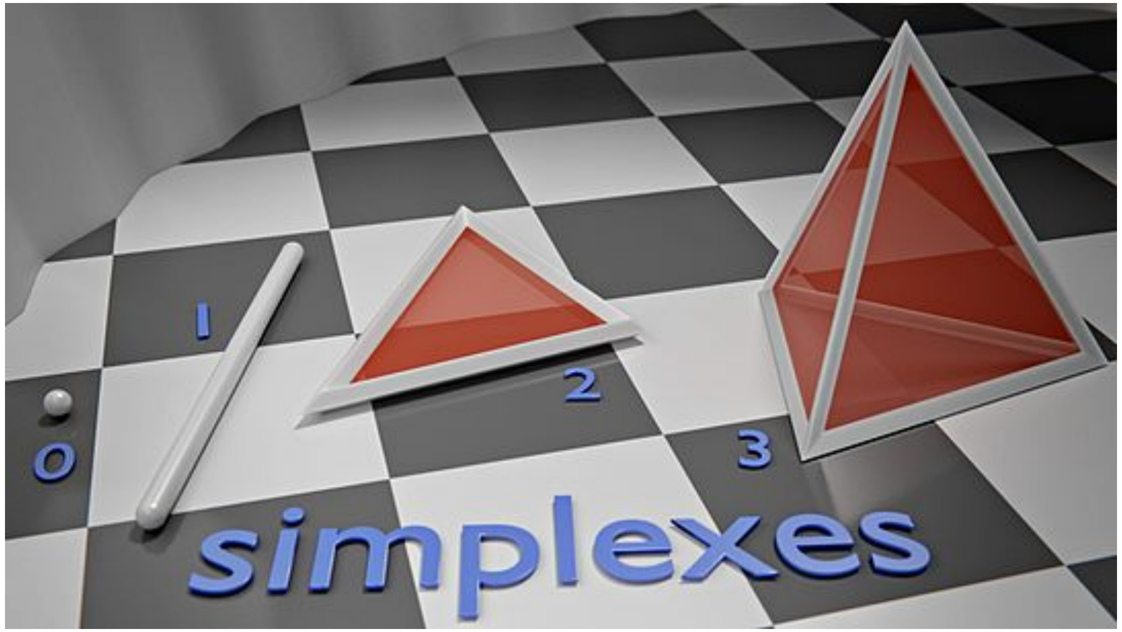
Quantum Finite Elements
Lattice Field Theory on Curved Manifolds



Dublin Institute for Advance Studies, Nov 24, 2022

Richard Brower, Boston University

Outline

- Introduction — Great Expectations
- Finite Elements Method(FEM) Free CFT
- Quantum Finite Elements — UV divergence for phi^4
- Ising Model on Affine Plane — Strong Lattice IR Ren Group
- Future — back to Great Expectations.

Lattice QCD IS BIG SCIENCE: 1/5 EXASCALE Oak Ridge 200,000,000,000,000,000 Floats/sec
9,216 IBM POWER9 CPUs and 27648 NVIDIA GPUs Each GPU has 5120 Cores

Thanks to my collaborators and co-authors

- George T. Fleming, Yale University/FNAL
- Anna-Marie Gluck, Yale/Heidelberg University

- Venkitesh Ayyar, Boston University
- Evan Owen, Boston University
- Cameron Cogburn, Boston University

- Timothy G. Raben, Michigan State University

- Chung-I Tan, Brown University

Making Steady Progress Going Backward!

- 2013: Lattice Radial Quantized: 3D Ising (R x S2)
- 2017: Lattice Dirac on S2 Simplicial Riemann Manifold (S2:Free CFT)
- 2018: phi^4 test of 2-d Ising CFT on S2 (S2)
- 2019: Lattice Setup for Quantum Field Theory in AdS2
- 2021: Radial Lattice Quantization of 3D phi^4 Field Theory (R x S2)
- 2022: Lattice AdS3 for Scalar Field Theory (w. C. Cogburn, E. Owen)
- 2022: Ising Model on the Affine Plane (w. E. Owen) (2D Torus!)

See References in Back up Slides

Ambitious goal*

$$\{\mathbb{R}^d, \delta_{\mu\nu}\} \quad \Longrightarrow \quad \{\mathcal{M}, g_{\mu\nu}\}$$

- Prove that Lattice can define any UV complete QFT on any Smooth Manifolds

- On maximally symmetric manifolds computational parallel efficiency should be comparable to lattice QCD adoption the state of art parallel algorithm on Exascale Platforms.

*Note all flat space Renormalizable QFT are generally believed to be perturbatively Renormalizable on Smooth Riemann Manifold: see M. Luscher, H. Osborn in Literature in 1990's et al.

SPHERES AND CYLINDERS ARE NICE*

* MAXIMALLY SYMMETRIC SPACES

- Conformal Field Theories are more easily studied on Sphere, Cylinders (Radial Quantization) and Hyperbolic Spaces (Gauge/Gravity Duality)

$\mathbb{S}^d$ $\qquad \mathbb{R} \times \mathbb{S}^{d-1}$ $\qquad \mathbb{A}d\mathbb{S}^{d+1}$

$\mathbb{R}^d \to \mathbb{S}^d$

$$ds^2_{flat} = \sum_{\mu=1}^{d} dx^\mu dx^\mu = e^{2\sigma(x)} d\Omega_d^2 \xrightarrow{Weyl} d\Omega_d^2 \,.$$

$\mathbb{R}^d \to \mathbb{R} \times \mathbb{S}^{d-1}$

$$ds^2_{flat} = \sum_{\mu=1}^{d} dx^\mu dx^\mu = e^{2\tau}(d\tau^2 + d\Omega_d^2) \xrightarrow{Weyl} (d\tau^2 + d\Omega_d^2) \,.$$

$\mathbb{R}^{d+1} \to \mathbb{A}d\mathbb{S}^{d+1}$

$$ds^2_{flat} = \sum_{\mu=1}^{d+1} dx^\mu dx^\mu \xrightarrow{Weyl} z^{-2}(dz^2 + d\vec{x} \cdot d\vec{x})$$

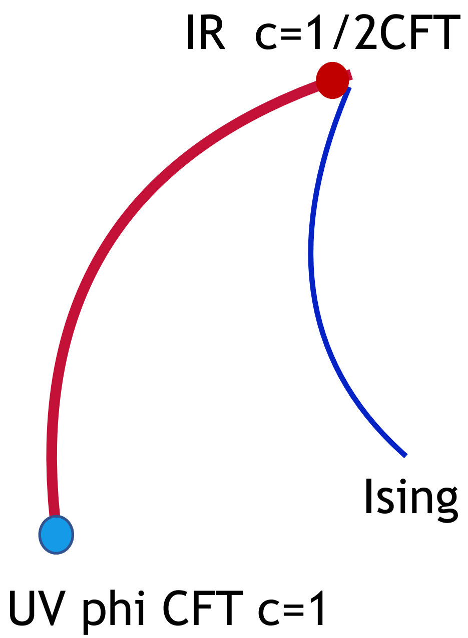
Ok, BE REALISTIC TO GET GOING!

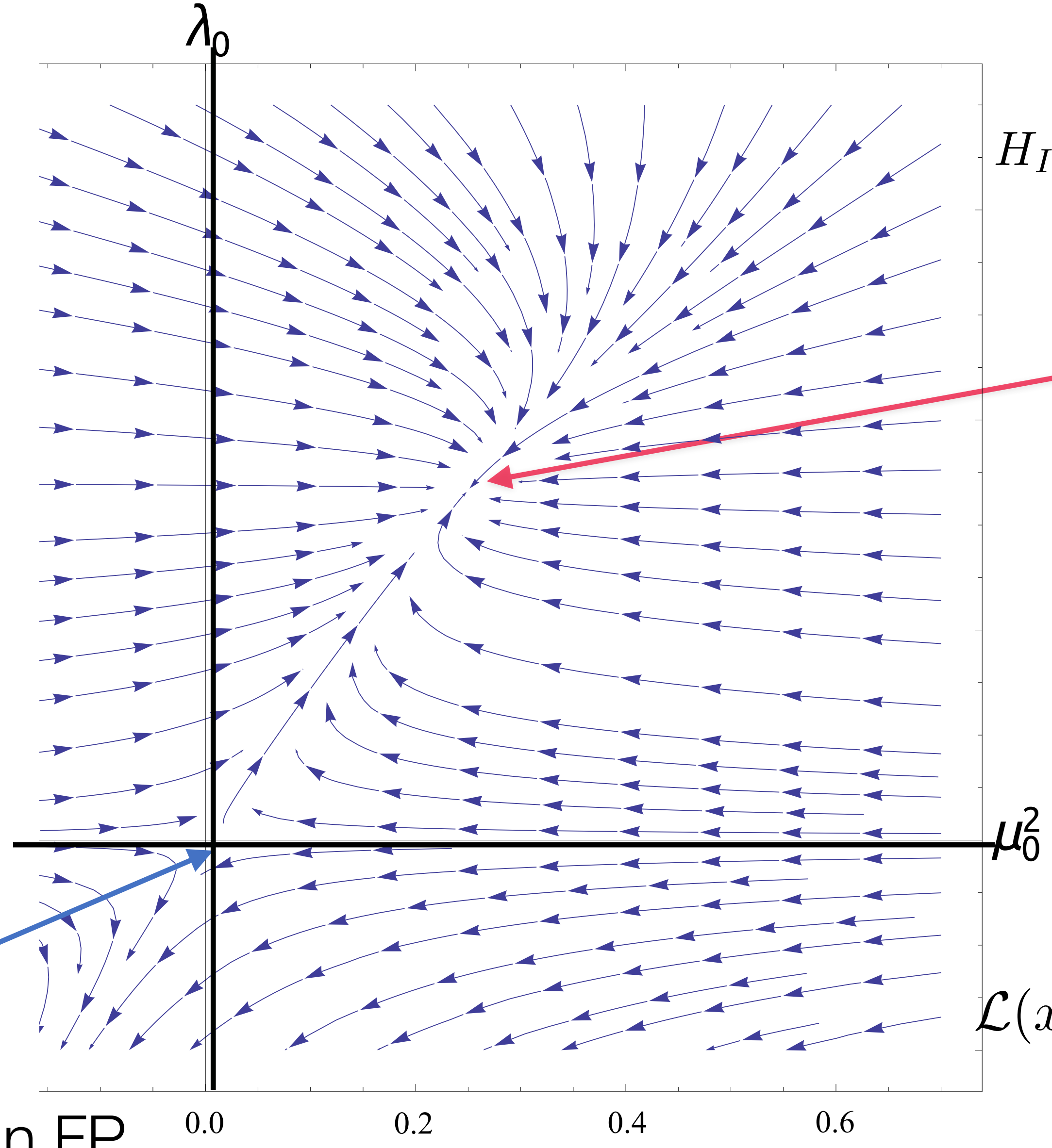
The art of doing mathematics consists finding that special case which contains all the germs of generality.

David Hilbert
Mathematician, Physicist, Philosopher

Author of Geometry and the Imagination

Scalar Phi4/Ising Model





$$H_{Ising} = -\sum_{\langle i, \rangle j} K s_i s_j = \frac{K}{2} \sum_{\langle i, \rangle j} (s_i - s_j)^2$$

IR: Wilson-Fisher FP

UV: Gaussian FP

$$\mathcal{L}(x) = -\frac{1}{2}(\nabla\phi)^2 + \lambda(\phi^2 - \mu^2/2\lambda)^2$$

First step: Construct the Classical Simplicial Action

$$S = \frac{1}{2} \int_{\mathcal{M}} d^d x \sqrt{g(x)} \left[g^{\mu\nu}(x) \partial_\mu \phi(x) \partial_\nu \phi(x) + m^2 \phi^2(x) + \lambda \phi^4(x) \right]$$

$g_{\mu\nu}(x)$

Regge Calc Geometry

Quantum field $\phi(x)$

Finite Element Method

Classical Simplicial Action

$$S_{FEM} = \frac{1}{2} \Big[\sum_{y \in \langle x, y \rangle} K_{xy} (\phi_x - \phi_y)^2 + \sqrt{g_x} [\xi \, Ric \, \phi_x^2 + m_0^2 \phi_x^2 + \lambda_0 \phi_{t,x}^4] \Big]$$

1985: Cardy's Radial Quantization Challenge

"It would therefore be very useful to generalize this result (in 2D) to dimensionality D > 2" "Unfortunately the result appears to be difficult to utilize for numerical work"

Last Sentence in 3 page article says

"Whether this will provide a useful numerical approach to critical exponents remains to be seen"

YES INDEED

Antipodal (CM) 4-pt function or $\mathbb{R} \times \mathbb{S}^2 \ni (t, \vec{x})$

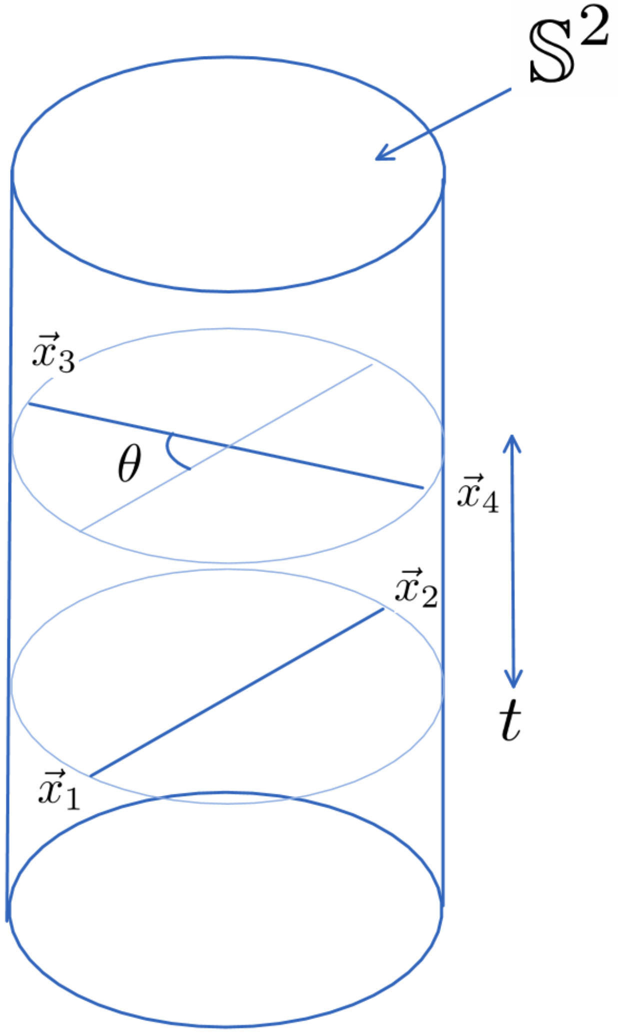
Conformal Block Expansion

$$\frac{\langle\sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\rangle}{\langle\sigma(x_1)\sigma(x_2)\rangle\langle\sigma(x_3)\sigma(x_4)\rangle} = 1 + \sum_{\mathcal{O}} f^2_{\sigma\sigma\mathcal{O}} G_{\mathcal{O}}(\Delta_{\mathcal{O}}; x_1, x_2, x_3, x_4)$$

Partial Wave Expansion

$$G_{\mathcal{O},l} = \sum_{n=0,2,4,\ldots} \sum_j e^{-(\Delta_{\mathcal{O}}+n)t} B_{n,j}(\Delta_{\mathcal{O}}) P_j(\cos(\theta))$$

$$\langle\sigma(x_1)\sigma(x_2)\rangle \frac{1}{|x_1 - x_2|^{2\Delta_\sigma}} \to \frac{1}{[\cosh(t) - \cos(\theta)]^{2\Delta_\sigma}}$$

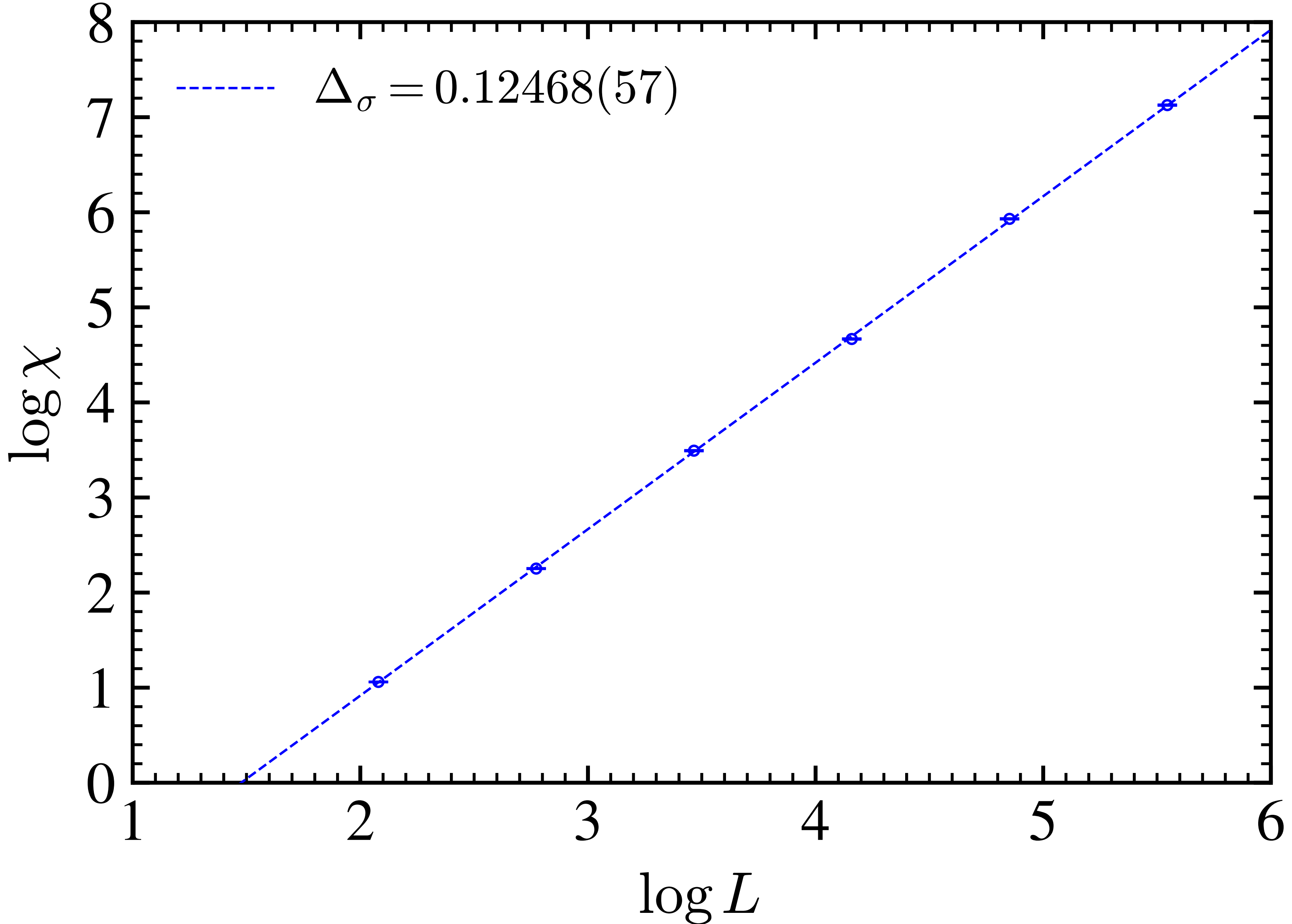


Free CFT

$$G^{(free)}_{CFT}(t,\theta) = 1 + \frac{1}{[\cosh(t) - \cos(\theta)]^{4\Delta}} + \frac{1}{[\cosh(t) + \cos(\theta)]^{4\Delta}}$$

s-channel   t-channel   u-channel

2D Radial Ising on $\mathbb{R} \times \mathbb{S}^1$ Trivial but very useful



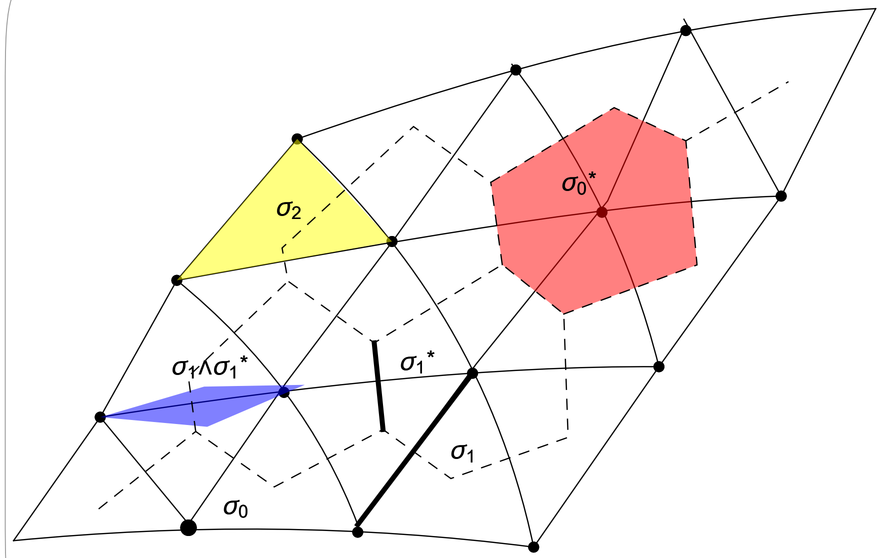
Part I:

