions: $\kappa_{q+1} = p(1-p)d\kappa_q/dp$

If we consider the limit $N \to \infty$ and keep $Np = \lambda$ finite, we get

$$P_{\mathrm{eB}} \xrightarrow[N\to\infty]{Np=\lambda} \frac{2e^{\lambda}\lambda^k}{(e^{2\lambda}+1)\,\Gamma(k+1)} = \mathrm{sech}(\lambda)\frac{\lambda^k}{k!}, \qquad k \in \text{non-negative even integer}$$

We call this as ‘even-Poisson’ (EP) distribution. λ is the average success times, or equivalently, the average number of vortices $\langle n \rangle$.

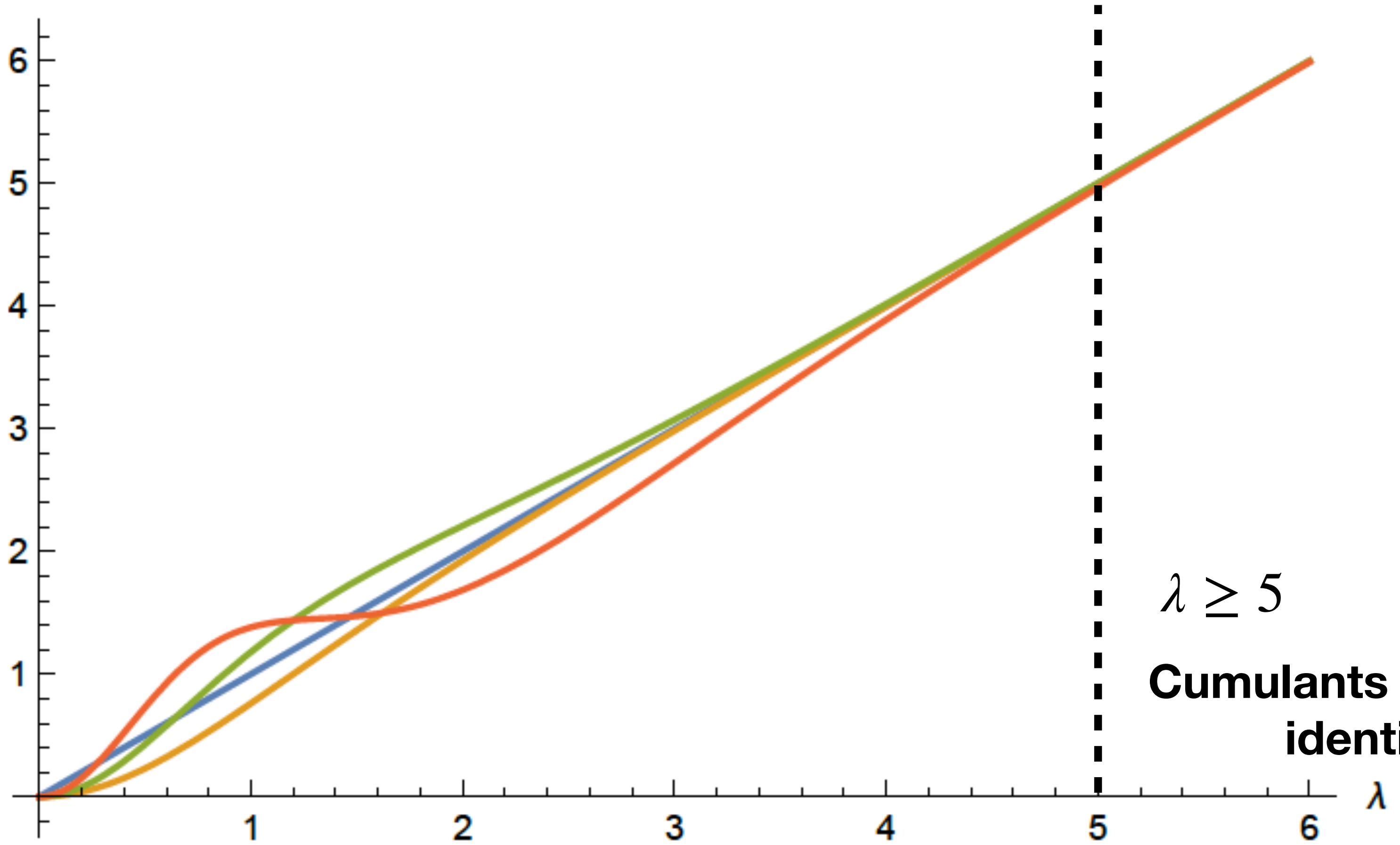
Besides, the first three cumulants in these limits will become,

$$\kappa_1 \xrightarrow[N\to\infty]{Np=\lambda} \lambda \tanh(\lambda),$$

$$\kappa_2 \xrightarrow[N\to\infty]{Np=\lambda} \lambda \left(\tanh(\lambda) + \lambda \mathrm{sech}^2(\lambda)\right),$$

$$\kappa_3 \xrightarrow[N\to\infty]{Np=\lambda} \lambda \left(\tanh(\lambda) + \lambda(3 - 2\lambda \tanh(\lambda))\mathrm{sech}^2(\lambda)\right).$$