FINITE ELEMENT for Free CFT

Simplicial Complex

REGGE: Piecewise linear metric

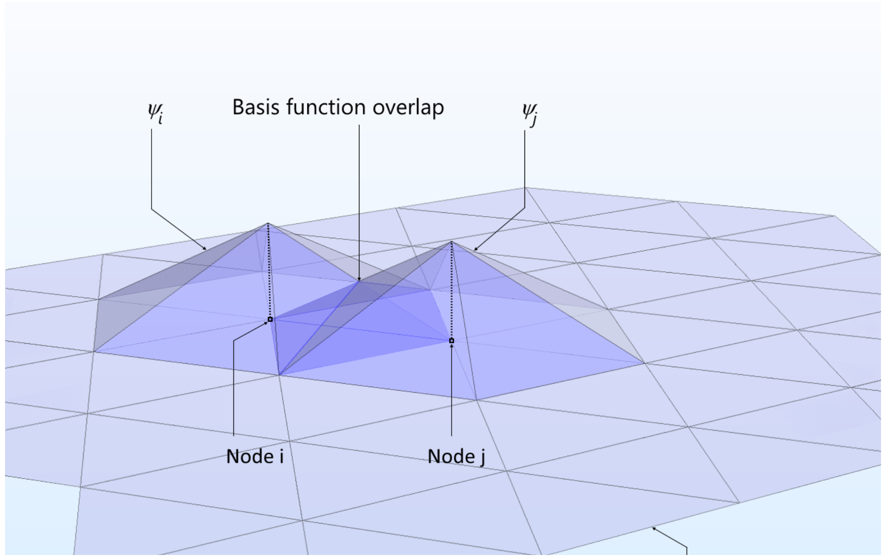
$$(\mathcal{M}, g_{\mu\nu}(x)) \leftrightarrow (\mathcal{M}_\sigma, g_\sigma = \{l_{ij}\})$$



Simplicial Complex/Delaunay Dual Complex + Regge flat metric on each Simplex

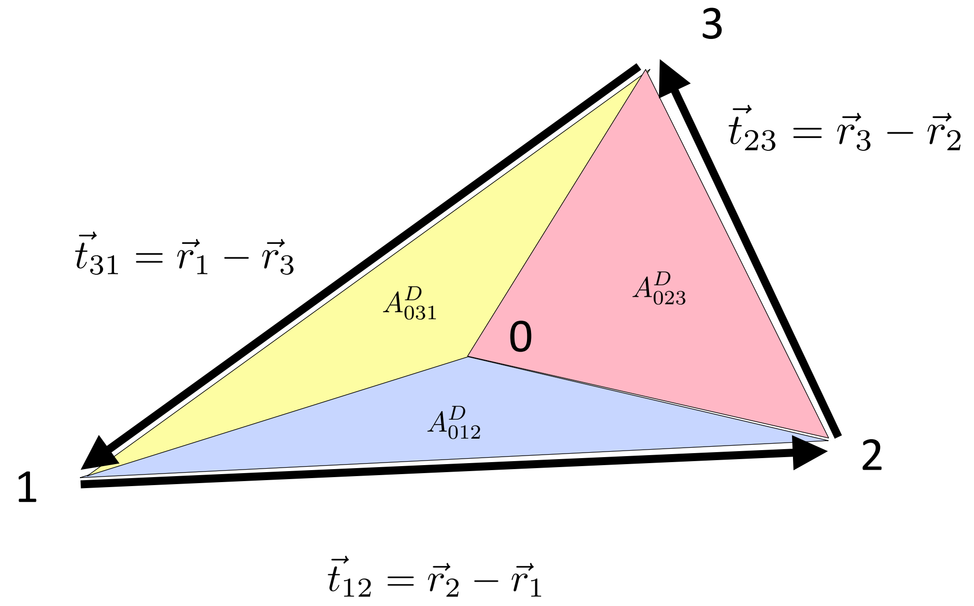
FEM: Piecewise linear fields

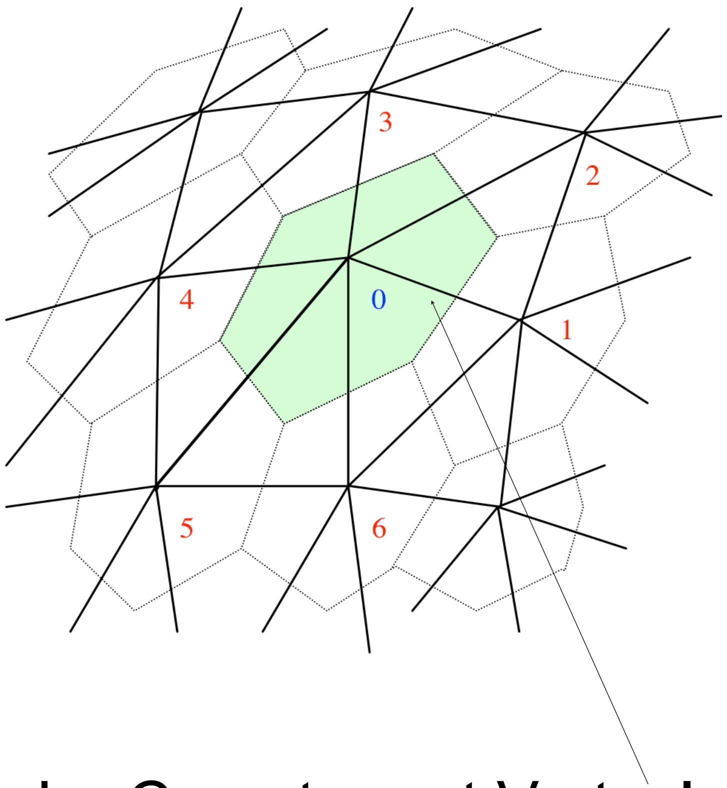
$$\phi(x) \leftrightarrow \phi = \sum_i \phi_i W_i(\xi)$$



Actually fancier methods: **Discrete Exterior Calculus** (scalar), Spin connection (Fermion), Wilson links (gauge) , etc.

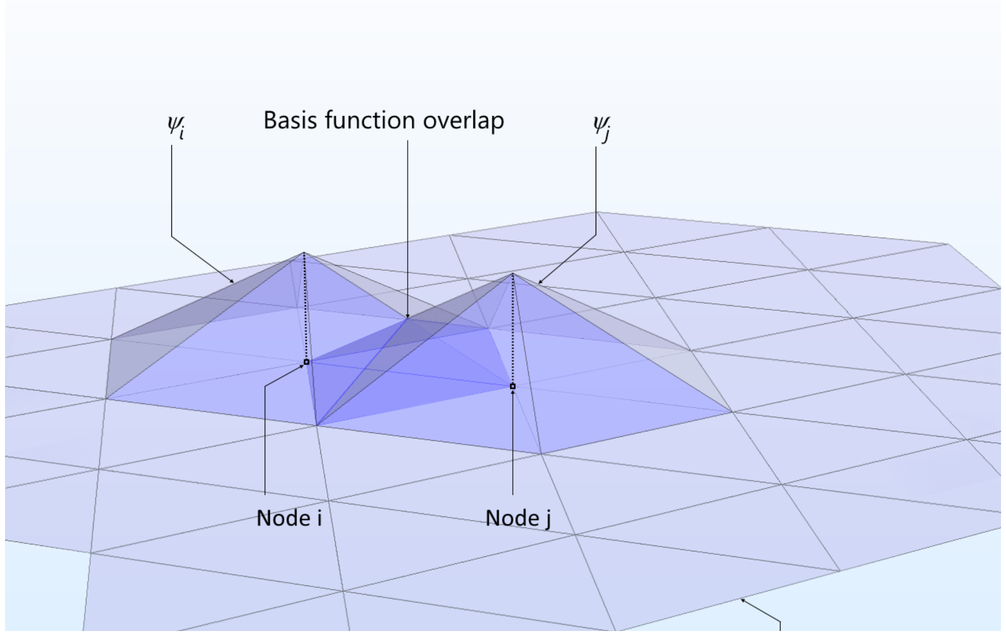
FEM geometry on edges.



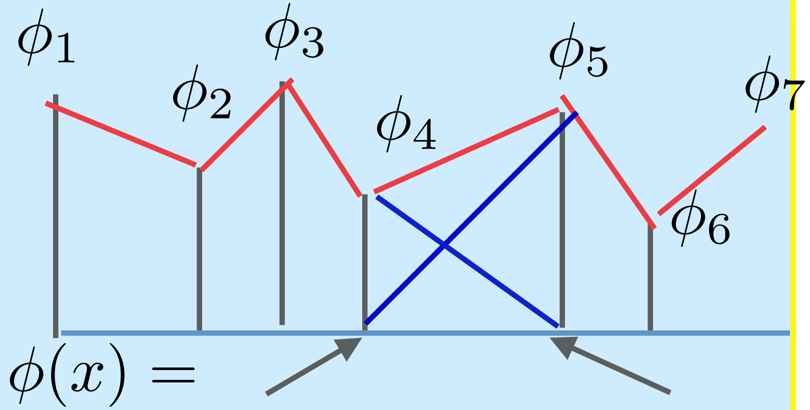


Singular Curvature at Vertex!

The l's fix metric and the local co-ordinates (diffeomorphism) and the angles the intrinsic curvature.



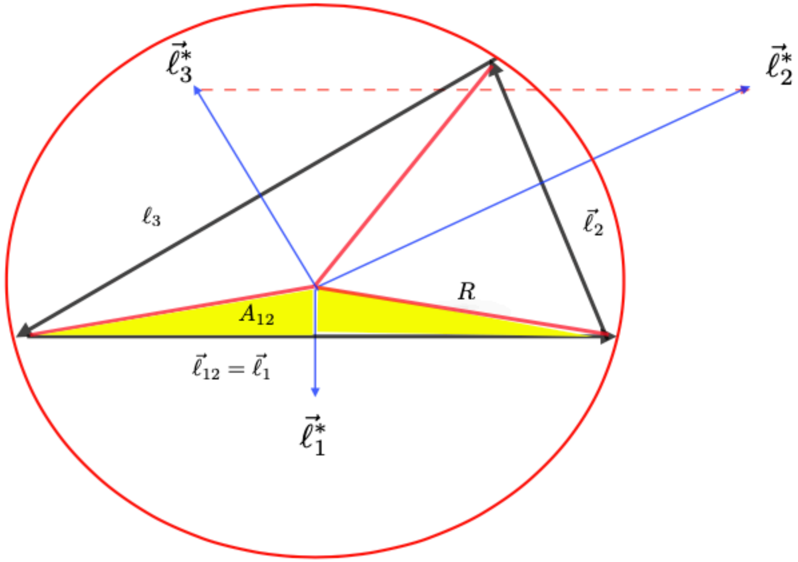
$$\phi(x) \leftrightarrow \phi = \sum_i \phi_i W_i(\xi)$$

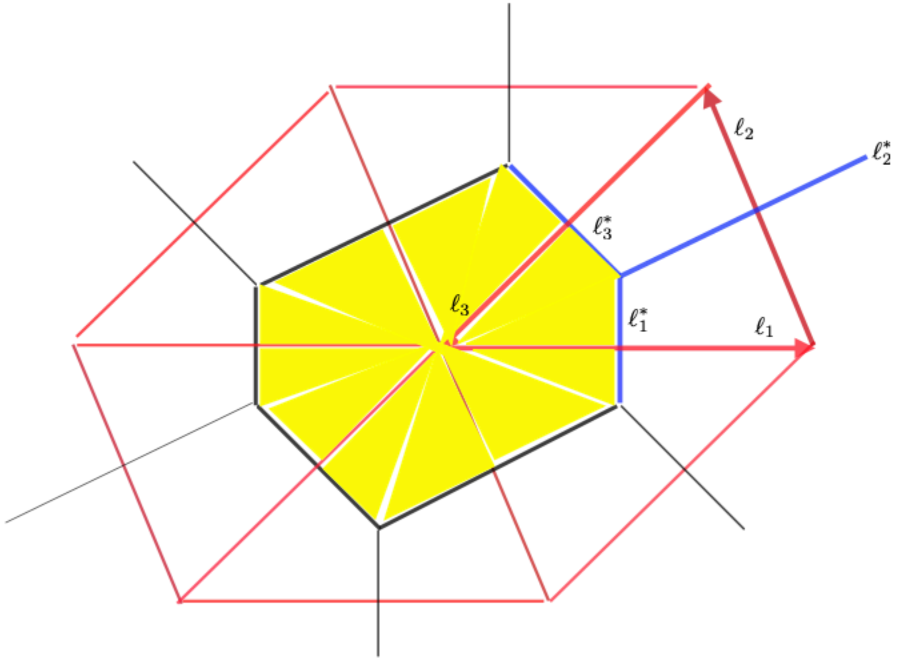


$$x = \xi_1 x_4 + \xi_2 x_5 \quad \text{with} \quad \xi_1 + \xi_2 = 1$$

$$\phi(x) = \xi_1 \phi_4 + \xi_2 \phi_5 \quad x \in [x_5, x_4]$$

1 D Linear FEM is essentially the “trapezoidal” rule





$$S_\triangle = \frac{\ell_{23}^2 + \ell_{31}^2 - \ell_{12}^2}{8A_\triangle}(\phi_1 - \phi_2)^2 + (23) + (31) = \frac{\ell_{12}^*}{4\ell_{12}}(\phi_1 - \phi_2)^2 + (23) + (31)$$

PIECE WISE LINEAR FEM

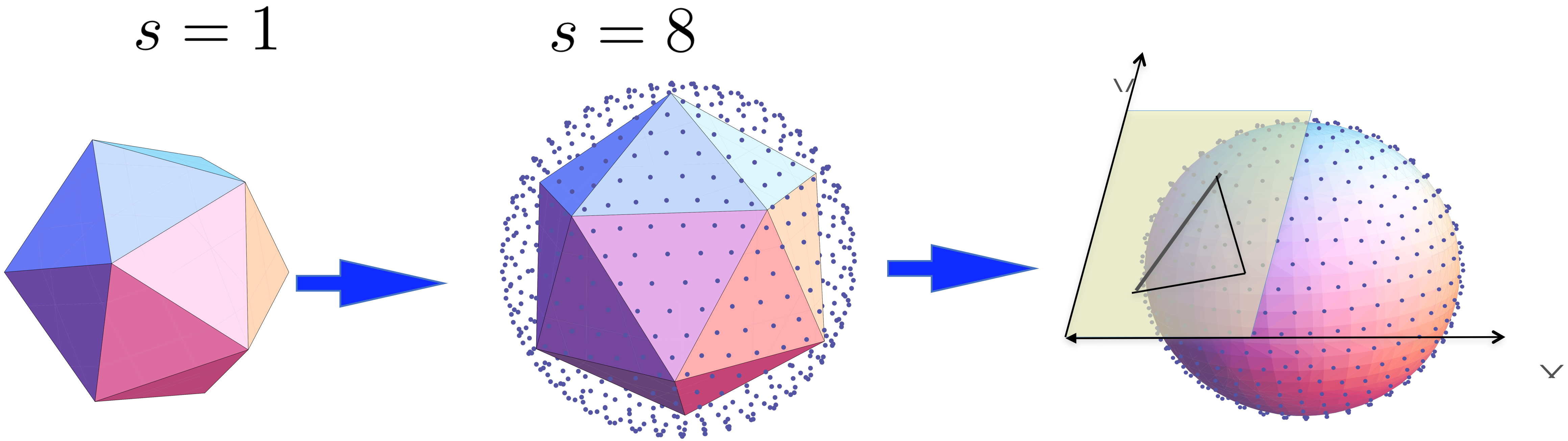
(Negative sign is Not problem in spectrum)

Discrete Exterior Calculus (DE)

$$\langle\, \sigma_n | d\omega \rangle = \langle\, \partial\sigma_n | \omega \rangle$$

$$*d * d\phi_i = * \frac{1}{|\sigma_0^*(i)|} \int_{\sigma_0^*} d[*(\phi_i - \phi_j)/l_{ij}] = \frac{1}{\sqrt{g_i}} \sum_{j \in \langle\, i,j \rangle} \frac{V_{ij}}{l_{ij}} \frac{\phi_i - \phi_j}{l_{ij}}$$

Free scalar on Two Sphere



Start with maximum regular Tesselation: preserve Icosahedral group upon refinement

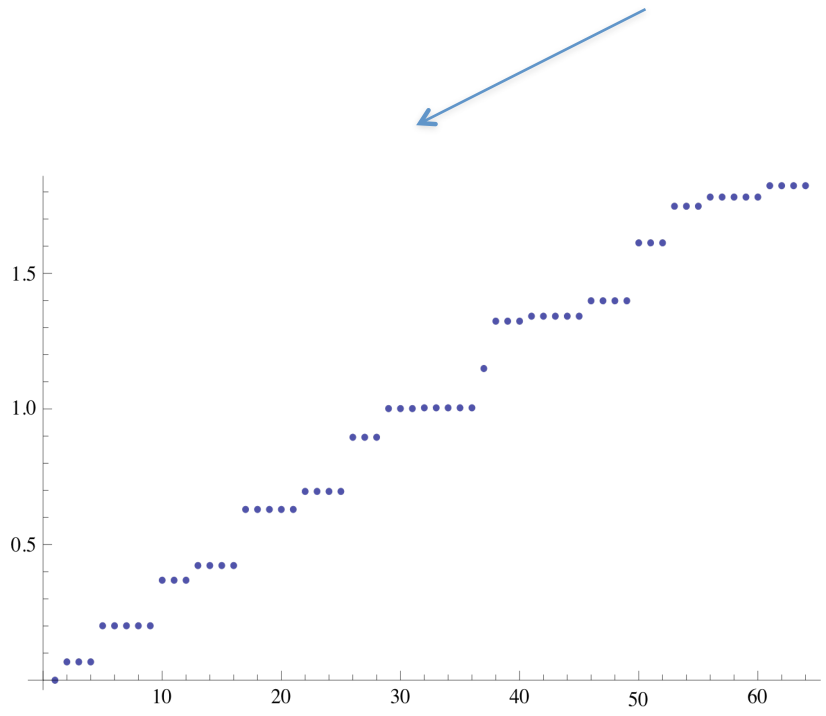
l = 0 (A),1 (T1) , 2 (H) are irreducible 120 Icosahedral subgroup of O(3)

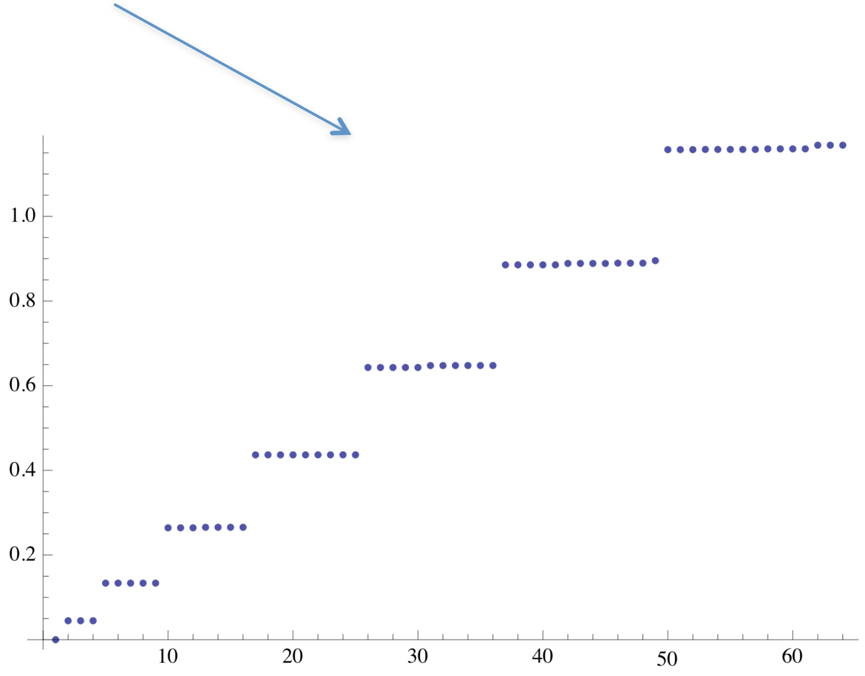
FEM FIXES SPECTRAL DEFECTS OF LAPLACIAN ON SPHERE

For s = 8 first (l+1)*(l +1) = 64 eigenvalues

BEFORE FEM

AFTER FEM





l, m

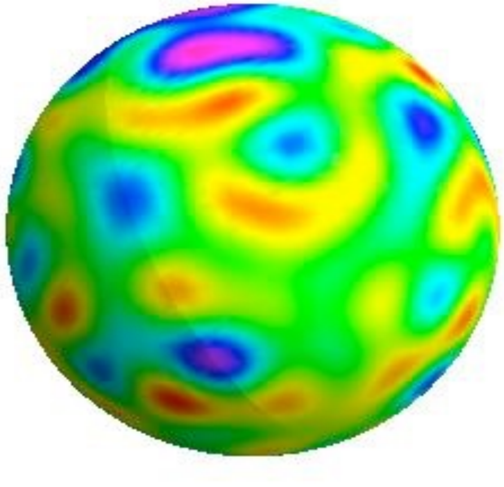
Spectral Splitting of 120 element Icoshedral Group

Part II:

QUANTUM FINITE ELEMENTS
with UV counter terms

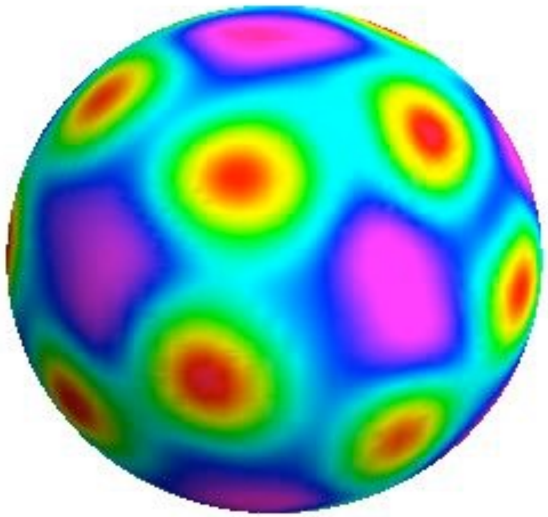
Now add $\lambda\phi^4$ term: What happens to FEM?

$\phi^2(x)$

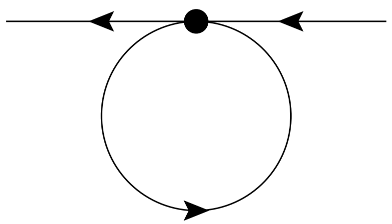


one configuration

$\langle \phi^2(x) \rangle$



Average of config.



$$\delta m^2 = \lambda \langle \phi(x)\phi(x) \rangle \to \frac{1}{K_{xx}}$$

Perturbative CT on the Sphere

$$\Delta m_i^2 = 6\lambda \left[K^{-1}\right]_{ii} \simeq \frac{\sqrt{3}}{8\pi}\lambda \log(1/m_0^2 a_i^2) = \frac{\sqrt{3}}{8\pi}\lambda \log(N_s) + \frac{\sqrt{3}}{8\pi}\lambda \log(a^2/a_i^2)$$

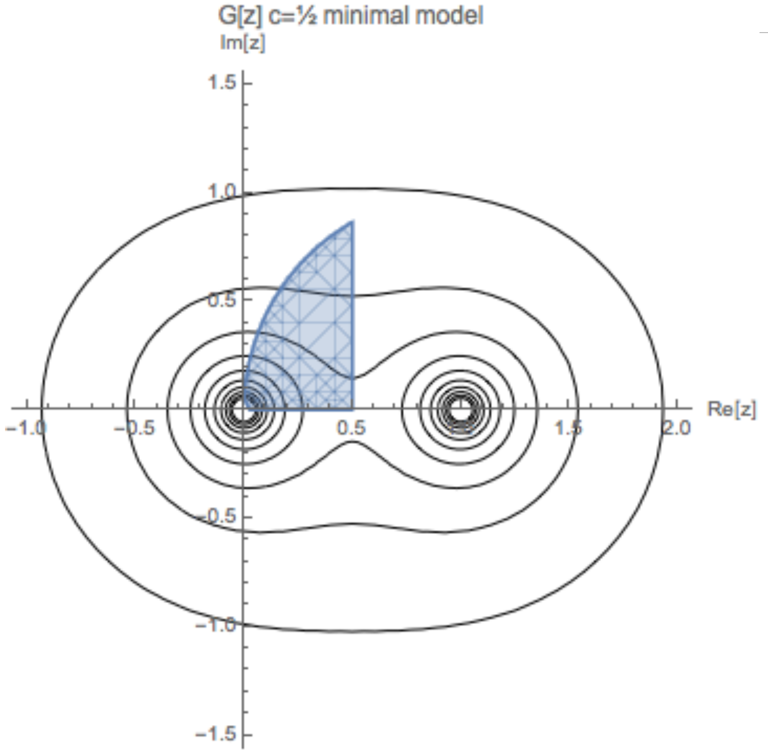
$$\delta\mu_i^2 = -6\lambda(\left[K^{-1}\right]_{ii} - \frac{1}{N_s}\sum_{j=1}^{N_s}\left[K^{-1}\right]_{jj})$$

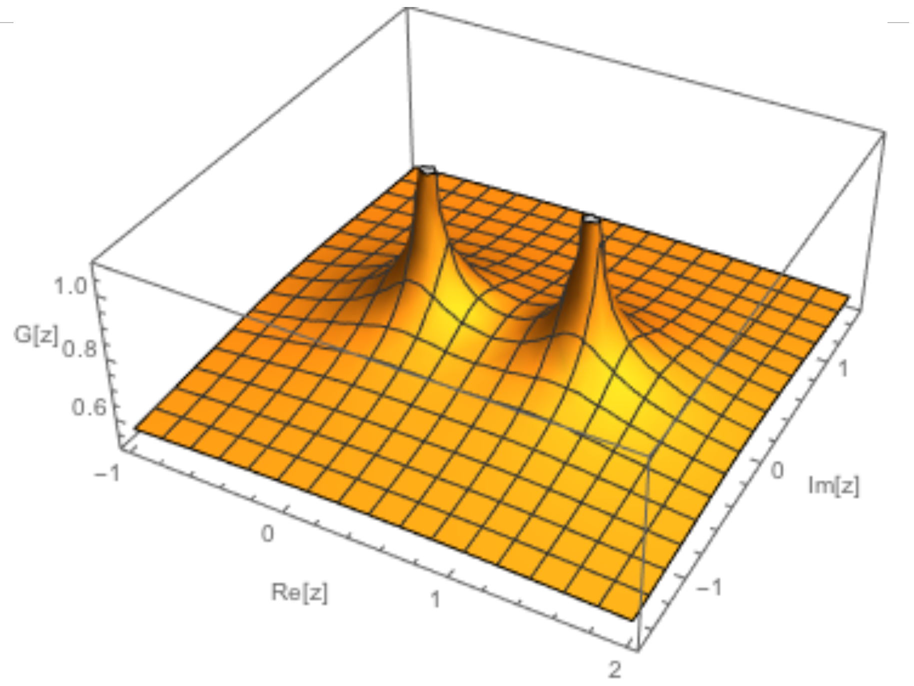
NUMERICAL TEST against Exact c=1/2 Ising CFT

μ^2	s	$r_{\min} \leq r \leq r_{\max}$	norm	Δ_ϵ	λ_ϵ^2	c
1.82241	9	$0.25 \leq r \leq 0.75$	0.2900	1.075	0.2536	0.4668
1.82241	9	$0.30 \leq r \leq 0.70$	0.2901	1.075	0.2533	0.4704
1.82241	9	$0.35 \leq r \leq 0.65$	0.2902	1.077	0.2533	0.4738
1.82241	9	$0.40 \leq r \leq 0.60$	0.2902	1.016	0.2427	0.4747
1.82241	18	$0.25 \leq r \leq 0.75$	0.2051	1.068	0.2563	0.4866
1.82241	18	$0.30 \leq r \leq 0.70$	0.2051	1.056	0.2544	0.4878
1.82241	18	$0.35 \leq r \leq 0.65$	0.2051	1.050	0.2535	0.4904
1.82241	18	$0.40 \leq r \leq 0.60$	0.2051	1.046	0.2526	0.4884
1.82241	36	$0.25 \leq r \leq 0.75$	0.1457	1.031	0.2528	0.4926
1.82241	36	$0.30 \leq r \leq 0.70$	0.1458	1.026	0.2519	0.4932
1.82241	36	$0.35 \leq r \leq 0.65$	0.1458	1.018	0.2508	0.4931
1.82241	36	$0.40 \leq r \leq 0.60$	0.1458	1.007	0.2486	0.4933

Lattice Sizes: N = 32+ 10 s^2 sites

OPE Expansion: $\phi \times \phi = \mathbf{1} + \phi^2$ or $\sigma \times \sigma = \mathbf{1} + \epsilon$





$$G_s(r,\theta) \propto 1 + \lambda_\epsilon^2\, g_{\epsilon,0}(r,\theta) + \lambda_T^2\, g_{T,2}(r,\theta)$$

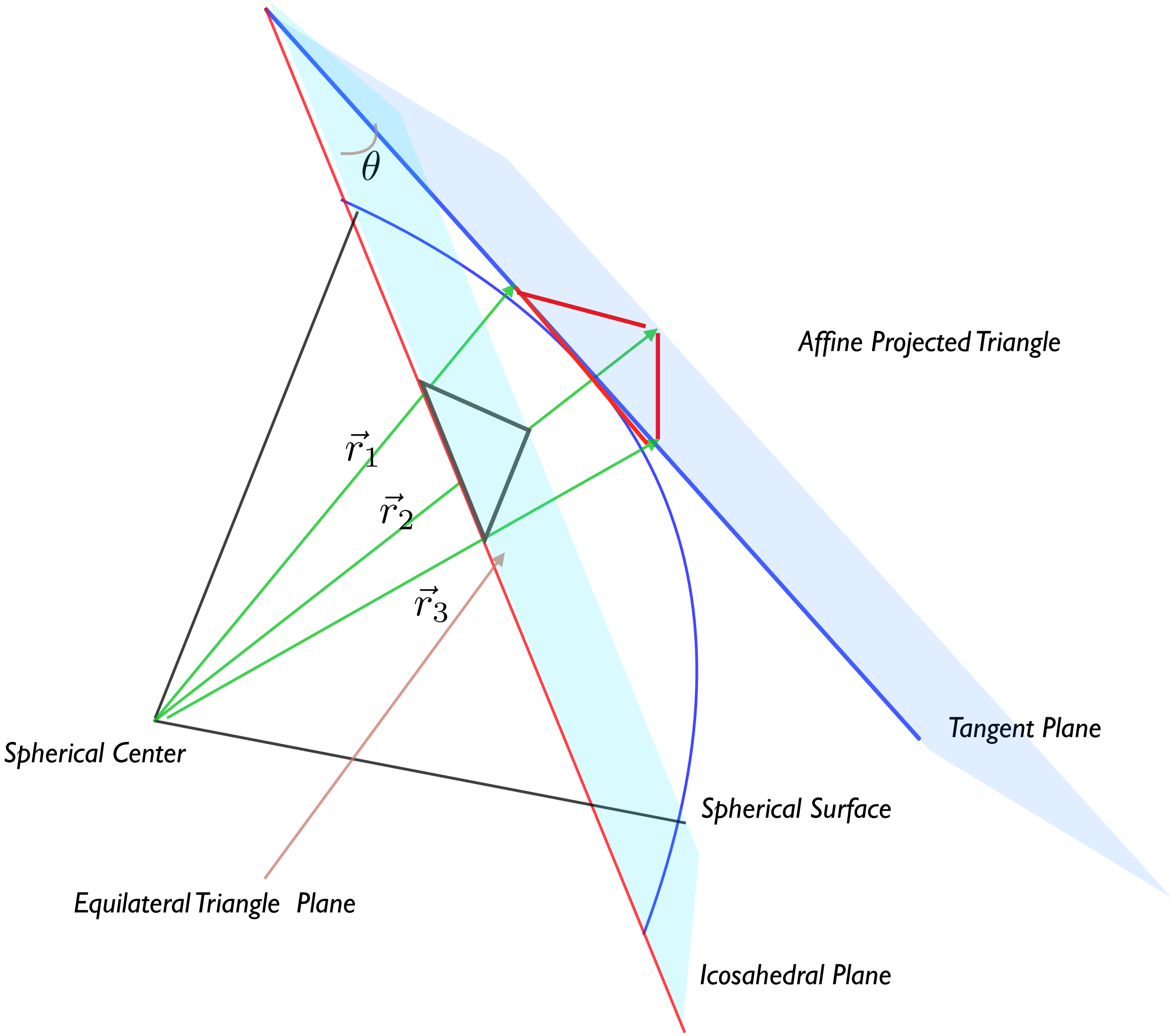
$\lambda_T^2 = \dfrac{\Delta_\sigma^2 d^2 |z|^{d-2}}{C_T (d-1)^2} \to \dfrac{1}{16 C_T}$ for $d = 2$, $\quad g_{T,2}(z) = -3\left(1 + \frac{1}{z}\left(1 - \frac{z}{2}\right)\log(1-z)\right) + \text{c.c.}$

Part III:

Ising Model on the Affine Plane

$$\mathbb{R}^2$$

To O(a^2) the tangent plane is an Affine lattice on each tangent plane.