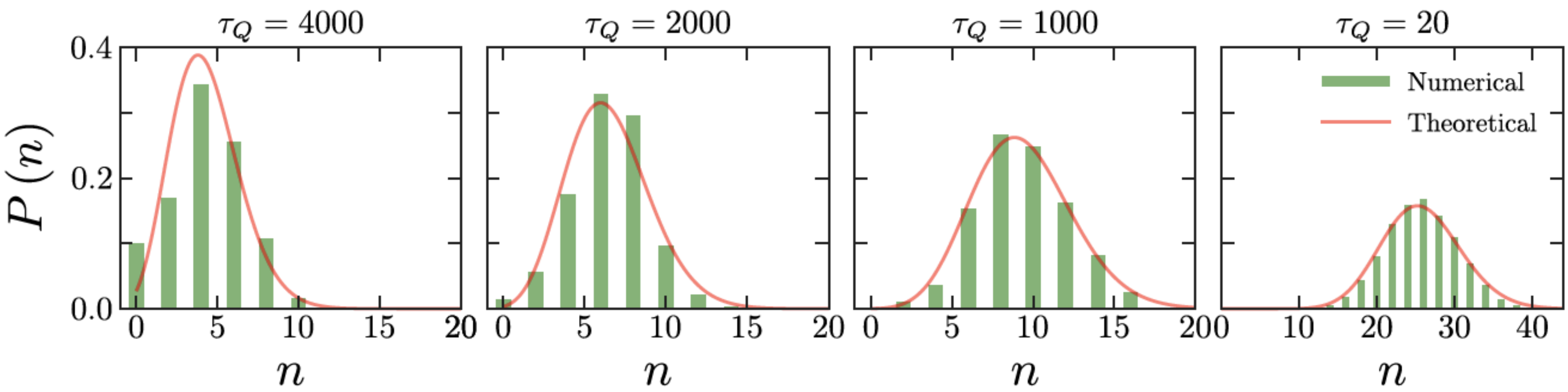
Plots of κ_q for even-Poisson



$\lambda \geq 5$

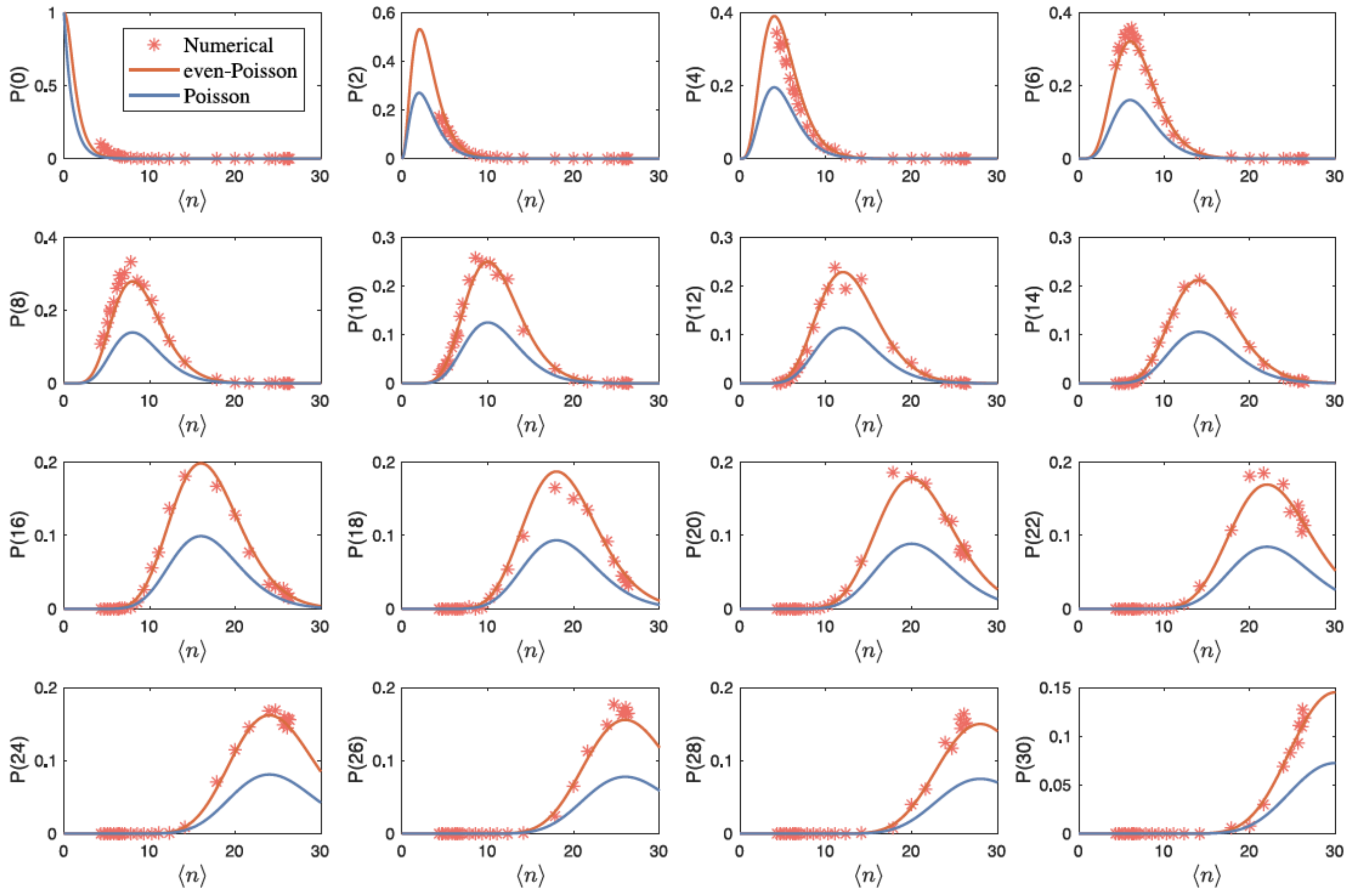
Cumulants are almost identical !