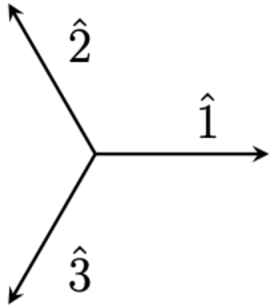


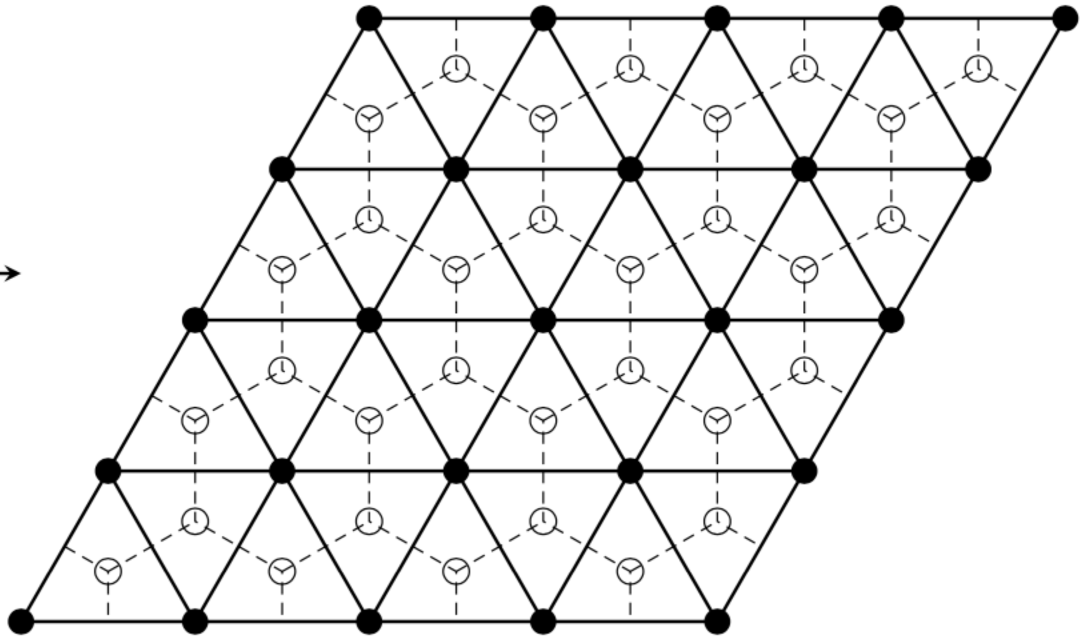
Affine Parameters:
2d Affine transformation takes circle to ellipse:

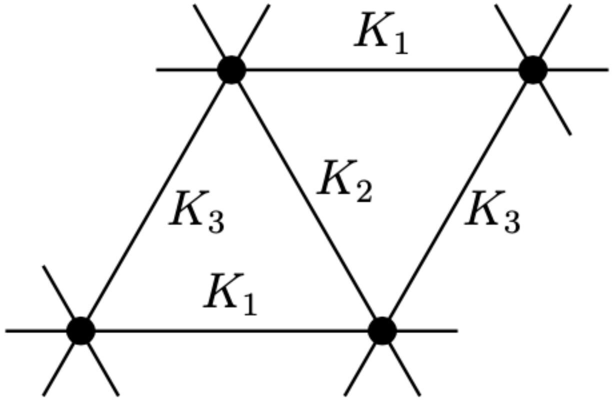
$$\langle \phi(x,y)\phi(0)\rangle = \frac{1}{(x^2+y^2)^{\Delta_\phi}} \leftrightarrow \frac{1}{(ax^2+bxy+cy^2)^{\Delta_\phi}}$$

- d = 2 Poincare 1 rotation 2 translation
- New Affine plus 1 major/minor + 1 orientation + 1 scaling
- General Poincare d(d+1)/2 plus d(d+1)/2 the number of edge in d-simplex - local metric

Ising Model on the Affine Plane







$$Z^{\triangle} = \sum_{s_n = \pm 1} e^{\,K_1 s_n s_{n+\hat{1}} + K_2 s_n s_{n+\hat{2}} + K_3 s_n s_{n+\hat{3}}} \;,$$

Prove the Emergent Geometry Requires

- Critical Ising at $p_1 p_2 + p_2 p_3 + p_3 p_1 = 1 \quad \text{with} \quad p_i = \exp(-2K_i)$

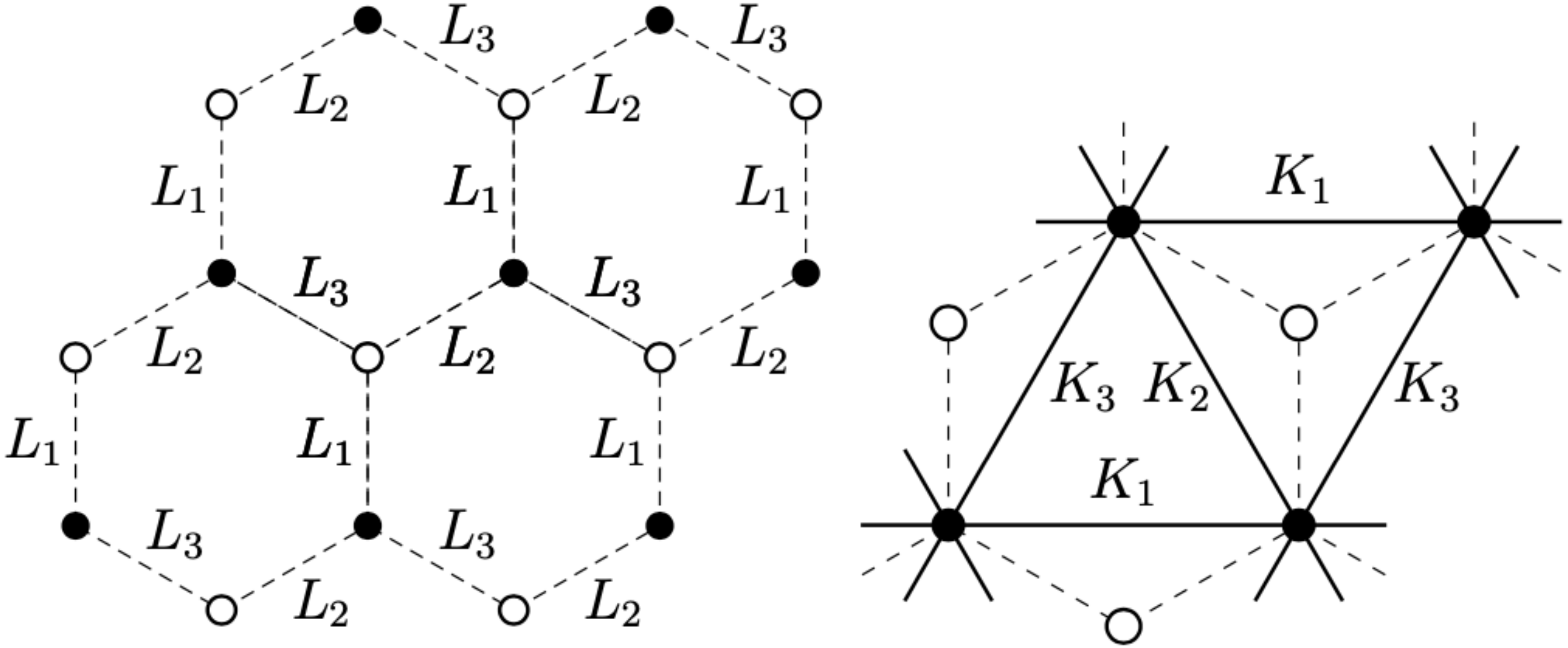
$$\sinh(2K_1) = \ell_1^*/\ell_1 \quad , \quad \sinh(2K_2) = \ell_2^*/\ell_2 \quad , \quad \sinh(2K_3) = \ell_3^*/\ell_3$$

- Free (FEM) scalar CFT.

$$S_{\text{free}} = \frac{1}{2}\sum_n [K_1(\phi_n - \phi_{n+\hat{1}})^2 + K_2(\phi_n - \phi_{n+\hat{2}})^2 + K_3(\phi_n - \phi_{n+\hat{3}})^2]$$

$$2K_1 = \ell_1^*/\ell_1 \quad , \quad 2K_2 = \ell_2^*/\ell_2 \quad , \quad 2K_3 = \ell_3^*/\ell_3 \ .$$

Step I : Star Triangle ID: Hex to Triangle Map



$$h \sinh(2K_1) \sinh(2L_1) = h \sinh(2K_2) \sinh(2L_2) = h \sinh(2K_3) \sinh(2L_3) = 1$$

$$h(K_1, K_2, K_3) = \frac{(1 - v_1^2)(1 - v_2^2)(1 - v_3^2)}{4\sqrt{(1 + v_1 v_2 v_3)(v_1 + v_2 v_3)(v_2 + v_3 v_1)(v_3 + v_1 v_2)}} \qquad \text{with } v_i = \tanh(K_i)$$

Step II: Map Hexagonal Loop Expansion is easy to map to free Ising to Free Wilson-Majorana Fermion*

$$Z_N^\psi = \prod_n \iint d\psi_n^1 d\psi_n^2 \; e^{-S[\bar\psi,\psi]} = \prod_n \int d^2\psi_n e^{-\frac{1}{2}\sum_n \bar\psi_n\psi_n} \prod_{n,i} [1 + \kappa_i \bar\psi_n P(\hat e_i)\psi_{n+\hat i}]$$

$$S[\psi] = \frac{1}{2}\sum_n \bar\psi_n\psi_n - \frac{1}{2}\sum_{n,i} \kappa_i \bar\psi_n (1 + \hat e_i \cdot \vec\sigma)\psi_{n+\hat i} \; .$$

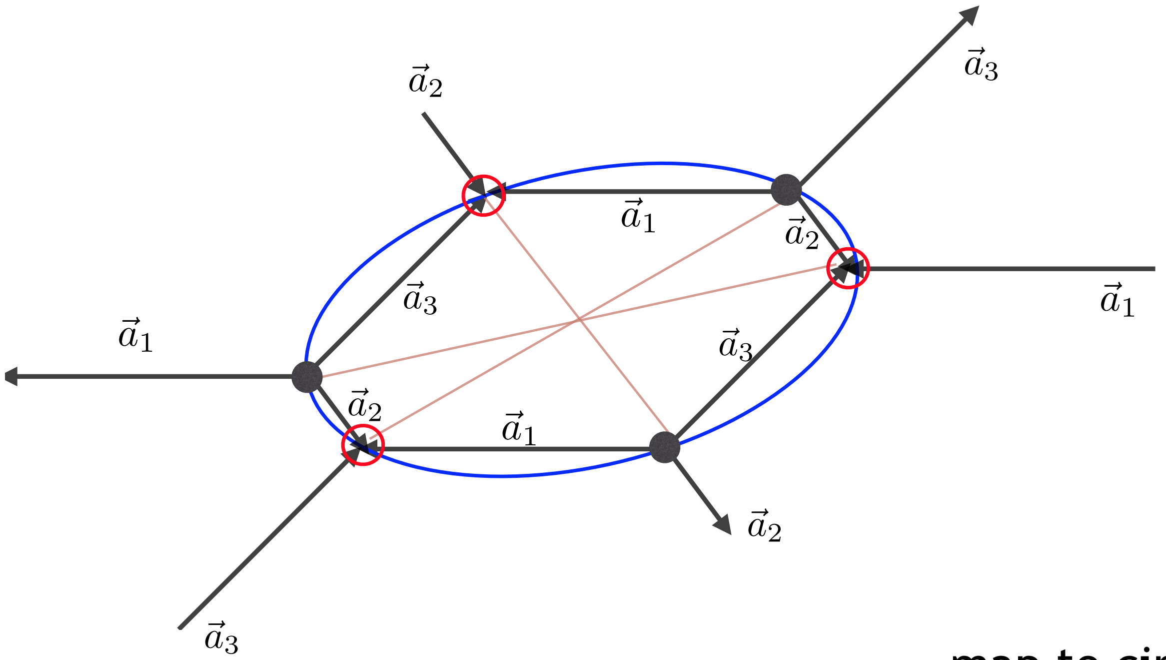
Horrible algebra (unless you are Baxter?)
but beautiful Geometry in spirit of Pascal's theorem**

***Generalizing very nice paper by Ulli Wolff.**

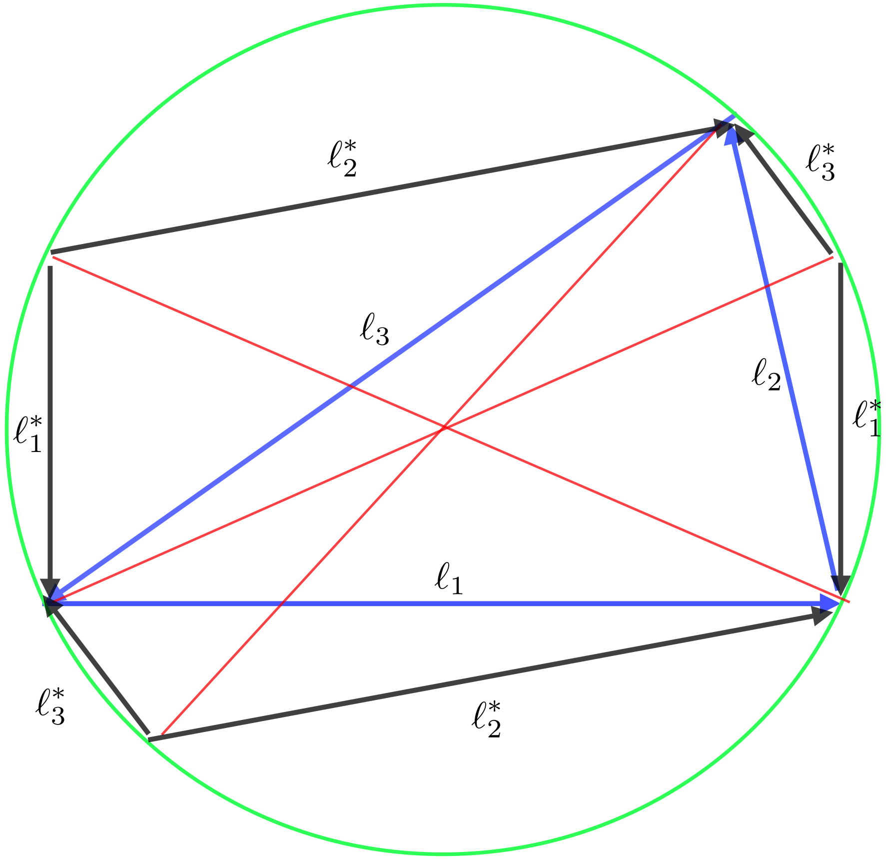
Ising model as Wilson-Majorana Fermions. Nucl. Phys. B, 955:115061, 2020.

**Blaise Pascal. Essay pour les conique (1640).

Elliptical Hexagon to a Circular Hexagon



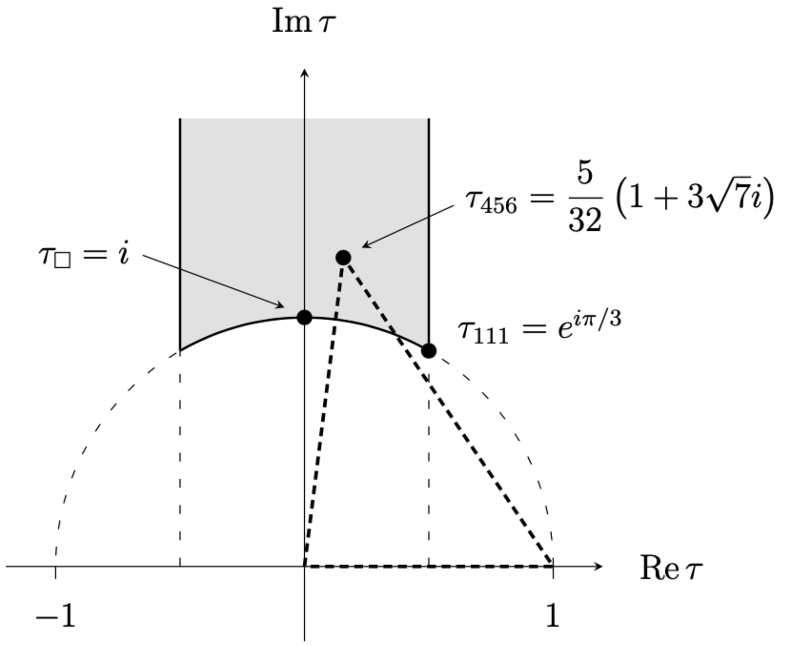
map to circle ==>

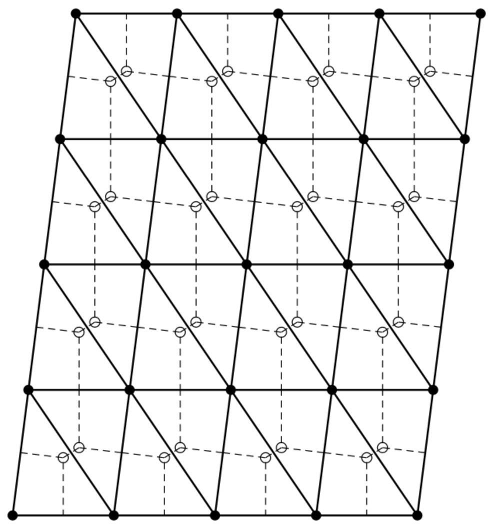


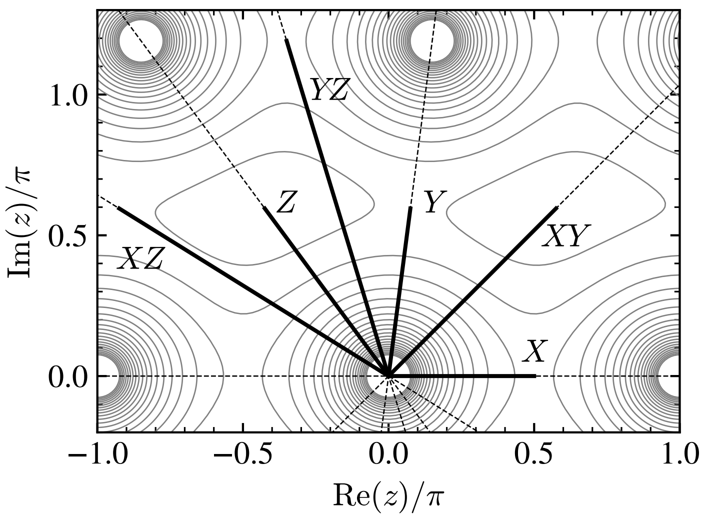
Basic algebra of Projective Geometry going back to Pascal in 1640!

- Blaise Pascal. Essay pour les conique. (facsimile) Nieders¨achsiche Landesbibliothek, Gottfried Wilhelm Leibniz Bibliothek, 1640.

Calculation Modular dependent on the torus





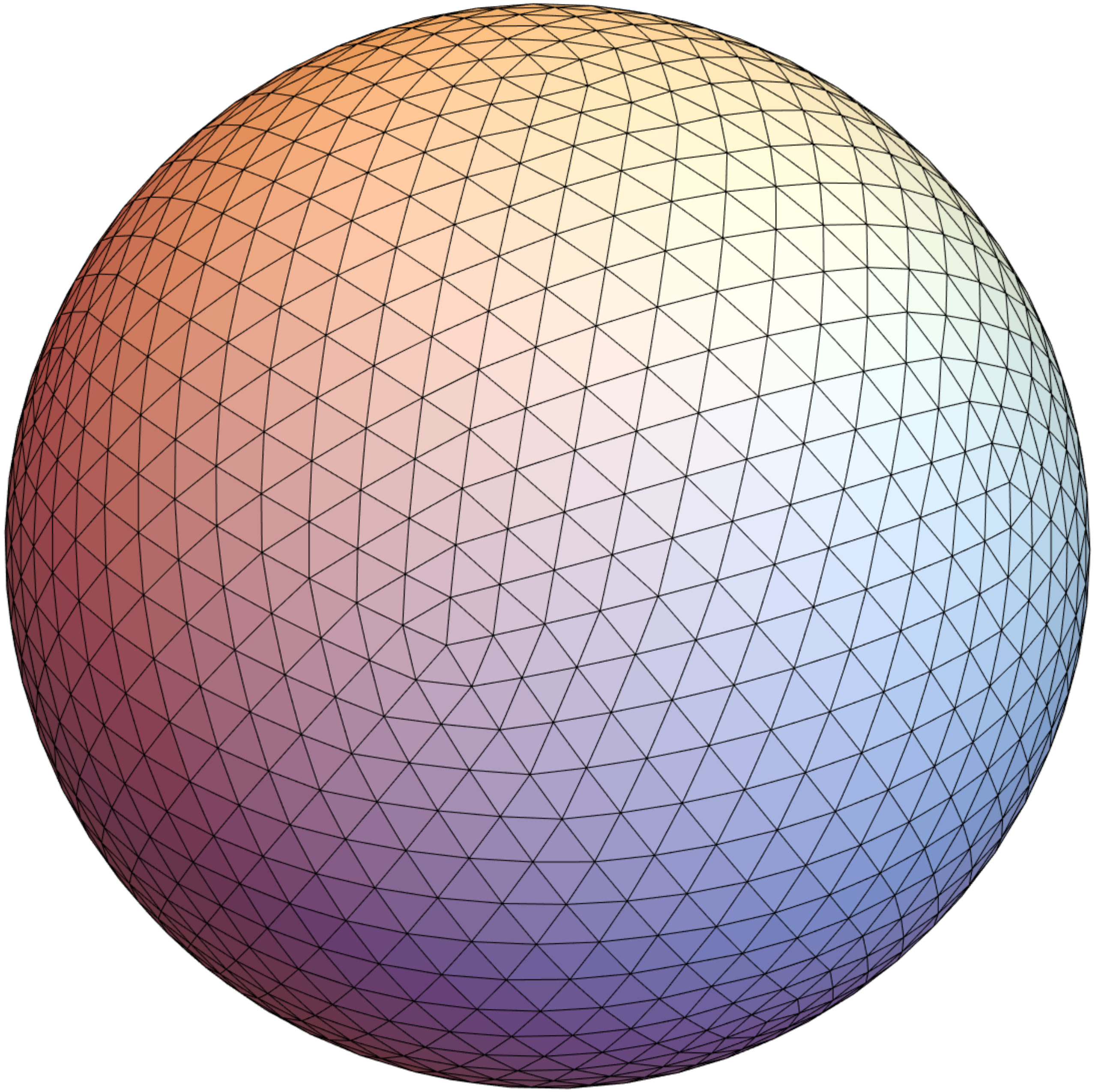


$$\langle \sigma(0)\sigma(z)\rangle = \left|\frac{\vartheta_1'(0|\tau)}{\vartheta_1(z|\tau)}\right|^{1/4} \frac{\sum_{\nu=1}^{4}|\vartheta_\nu(z/2|\tau)|}{\sum_{\nu=2}^{4}|\vartheta_\nu(0|\tau)|}$$

Back to Putting critical 2d Ising on the sphere?