Numerical results (1): PDF

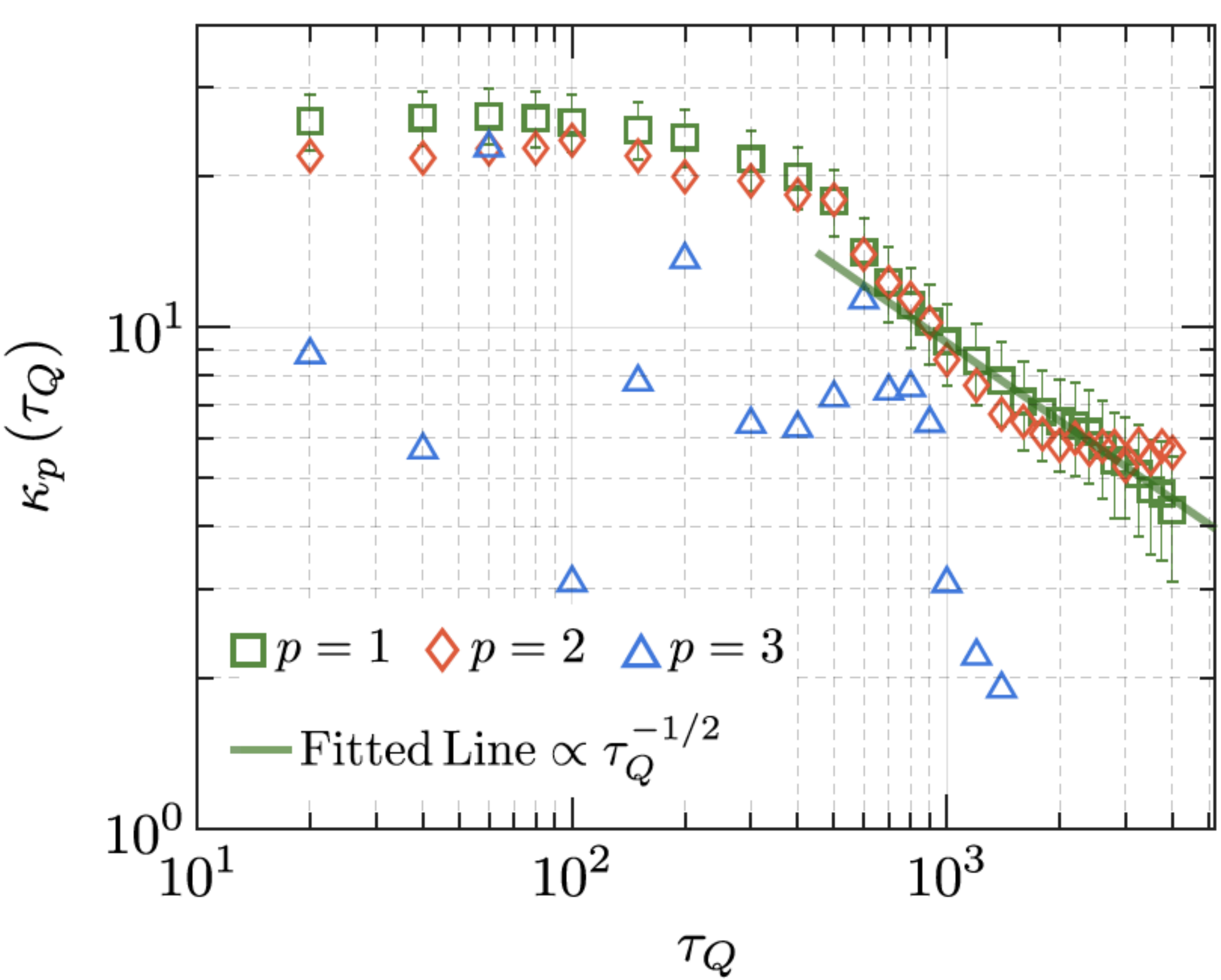


Red lines are the theoretical even-Poisson distribution

Numerical results (2): P(n=even)



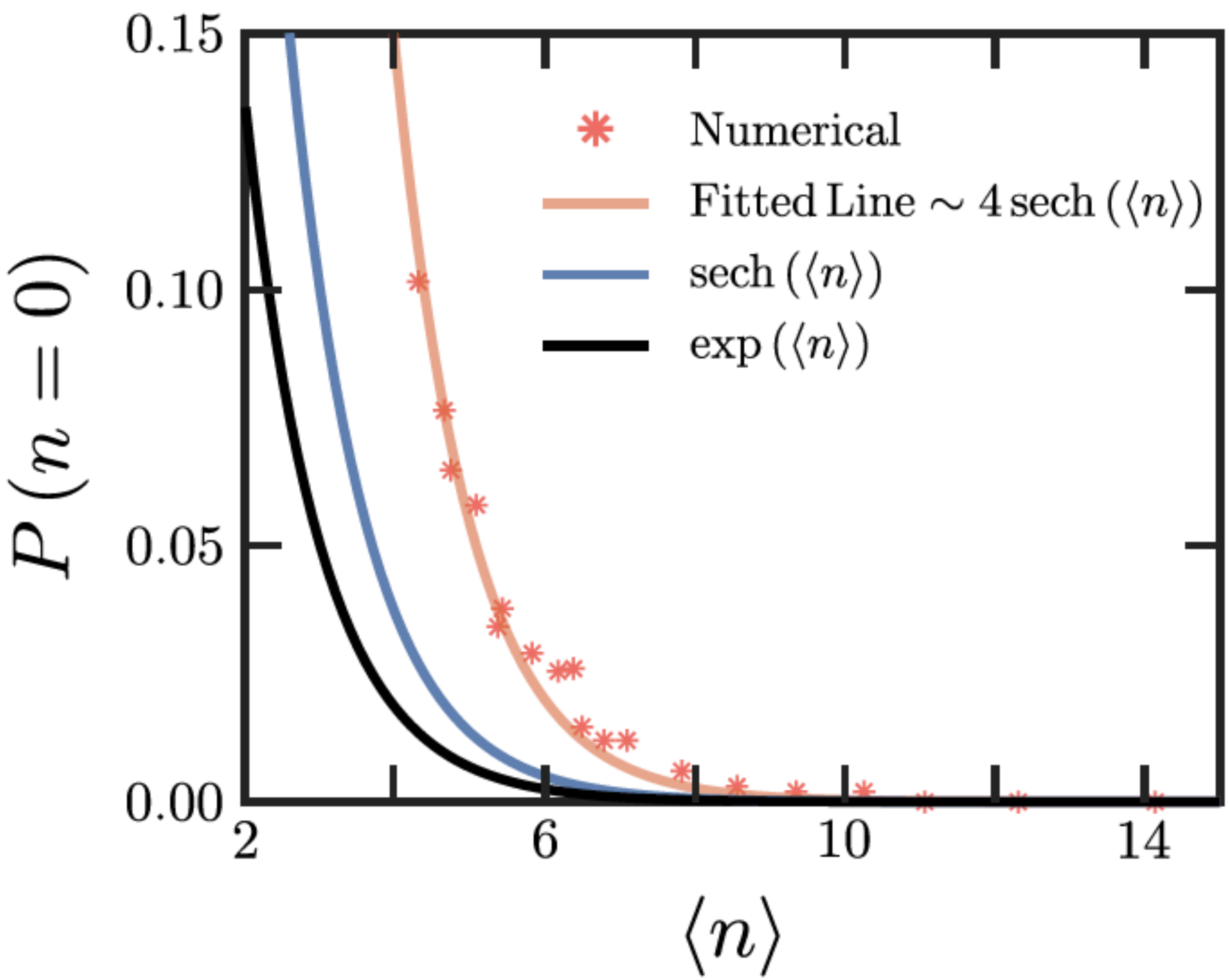
Numerical results (3): first three cumulants



Deviations of κ_3 implies that we need more simulations. Currently, for each τ_Q we have 1000 trajectories, due to the running time.