- Do affine project to local tangent plane

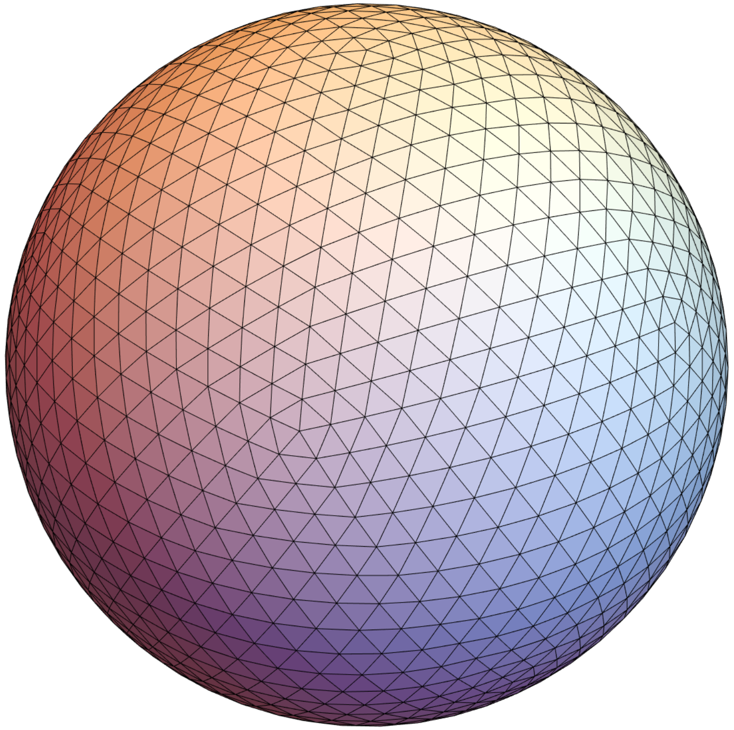
- Tune to local UV criticality



Critical Ising Model on S^2

- Back to our question: Can we simulate a critical Ising spin model on S^2?
- Determine critical couplings *locally* for each link
- Can be improved by accounting for curvature (non-trivial spin connection)[14]

$\mathbb{S}^2$



Critical Ising Model on S^2

- Conformal correlator on a sphere has the form

$$\langle \sigma(\hat{x})\sigma(\hat{y})\rangle \propto \frac{1}{(1-\hat{x}\cdot\hat{y})^{\Delta_\sigma}}$$

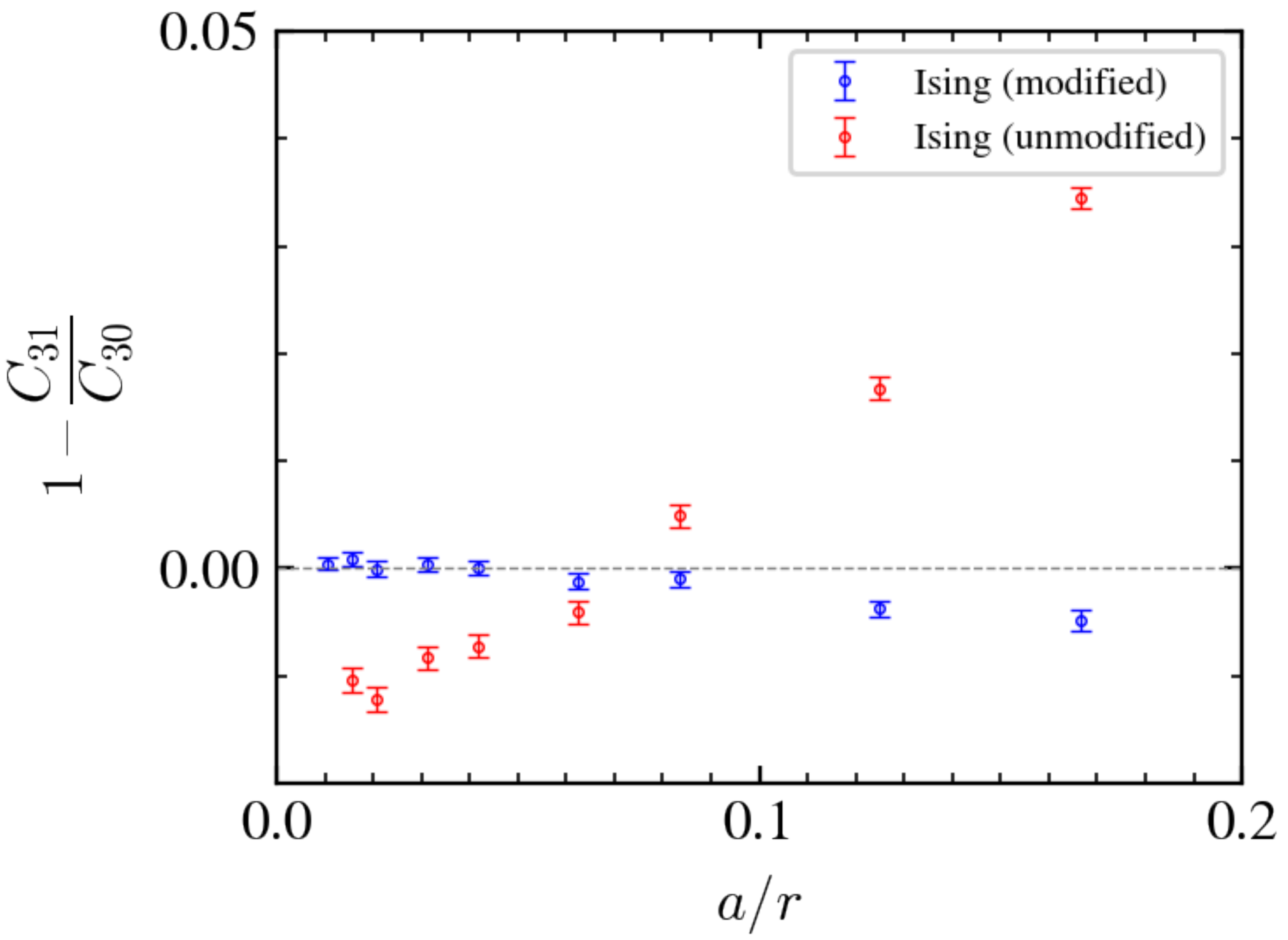
- Measure as a series in Legendre polynomials

$$\langle \sigma(\hat{x})\sigma(\hat{y})\rangle = \sum_\ell F_\ell P_\ell(\hat{x}\cdot\hat{y})$$

$$\frac{F_\ell}{F_0} = \frac{\Gamma(\Delta_\sigma+\ell)\Gamma(2-\Delta_\sigma)}{\Gamma(\Delta_\sigma)\Gamma(2-\Delta_\sigma+\ell)}$$

Application of Affine Coupling to Sphere

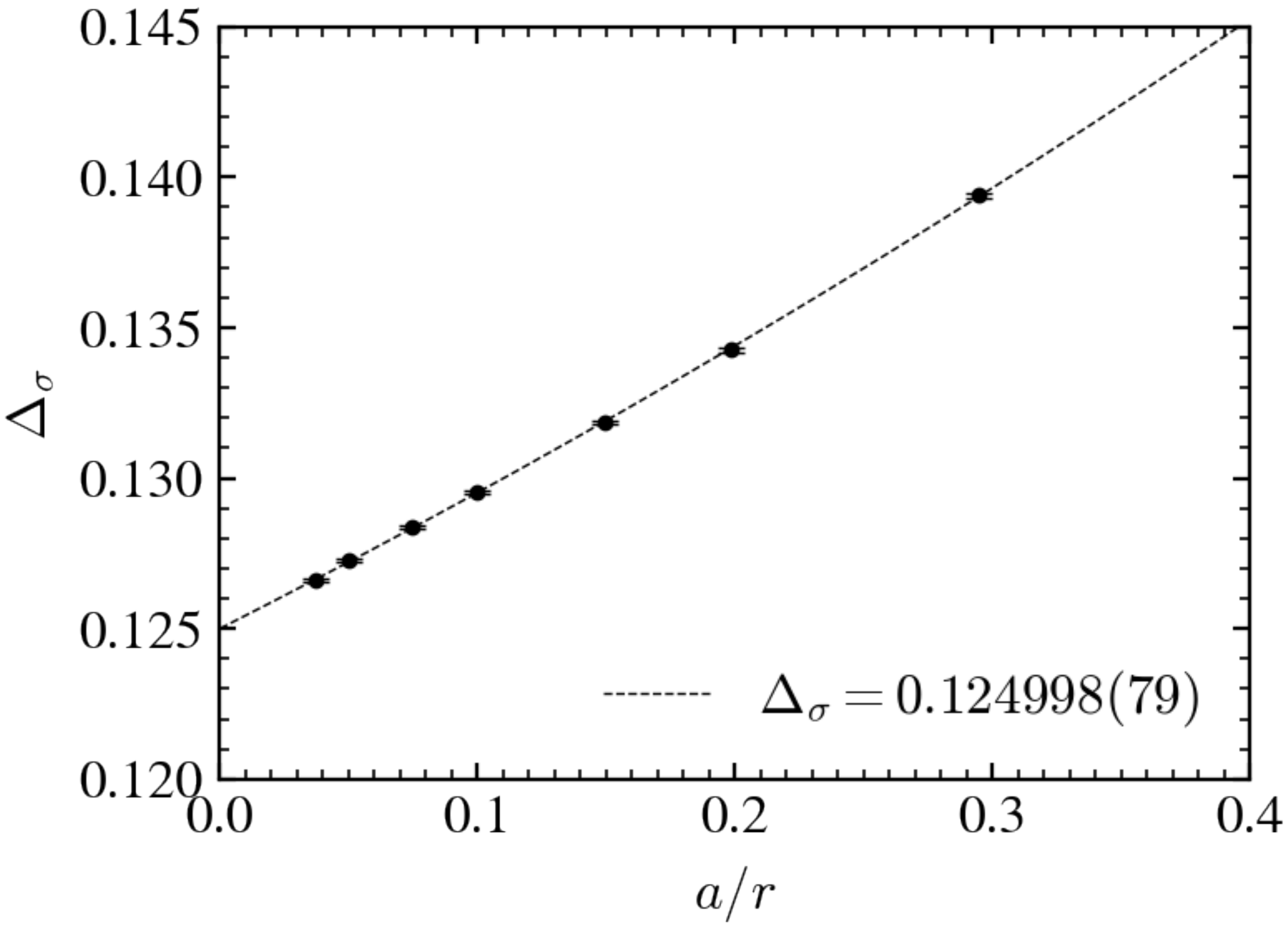
- Tune to equal dual areas



Critical Ising Model on S^2

SLIDE OF FROM EVAN OWEN

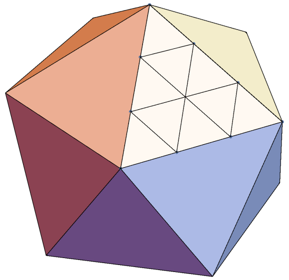
Continuum limit extrapolation of Δ_σ (quadratic fit, $\chi^2/\text{dof} = 0.29$)



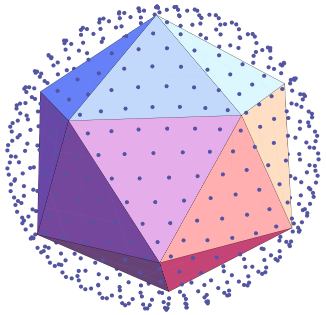
Part III:

Back to Phi 4th Radial Quantization on

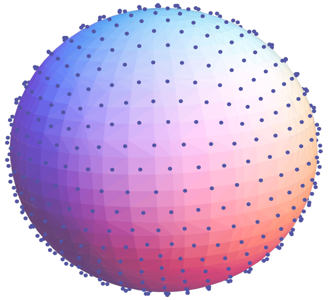
$$\mathbb{R} \times \mathbb{S}^2$$

$\mathbb{R} \times$ 

CFT on the Icosahedron

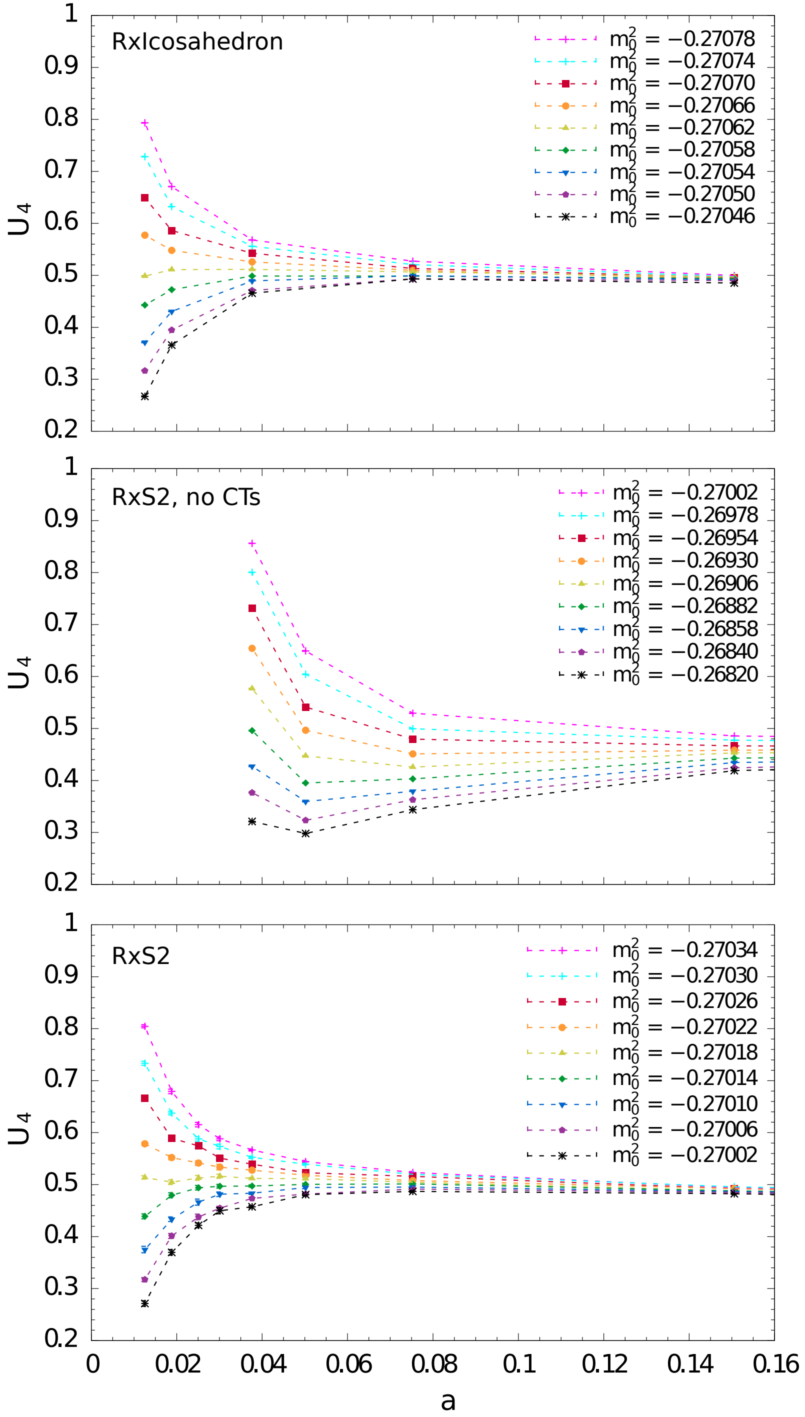
$\mathbb{R} \times$ 

FEM CFT on Sphere

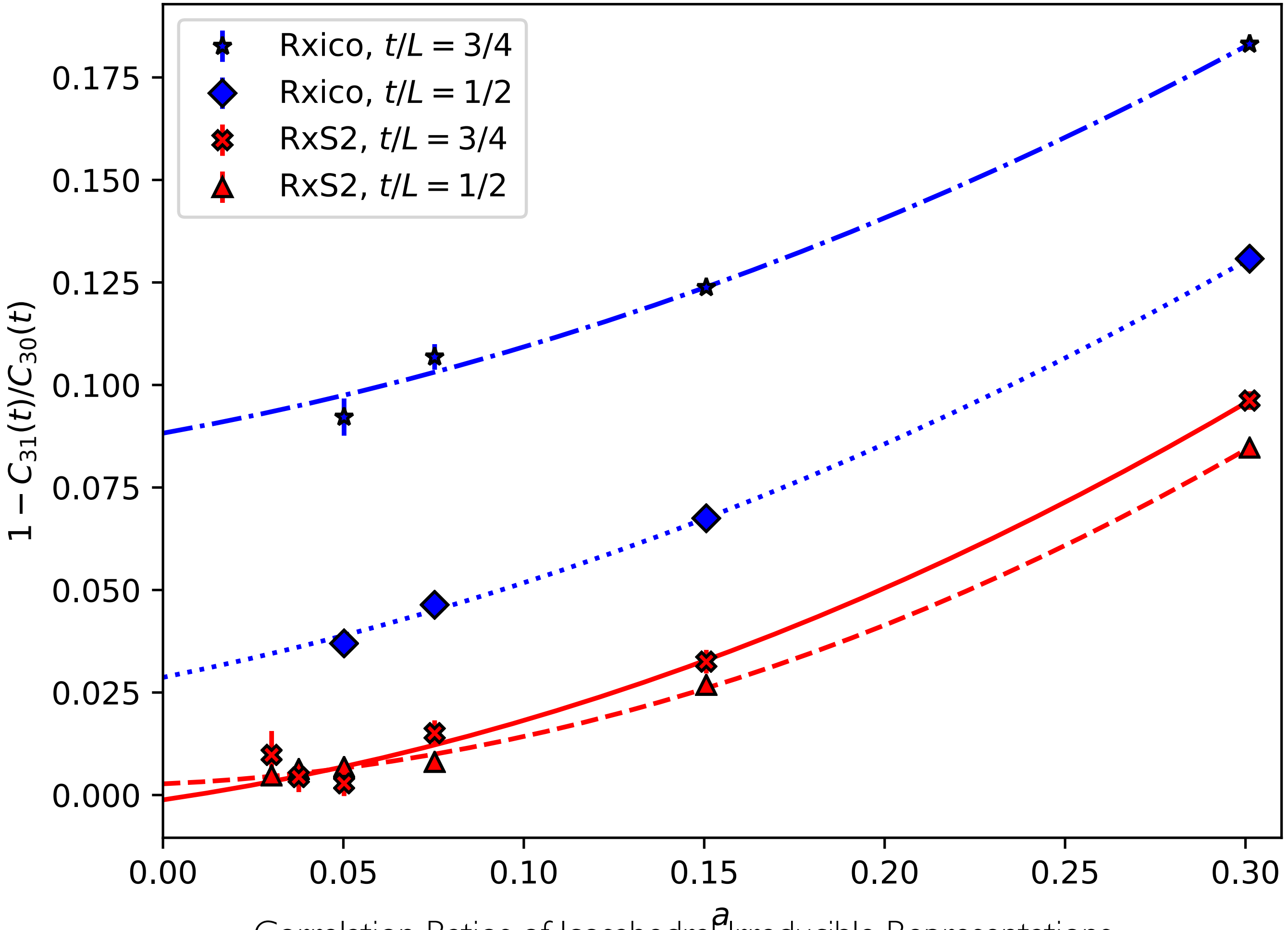
$\mathbb{R} \times$ 

QFE CFT on Sphere

$$U_4(L, \mu_0, \lambda_0) = \frac{3}{2}\left[1 - \frac{\langle M^4 \rangle}{3\langle M^2 \rangle^2}\right]$$