Numerical results (4): rare events

No vortices at all is a rare event away from adiabatic limit, $P_{EP}(n=0) = \mathrm{sech}(\langle n \rangle)$



The prefactor decreases as we increase the samplings. Therefore, we expect a better fitting of P_EP(n=0) if we have enough times of simulations.

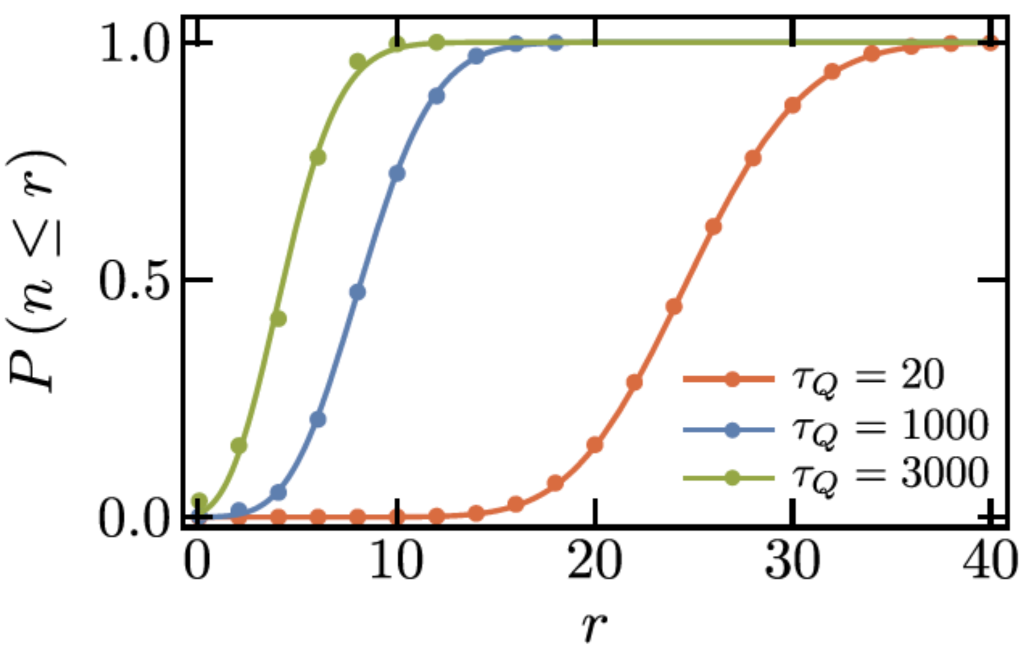
Numerical results (5): CDF

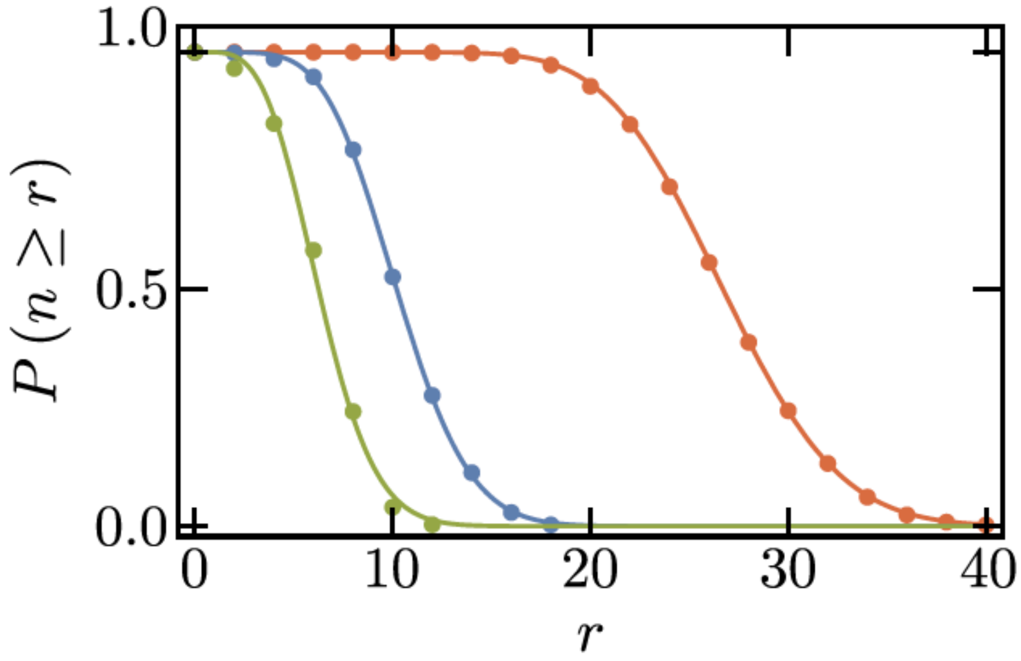
Cumulative distribution function (CDF)

$$P_{\mathrm{EP}}(n \leq r) = 1 - \frac{\operatorname{sech}(\lambda)\lambda^{2\lfloor \frac{r}{2} \rfloor + 2} \, {}_1F_2\left(1; \lfloor \frac{r}{2} \rfloor + \frac{3}{2}, \lfloor \frac{r}{2} \rfloor + 2; \frac{\lambda^2}{4}\right)}{\left(2\left(\lfloor \frac{r}{2} \rfloor + 1\right)\right)!},$$

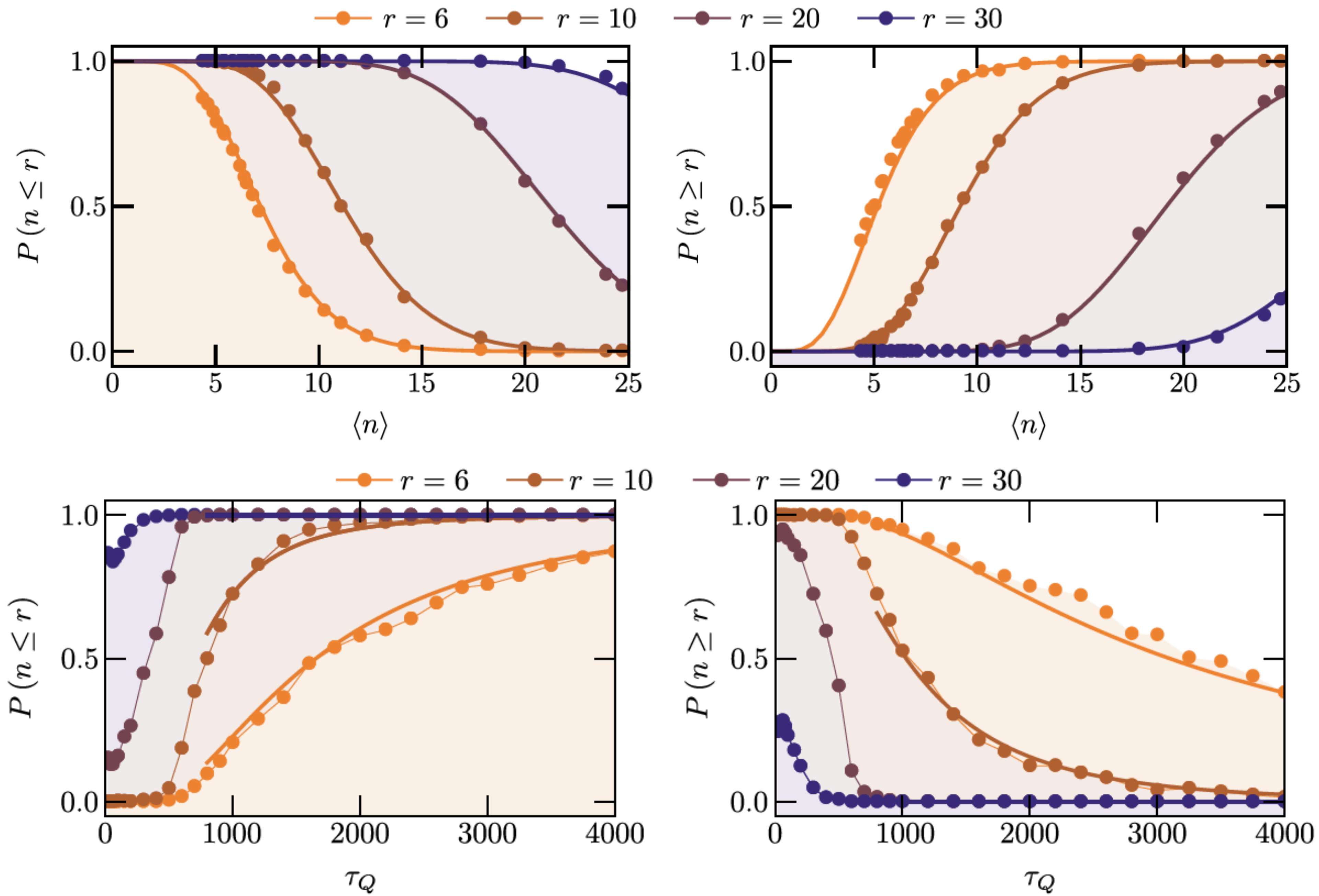
$$P_{\mathrm{EP}}(n \geq r) = \frac{\operatorname{sech}(\lambda)\lambda^{r} \, {}_1F_2\left(1; \frac{r}{2} + \frac{1}{2}, \frac{r}{2} + 1; \frac{\lambda^2}{4}\right)}{r!},$$

in which ${}_1F_2$ is a hypergeometric function; $\lfloor \, . \, \rfloor$ is the floor.





Numerical results (5):CDF



Extremal distribution: large deviation and the maxima in long sequences of realizations.

Fisher-Tippett-Gnedenko theorem: the extreme maximal values of the independently and identically distributed (iid) variables satisfy the generalized extreme value (GEV) distribution, (de Haan, Ferreira, `Extreme quantile and tail estimation, in Extreme Value Theory: An Introduction, Springer New York, U.S.A. (2006))

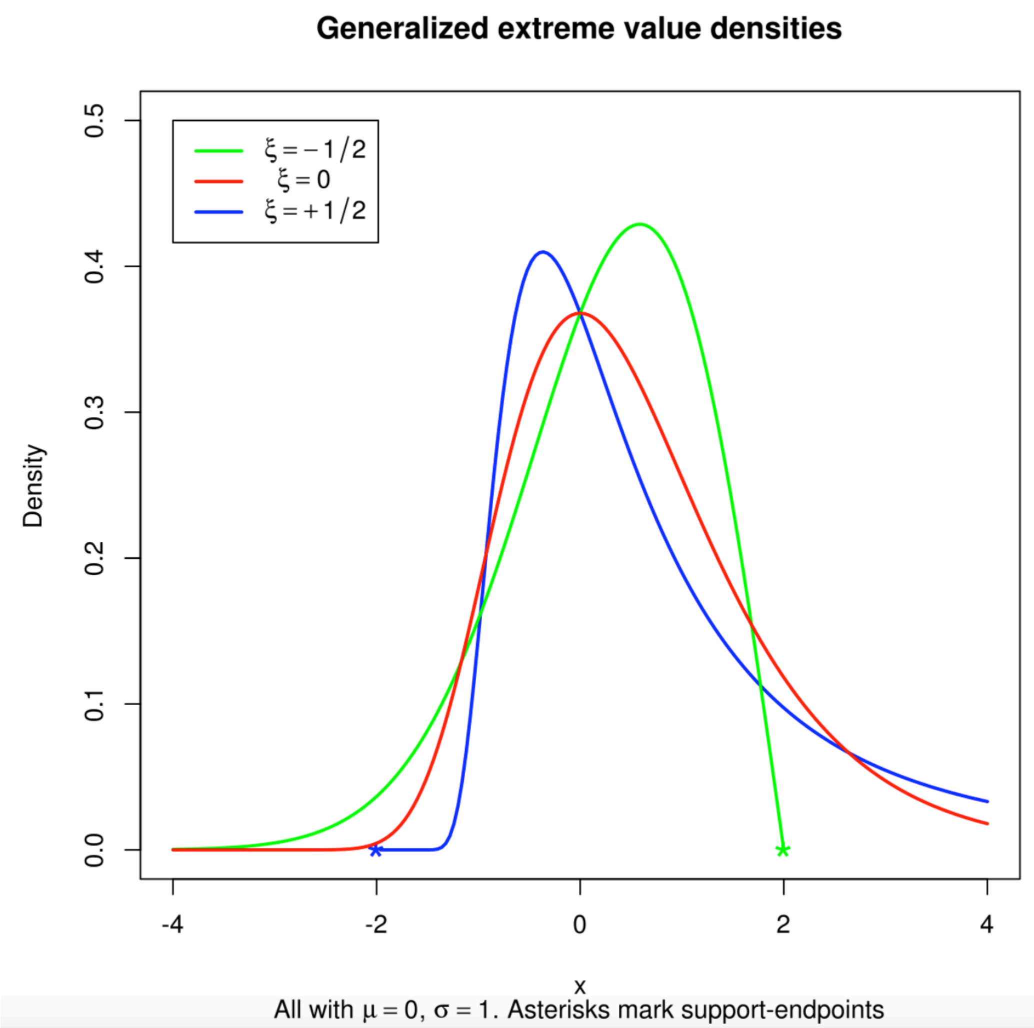
$$G(x;\mu,\sigma,\xi) = \begin{cases} \exp\left(-\left(1+\frac{x-\mu}{\sigma}\xi\right)^{-1/\xi}\right), & \xi \neq 0 \\ \exp\left(-\exp\left(-\frac{x-\mu}{\sigma}\right)\right), & \xi = 0. \end{cases}$$