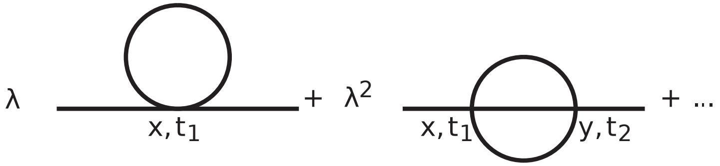


Test Restoration of Spherical Symmetry as a —> 0

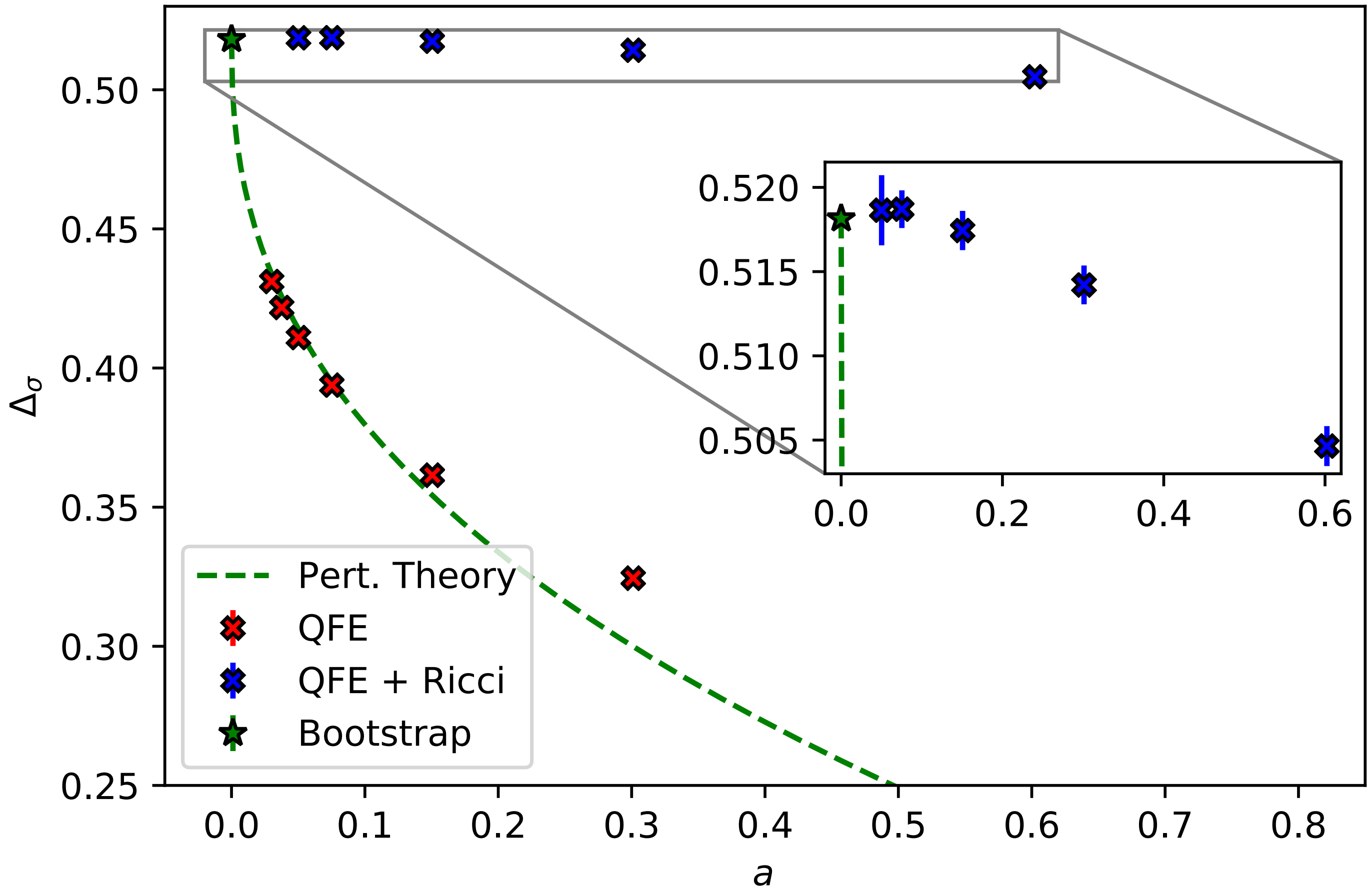


Correlation Ratios of Icosahedral Irreducible Representations

$$S_{FEM} \to S_{QFE} = S_{FEM} + \sum_x C(\lambda)\delta m_x^2 \phi_x^2$$



Lattice Test against very precise CFT Bootstrap constraint



$$S_{FEM} = \frac{a_t}{2}\Big[\sum_{y\in\langle x,y\rangle} \frac{l^*_{xy}}{l_{xy}}(\phi_{t,x}-\phi_{t,y})^2 + \frac{\sqrt{g_x}}{4R^2}\phi^2_{t,x}$$
$$+\sqrt{g_x}\Big[\frac{(\phi_{t,x}-\phi_{t+1,x})^2}{a_t^2} + m^2\phi^2_{t,x} + \lambda\phi^4_{t,x}\Big]\Big]$$

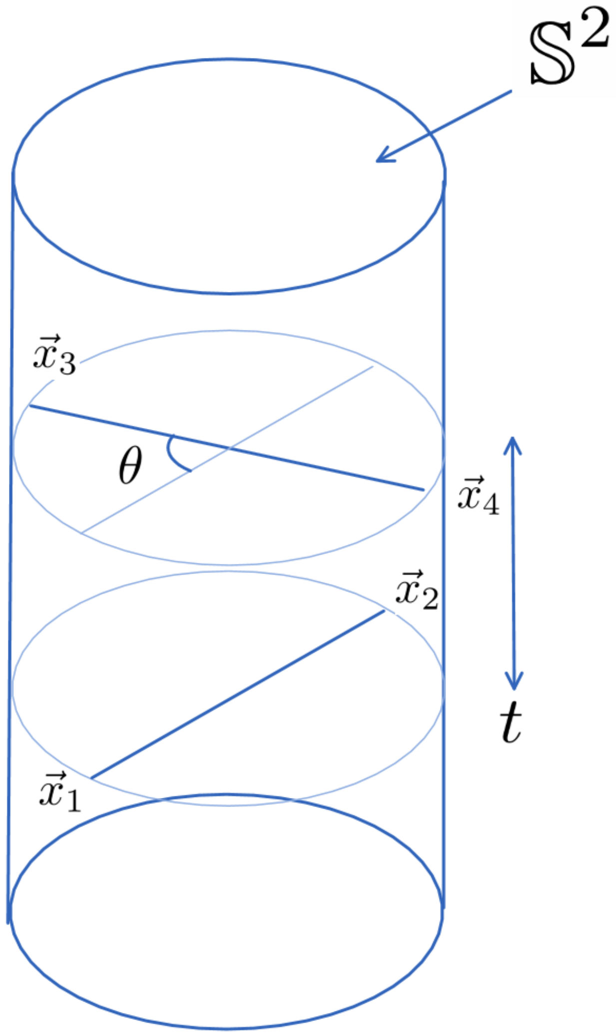
Exact Ricci term
at lambda = 0

Need for Ricci Term
Improvement Scheme

Antipodal 4-point function on $\mathbb{R} \times \mathbb{S}^2 \ni (t, \vec{x})$

$$\frac{\langle \sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\rangle}{\langle \sigma(x_1)\sigma(x_2)\rangle\langle \sigma(x_3)\sigma(x_4)\rangle} = 1 + \sum_{\mathcal{O}} f^2_{\sigma\sigma\mathcal{O}} G_{\mathcal{O}}(\Delta_{\mathcal{O}}; x_1, x_2, x_3, x_4)$$

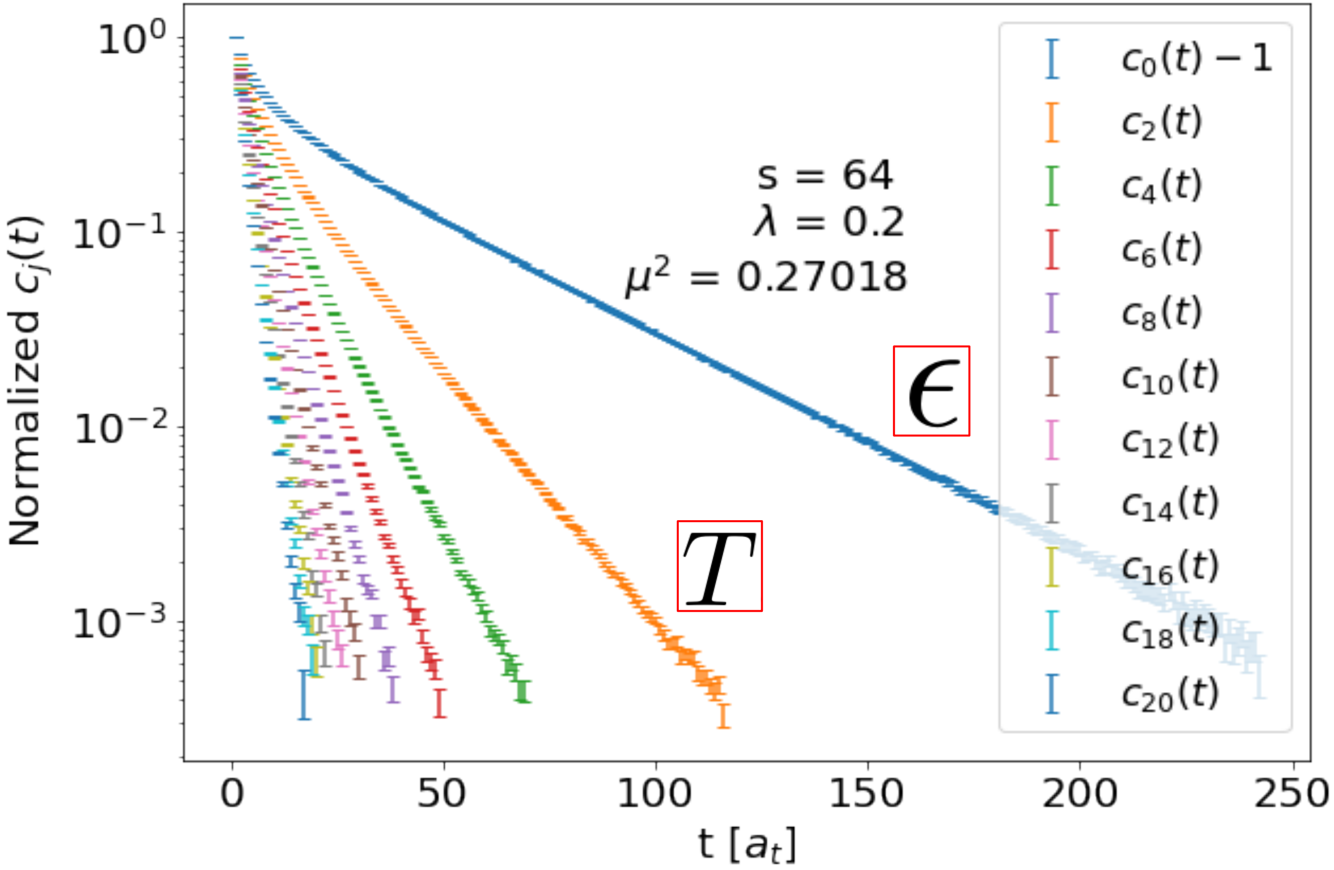
$$G_{\mathcal{O},l} = \sum_{n=0,2,4,\ldots} \sum_j e^{-(\Delta_{\mathcal{O}}+n)t} B_{n,j}(\Delta_{\mathcal{O}}) P_j(\cos(\theta))$$



Numerical results

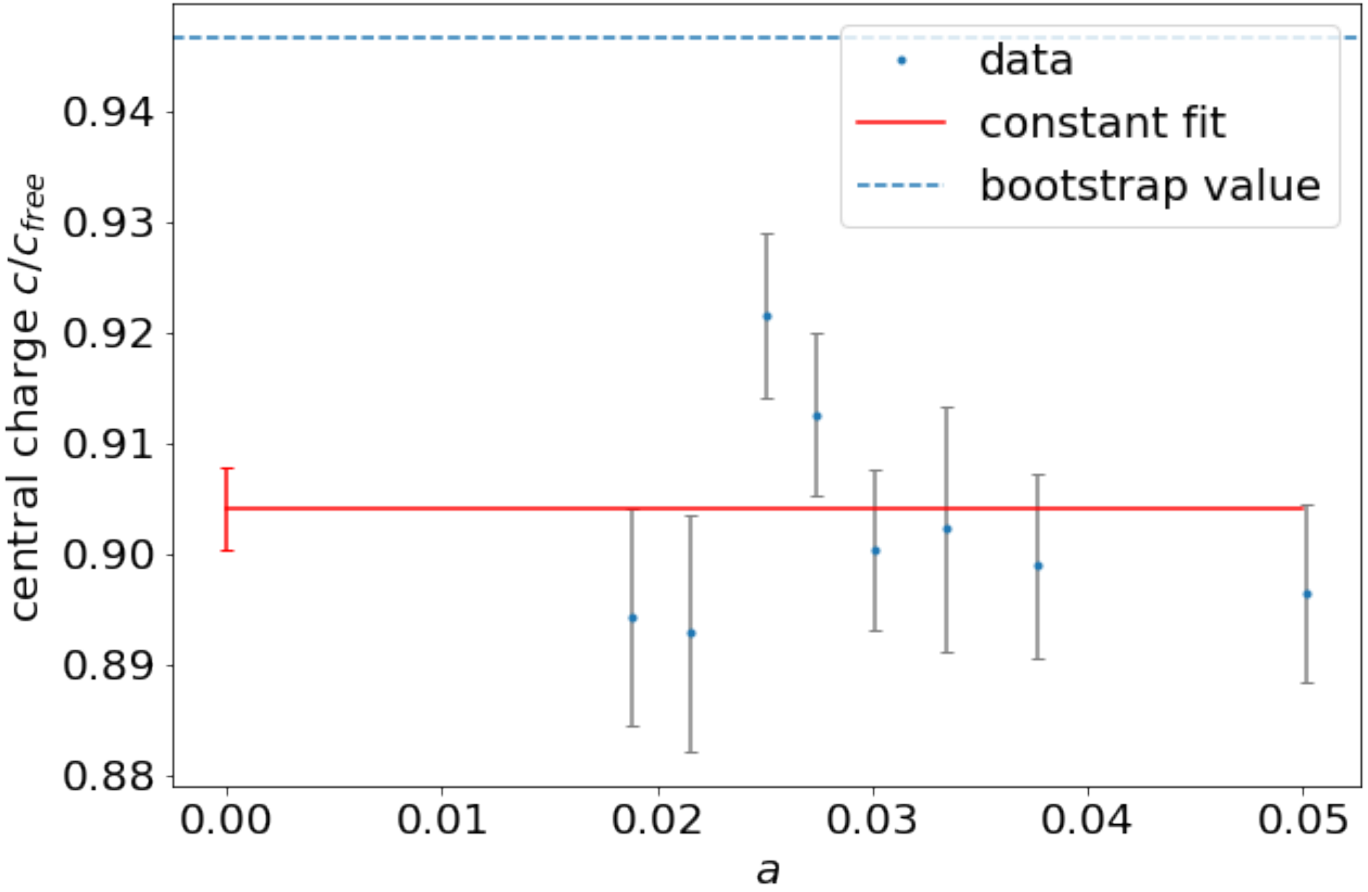
$$j \in \{\max(0, l-n), ..., l+n-2, l+n\}$$

$$\frac{\langle\sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\rangle}{\langle\sigma(x_1)\sigma(x_2)\rangle\langle\sigma(x_3)\sigma(x_4)\rangle} = \sum_{\text{even } j} c_j(\Delta t) P_j(\cos(\theta)) = 1 + \sum_{\mathcal{O}} f^2_{\sigma\sigma\mathcal{O}} \sum_{n=0,2,4,\ldots} \sum_j e^{-(\Delta_{\mathcal{O}}+n)c_R g t} B_{n,j}(\Delta_{\mathcal{O}}) P_j(\cos(\theta))$$



Simultaneous fits of c_0(t) and c_2(t) using primaries ϵ, T, ϵ', T' up to n=20

Fit to central charge anomaly



$$c/c_{free} = \frac{\Delta_\sigma^2 \Delta_T^2}{3 f_{\sigma\sigma T}^2}$$

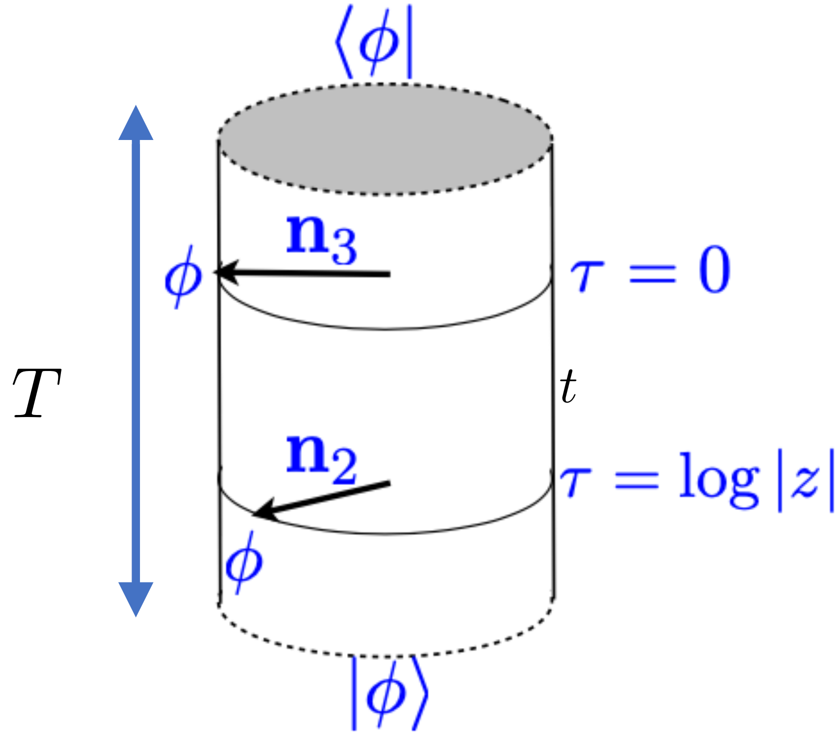
$$c_{bootstrap}/c_{free} \geq 0.946534(11)$$

(const in lattice spacing a)

$$c^{fit}/c_{free} = 0.9041(37)$$

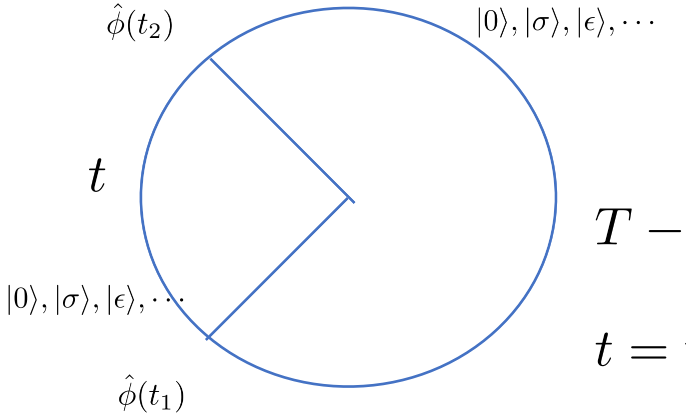
Something is not quite right: Statistics, Extrapolation in “a”, UV counter terms ?