μ is the location parameter, σ is the scale parameter and ξ is the shape parameter. Note: $\xi(x-\mu)/\sigma + 1 > 0$ and zero otherwise

GEV distribution function $G(x;\mu,\sigma,\xi)$ is the CDF, its PDF is

$$P(x;\mu,\sigma,\xi)=\begin{cases}\frac{1}{\sigma}\left(\frac{\xi(x-\mu)}{\sigma}+1\right)^{-\frac{1}{\xi}-1}\exp\left(-\left(\frac{\xi(x-\mu)}{\sigma}+1\right)^{-1/\xi}\right), & \xi\neq 0\\ \frac{1}{\sigma}\exp\left(-\frac{x-\mu}{\sigma}-\exp\left(-\frac{x-\mu}{\sigma}\right)\right), & \xi=0.\end{cases}$$

- $\xi<0$, Weibull distribution which is upper bounded;
- $\xi=0$, Gumbel distribution which has a light tail;
- $\xi>0$, Fréchet distribution which has a heavy tail and a lower bound.



In practice, to analyze the extreme value distributions for iid variables, it is customary to separate the data into several groups (or blocks), and then proceed to identify the maximum in each group.

The final list of maxima will tend to satisfy the above GEV distribution. This method is called ‘Block Maxima’ method.

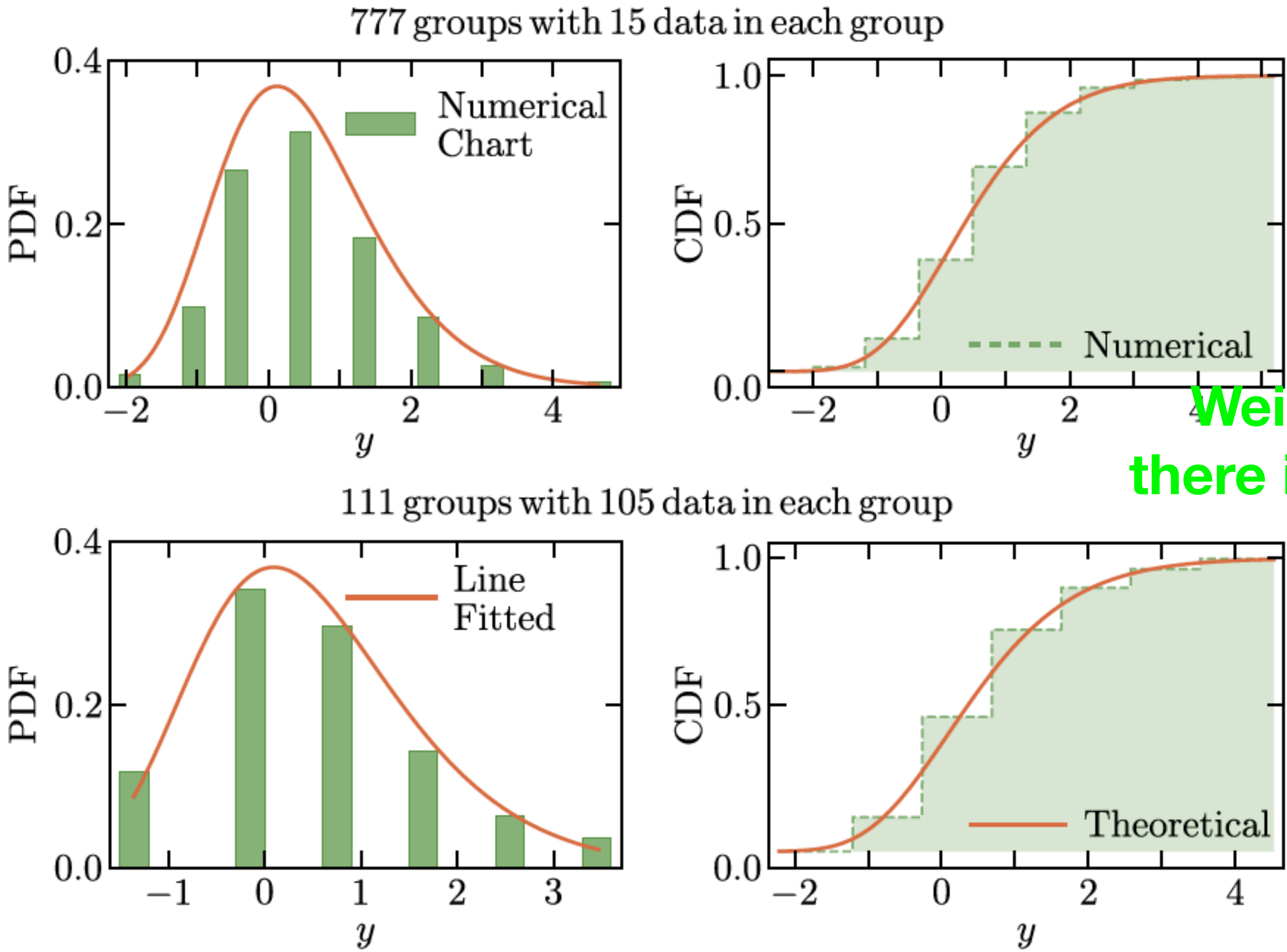
We adopt the ‘Block Maxima’ method to study the maximum value distributions for the vortex numbers in numerical simulations.