Finite Volume/Temperature Measurements



$$Tr[e^{-\beta\hat{H}+\hat{h}_x\hat{\phi}_x}] = e^{-F(\beta, h_x)}$$

Central charge enters finite temperature free energy and amplitudes:
Can trace from UV to IR



$t = t_2 - t_1$

$0 \leq t \leq T$ and periodic $t \to t + nT$

$$\hat{H}|0\rangle = 0 \text{ and } \hat{\phi}|0\rangle = |\sigma\rangle$$

$$\begin{aligned}
\langle \phi_\ell(t_2)\phi_{\ell_1}(t_1)\rangle_T &= Tr[\,\hat{\phi}_\ell(0)e^{-t\hat{H}}\hat{\phi}_\ell(0)e^{-(T-t)\hat{H}}\,] \\
&\equiv \sum_{\mathcal{O}} e^{-T\Delta_{\mathcal{O}}}\langle\mathcal{O}|\hat{\phi}_\ell(0)e^{-t(\hat{H}-\Delta_{\mathcal{O}})}\hat{\phi}_\ell(0)|\mathcal{O}\rangle \\
&\simeq e^{-t\Delta_{\sigma,\ell}} + e^{-(T-t)\Delta_{\sigma,\ell}} \\
&+ \quad f^2_{\epsilon\phi_\ell\sigma}e^{-\Delta_\sigma T}[e^{-t(\Delta_\epsilon-\Delta_\sigma)} + e^{-(T-t)(\Delta_\epsilon-\Delta_\sigma)}] + \cdots
\end{aligned}$$

Part IV : FUTURE DIRECTIONS

- Perturbative vs non-perturbative QFE counter terms
- Super Renormalizable vs Asymptotic Freedom
- 2 +1 QED on $\mathbb{R} \times \mathbb{S}^2$
- Three sphere $\mathbb{S}^3$ and $R \times \mathbb{S}^3$
- SUSY, AdS, Full 4D Non-Abelian Gauge theory ….

SIMPLICIAL EXTERIOR CALCULUS DOES ALMOST ALL FOR CLASSICAL

J = 0

$$S_{scalar} = \frac{1}{2}\sum_{\langle i,j\rangle} \frac{V_{ij}}{l_{ij}^2}(\phi_i - \phi_j)^2 \ ,$$

$$l_{ij}^2 = |\sigma_1(ij)|^2$$

J = 1/2

$$S_{Wilson} = \frac{1}{2}\sum_{\langle i,j\rangle} \frac{V_{ij}}{l_{ij}}(\bar{\psi}_i \hat{e}_a^{j(i)}\gamma^a \Omega_{ij}\psi_j - \bar{\psi}_j \Omega_{ji}\hat{e}_a^{i(j)}\gamma^a\psi_i)$$

J = 1

$$S_{gauge} = \frac{1}{2g^2 N_c}\sum_{\triangle_{ijk}} \frac{V_{ijk}}{A_{ijk}^2} Tr[2 - U_{\triangle_{ijk}} - U^\dagger_{\triangle_{ijk}}]$$

FFdual

$$\epsilon^{ijkl} Tr[U_{\triangle_{0ij}} U_{\triangle_{0kl}}] \simeq V_{ijkl}\epsilon^{\mu\nu\rho\sigma} Tr[F_{\mu\nu}(0)F_{\rho\sigma}(0)]$$

$$U_{\triangle_{ijk}} = U_{ij}U_{jk}U_{ki} \quad A_{ijk} = |\sigma_2(ijk)| \quad V_{ijk} = |\sigma_2(ijk) \wedge \sigma_2^*(ijk)|$$

$$U_{0ij} = U_{0i}U_{ij}U_{j0} \quad , \quad U^\dagger_{0ij} = U_{0j}U_{ji}U_{i0} \quad V_{ij} = |\sigma_1(ij) \wedge \sigma_1^*(ij)|$$

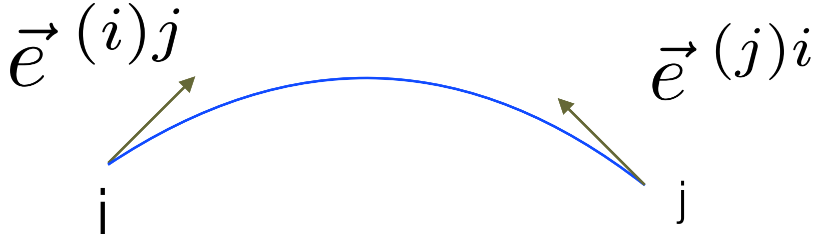
But Dirac needs Spin Connection (Kahler Dirac doesn't)

$$S = \frac{1}{2}\int d^D x \sqrt{g}\bar{\psi}[\mathbf{e}^\mu(\partial_\mu - \frac{i}{4}\boldsymbol{\omega}_\mu(x)) + m]\psi(x)$$

$\mathbf{e}^\mu(x) \equiv e^\mu_a(x)\gamma^a$ Verbein & Spin connection*

$\boldsymbol{\omega}_\mu(x) \equiv \omega^{ab}_\mu(x)\sigma_{ab}$, $\sigma_{ab} = i[\gamma_a, \gamma_a]/2$

$$S_{naive} = \frac{1}{2}\sum_{\langle i,j\rangle} \frac{V_{ij}}{l_{ij}}[\bar{\psi}_i \vec{e}^{\,(i)j} \cdot \vec{\gamma}\Omega_{ij}\psi_j - \bar{\psi}_j\Omega_{ji}\vec{e}^{\,(i)j} \cdot \vec{\gamma}\psi_i] + \frac{1}{2}mV_i\bar{\psi}_i\psi_i$$



Simplicial Tetrad Hypothesis

$$e^{(i)j}_a\gamma^a\Omega_{ij} + \Omega_{ij}e^{(j)i}_a\gamma^a = 0$$

See next talk + 3 by Shoto Aoki

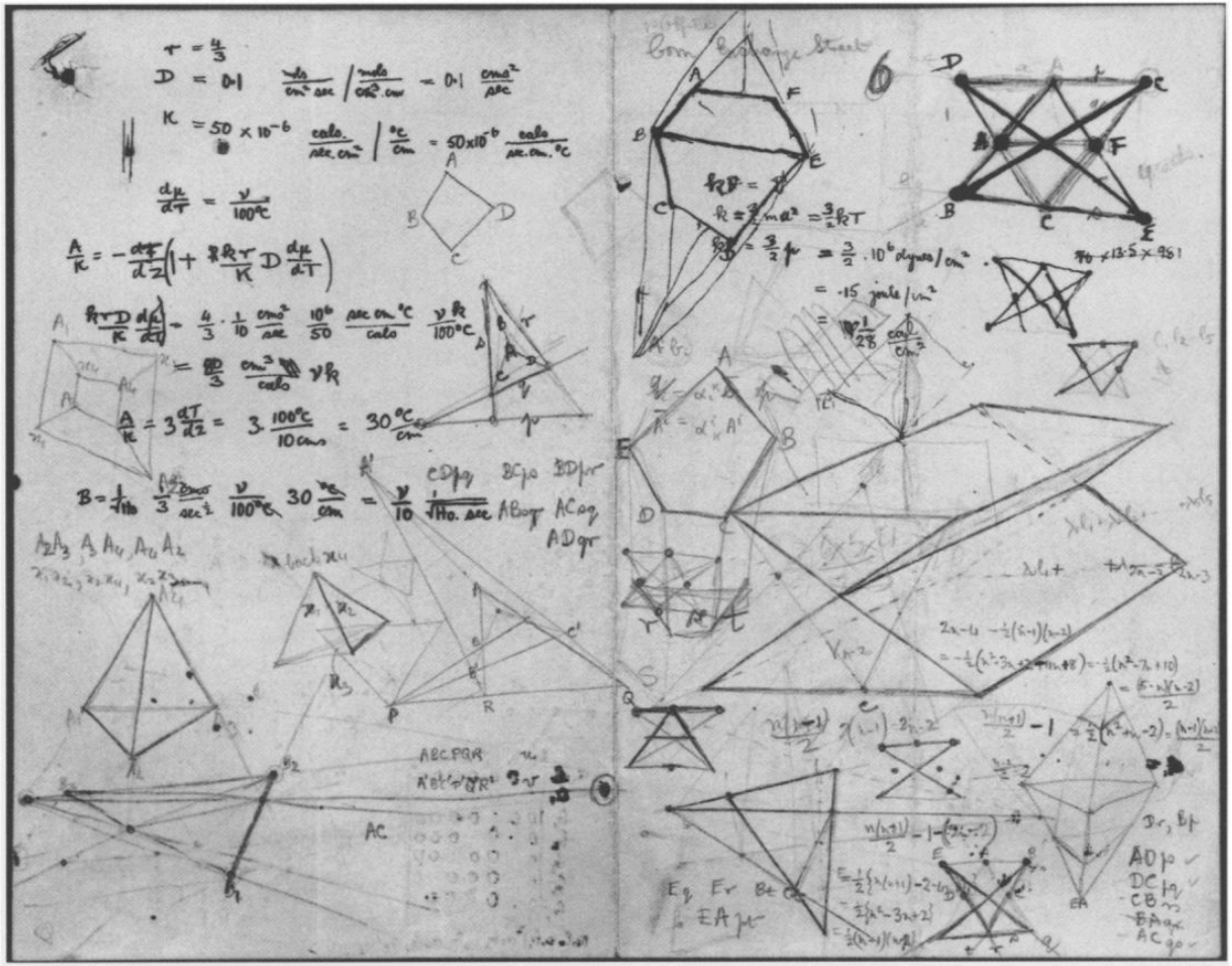
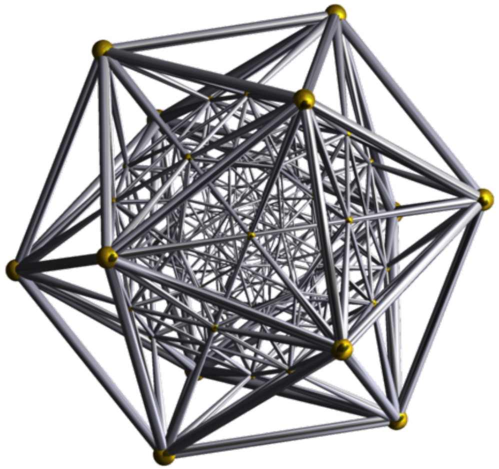


FIGURE 1. Paul Dirac, Geometrical Sketches, in the Paul A. M. Dirac Papers, Florida State University, Tallahassee, Florida; hereafter PDP. By permission of the Florida State University Libraries.

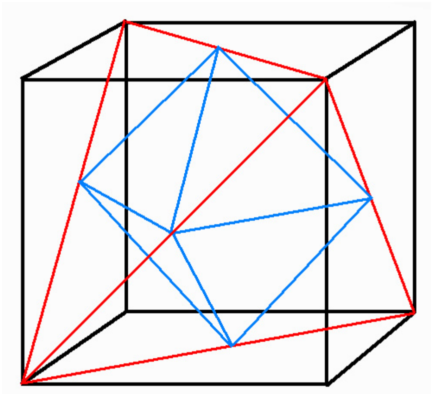
3 Spheres and 4D Radial Simplicial Lattices

$$\mathbb{S}^3 \quad \Longrightarrow \quad \mathbb{R} \times \mathbb{S}^3$$



Aristotle' s 2% Error!

$$(2\pi - 5ArcCos[1/3])/(2\pi) = 0.0204336$$



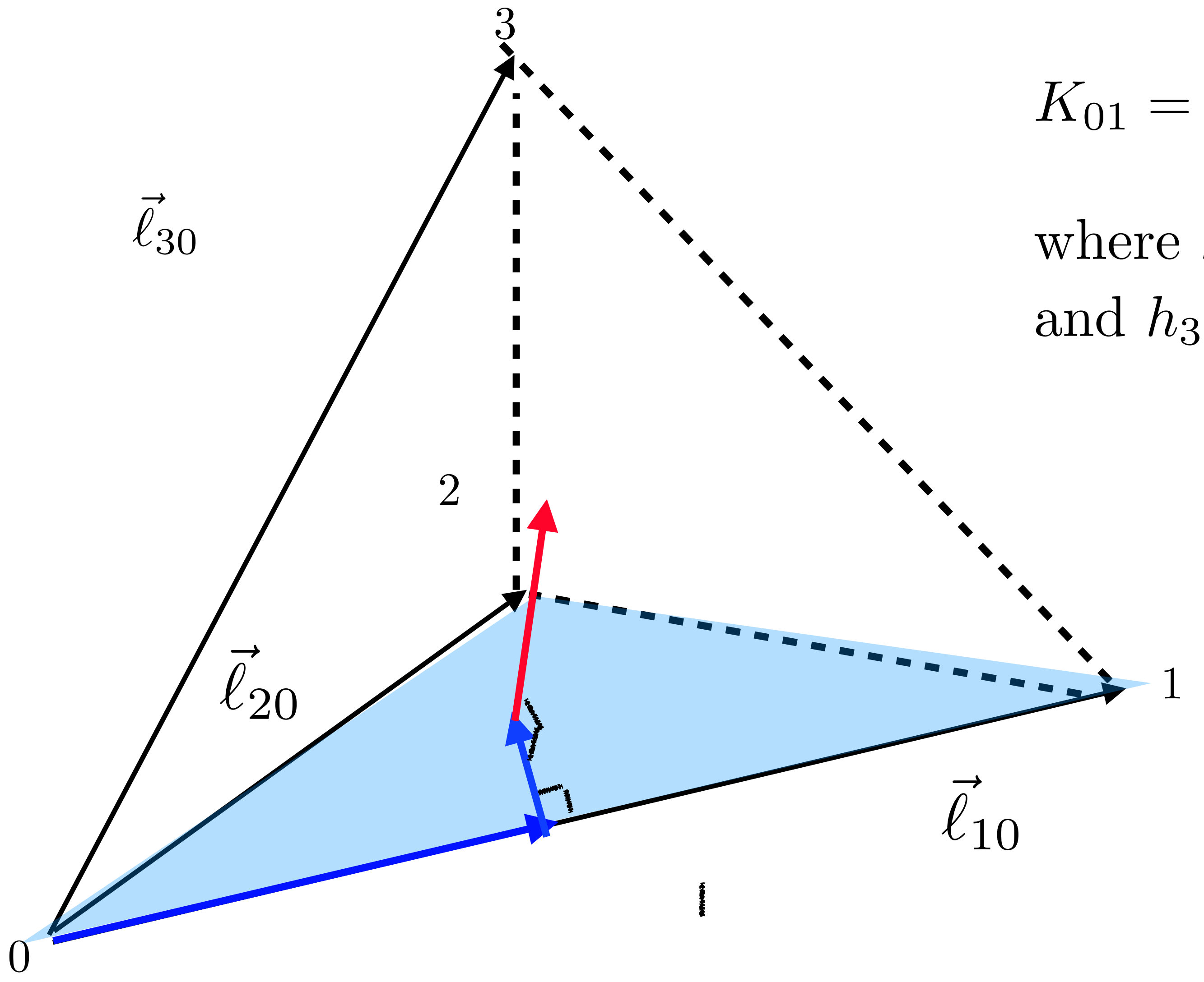
Fast Code Domains of
Regular 3D Grids on Refinement

600 cell: "Square of the icosahedron" –Symmetries 1440= 120 * 120 the 120 copies of icosahedron

$$O(4) \sim SU(2) \times SU(2)$$

The full symmetry group of the 600-cell is the Weyl group of H_4. This is a group of order 14400. It consists of 7200 rotations and 7200 rotation-reflections. The rotations form an invariant subgroup of the full symmetry group.