There are some arbitrary choices in the partition of the data. We partition the data into more than 100 groups, which is sufficient for the observed vortex-number maxima distribution to be identified with the GEV.

Numerical results (6): fast quench

Fast quench $\tau_Q = 20$

We have 11655 numerical data of the vortex number. Both PDF and CDF are shown as a function of the variable $y = (x - \mu)/\sigma$.



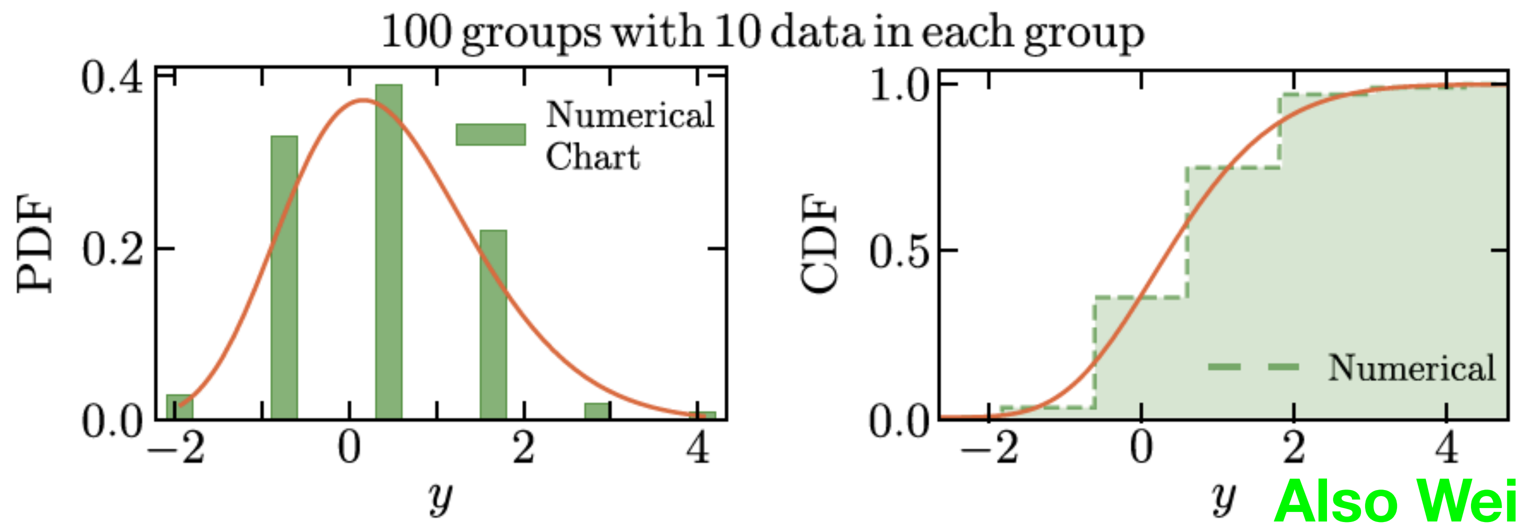
$\mu = 32.841$
$\sigma = 2.382$
$\xi = -0.113$

Weibull distribution, there is an upper bound.

$\mu = 36.558$
$\sigma = 2.114$
$\xi = -0.09$

Numerical results (6): slow quench

Slow quench $\tau_Q = 1000$

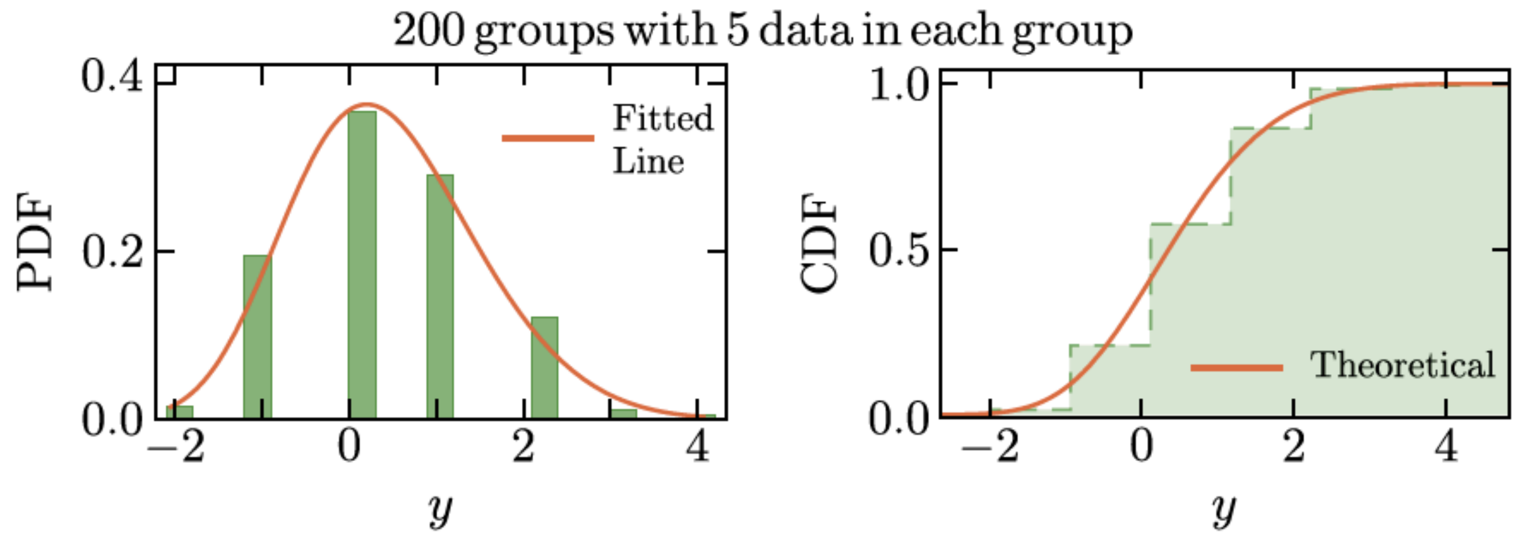


$\mu = 13.056$

$\sigma = 1.666$

$\xi = -0.149$

Also Weibull distribution



$\mu = 11.892$

$\sigma = 1.943$

$\xi = -0.184$

Chernoff bound: exponentially decreasing bounds on the tail distributions of vortex numbers. (Molloy, Reed, 'The Chernoff bound, in Graph Colouring and the Probabilistic Method', Springer Berlin Heidelberg, Germany (2002))

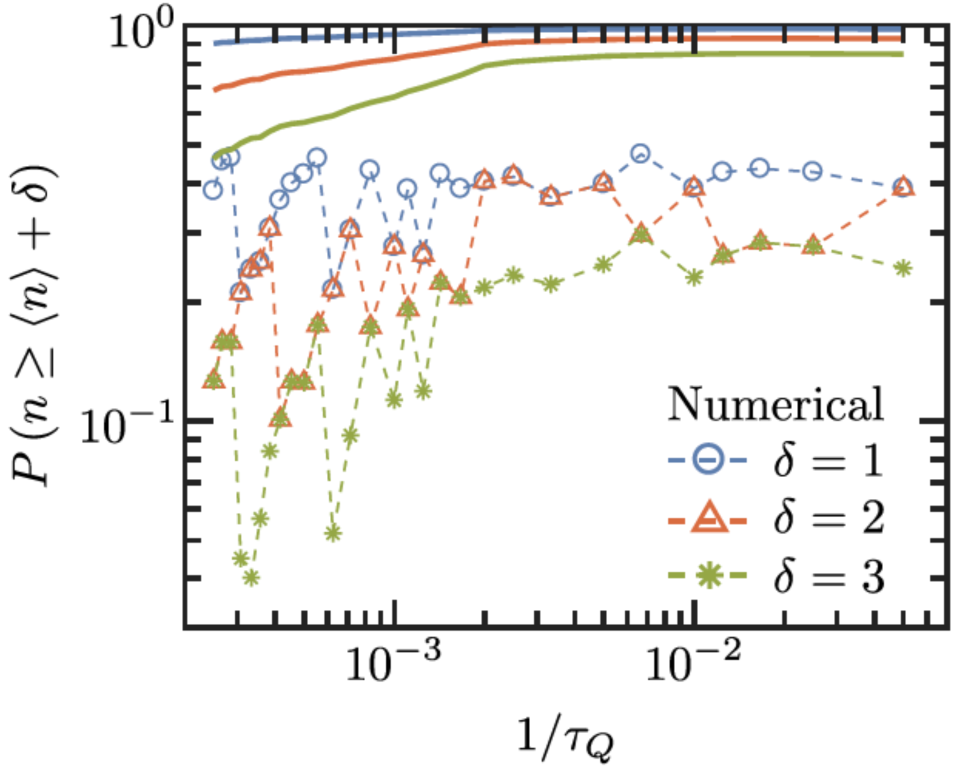
In its looser form, Chernoff bound can be written as

$$P(n \leq \langle n \rangle - \delta) \leq e^{-\frac{\delta^2}{2\langle n \rangle}}, \qquad \text{(Lower tail)}$$

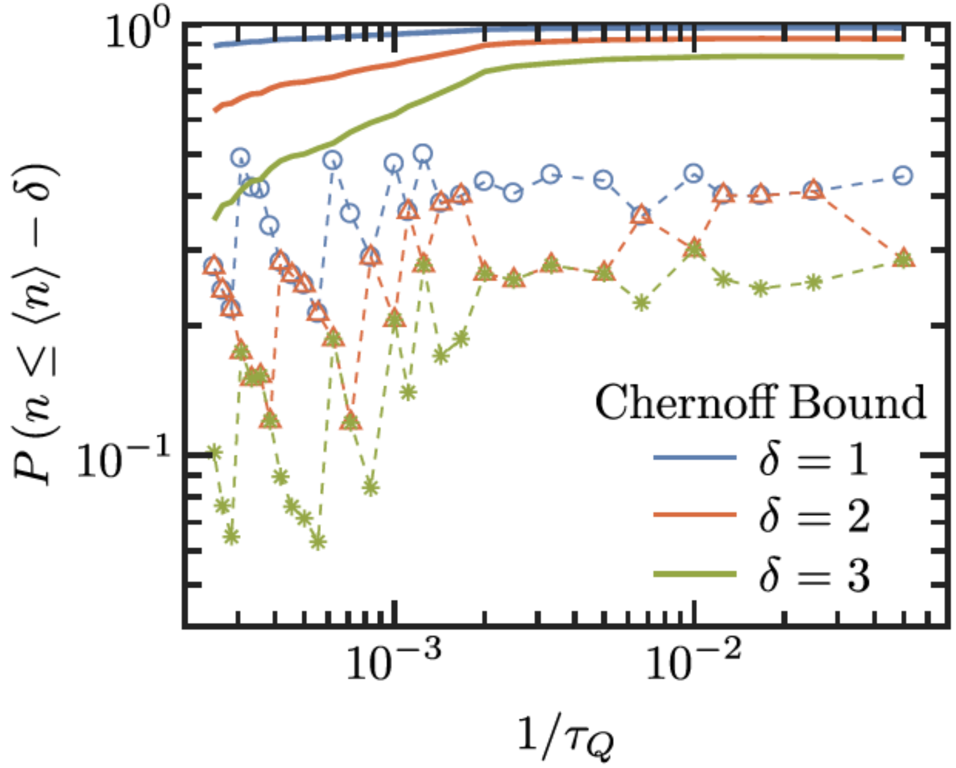
$$P(n \geq \langle n \rangle + \delta) \leq e^{-\frac{\delta^2}{2\langle n \rangle + \delta}}. \qquad \text{(Upper tail)}$$

In which, δ can be any positive real number.

Numerical results (7): Chernoff bound



Upper tail



Lower tail

Summaries

- **I have talked about various aspects of statistics of vortices from holographic realization;**
- **Mean and large fluctuations distributions;**
- **Mean number is Even-Poisson distributions (PDF); Its corresponding CDF is also verified numerically;**
- **Maximal values distribution is Weibull, has an upper bound;**
- **The tail distributions has a bound satisfying Chernoff bound.**

Thanks!

Contact me at: hqzhang@buaa.edu.cn

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