DEC contribution form the 01 edge of Tetrahedron

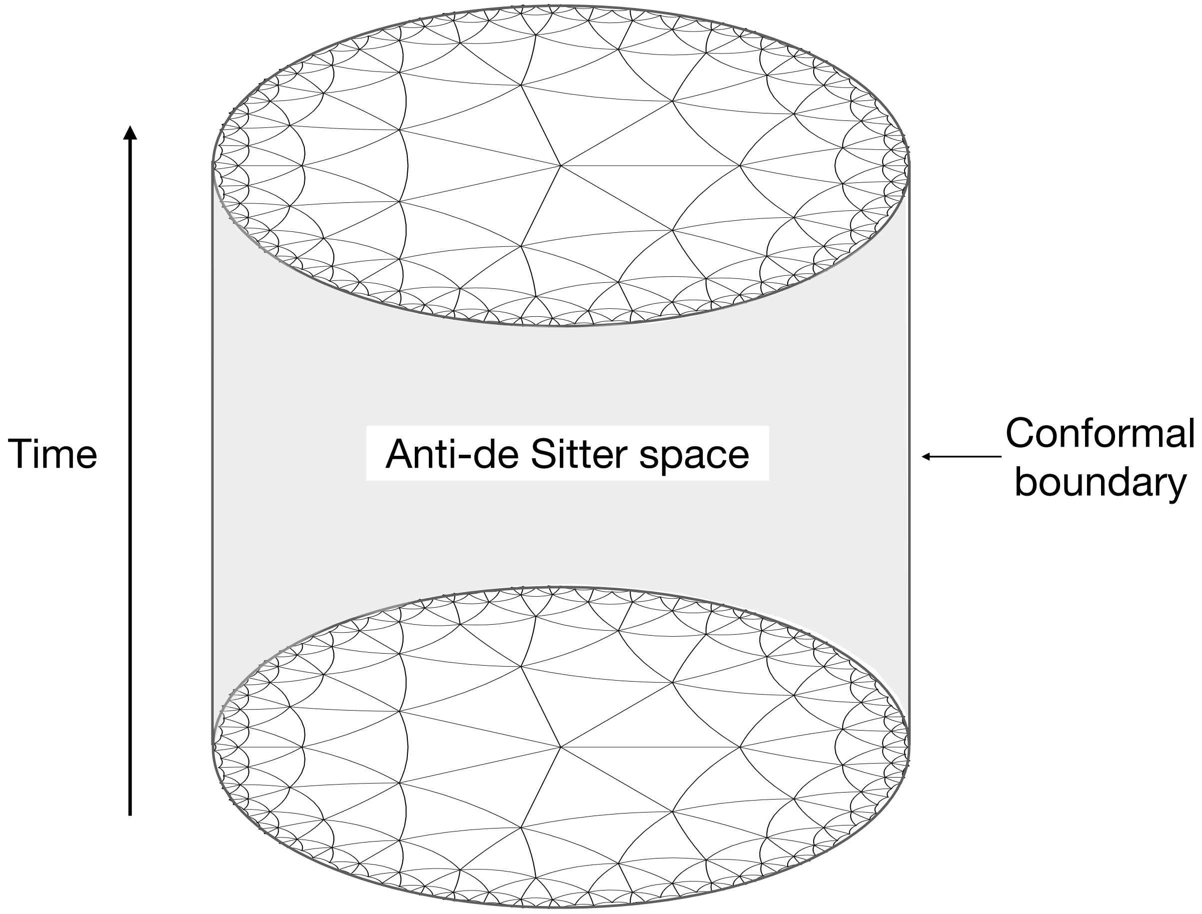


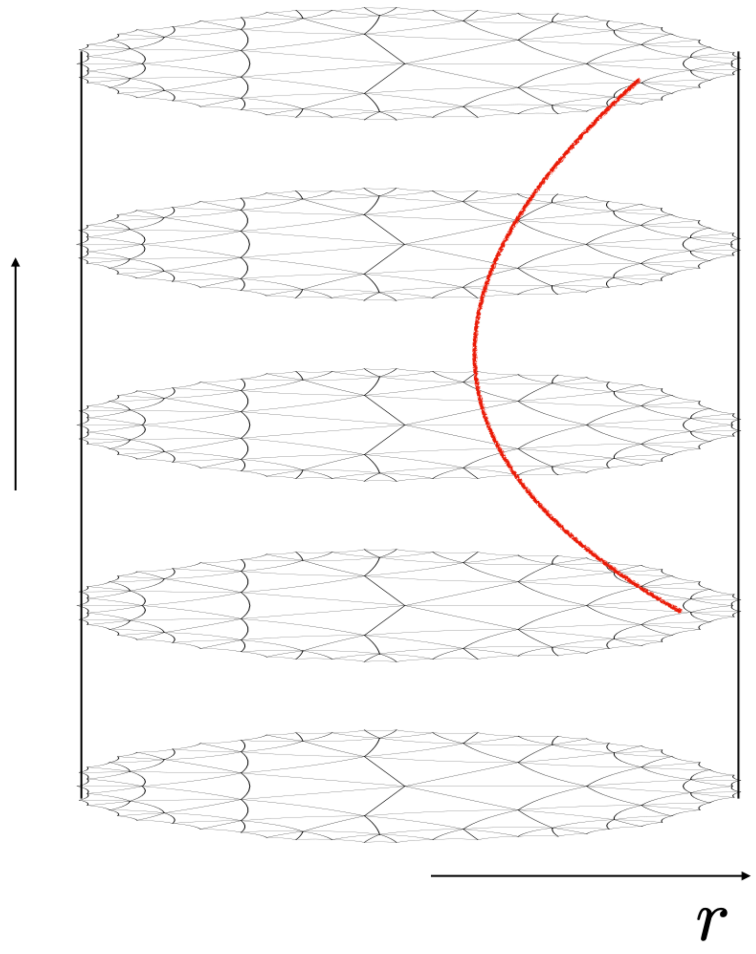
$$K_{01} = \ell_{01}^2 \frac{\ell_{02}^1 + \ell_{12}^2 - \ell_{01}^2}{4A(012)} h_2 + \ell_{01}^2 \frac{\ell_{03}^1 + \ell_{13}^2 - \ell_{01}^2}{4A(013)} h_3$$

where $h_2 = \epsilon[\xi_3]\sqrt{R_{cc}^2 - R^2(0,1,2)}$

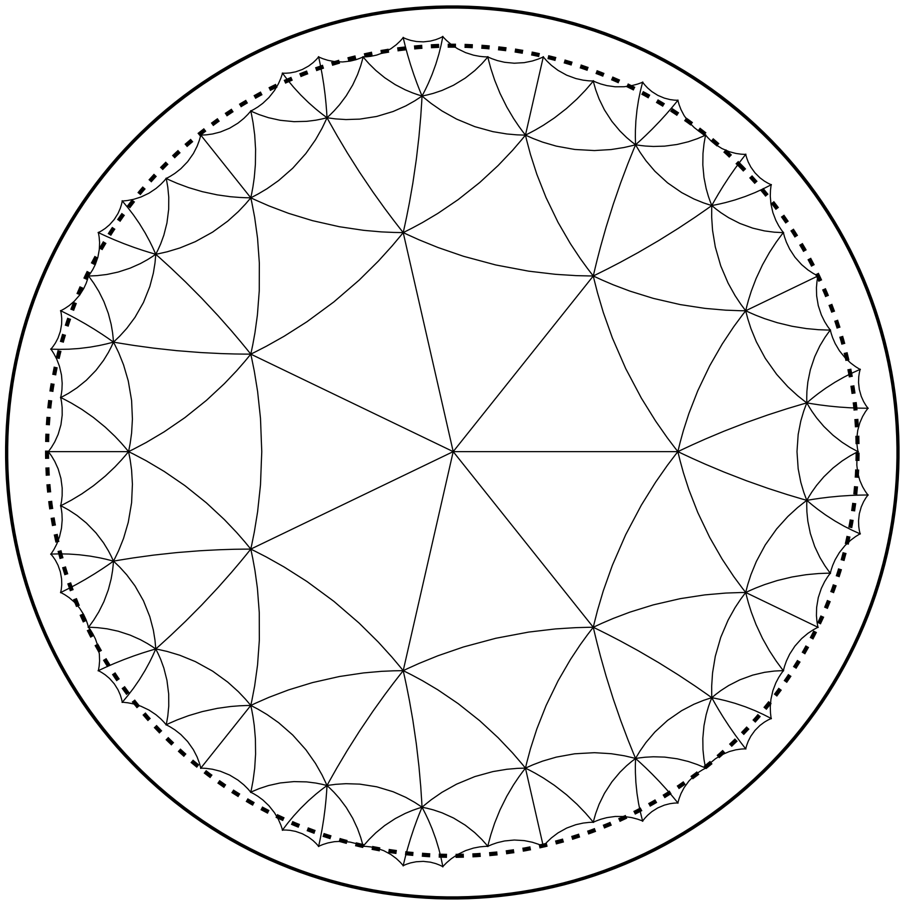
and $h_3 = \epsilon[\xi_2]\sqrt{R_{cc}^2 - R^2(0,1,3)}$

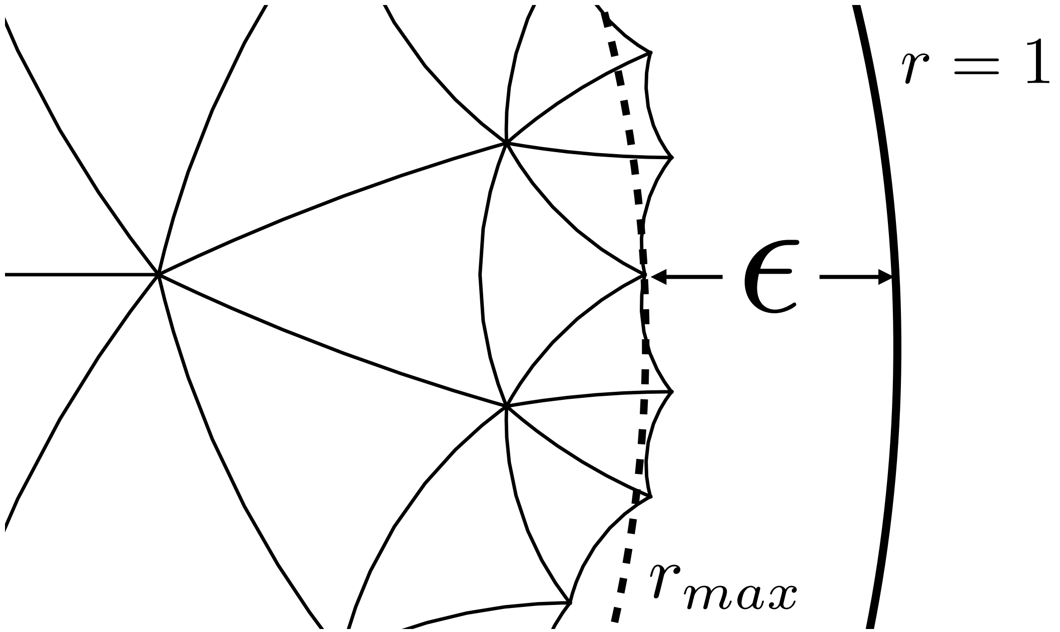
AdS3 Hamiltonian from



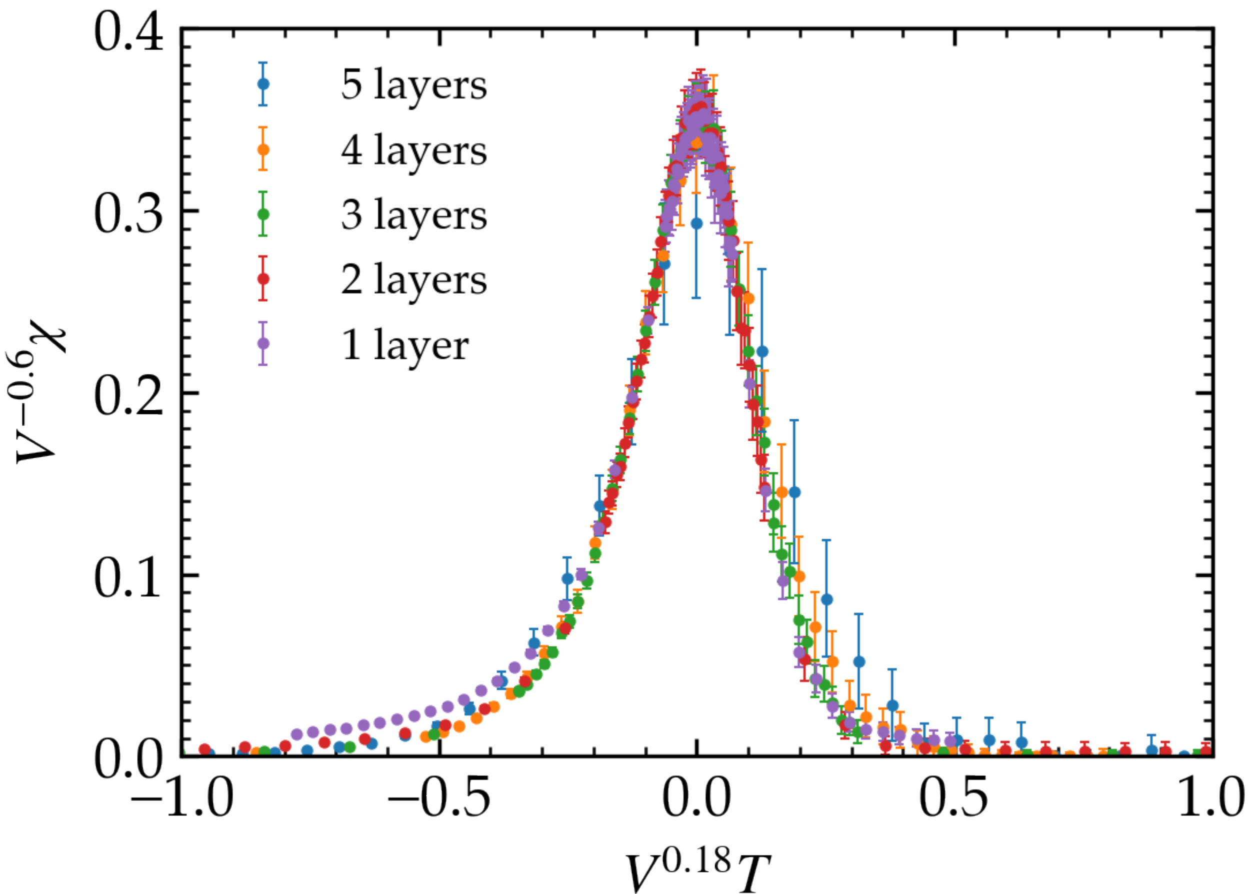


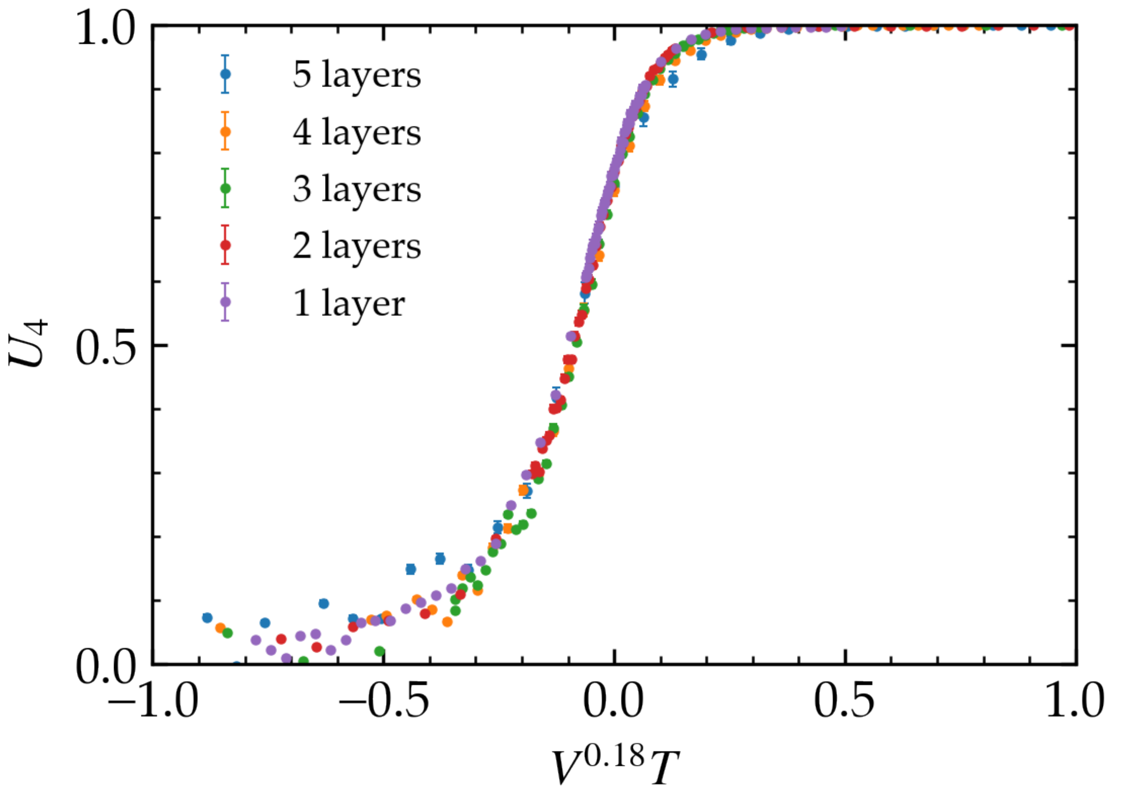
UV cut off problem





Bulk to Boundary Critical Phenomena





THE THEORIST EXPERIMENTAL LAB

Lattice QCD IS BIG SCIENCE: 1/5 EXASCALE Oak Ridge 200,000,000,000,000,000 Floats/sec
9,216 IBM POWER9 CPUs and 27648 NVIDIA GPUs Each GPU has 5120 Cores

BACK UP SLIDES

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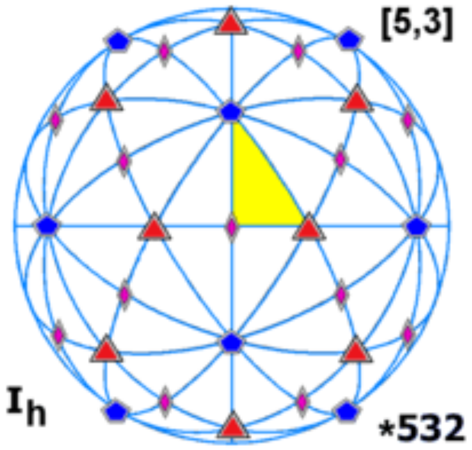
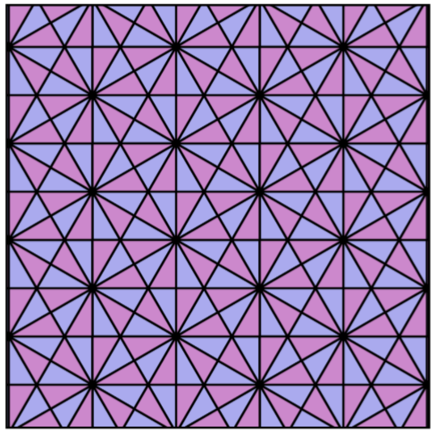
DISCRETE ISOMETRIES & THE TRIANGLE GROUP

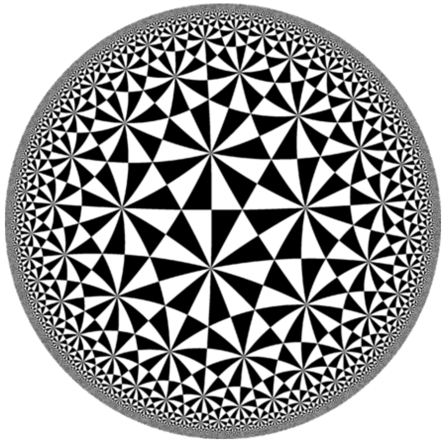
$$\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} \begin{cases} > \pi & \text{Postive curvature} \\ = \pi & \text{Zero curvature} \\ < \pi & \text{Negative Curvature} \end{cases}$$



$(2,3,5)$
120 element
Icosahedral in O(3)



$(2,3,6)$
Triangle Lattice
on Euclidean $\mathbb{R}^2$



$(2,3,7)$
Subgroup of Modular
Group on $\mathbb{H}^2$

https://en.wikipedia.org/wiki/(2,3,7)_triangle_group

A triangle is a 2d primitive simplex

Trick Question: How many different triangles are there?

Rene Descartes :

Geometric Algebra

FEM:

Euclide

Affine Geometry

Finite Elements (FEM) — keeping score! Goggle FEM about 696,000,000 results Google String Theory about 198,000,000 results

Triangle is a 2d primitive simplex

Trick Question: How many different triangles are there?

Rene Descartes: 2x3 = 6 real parameter

Geometric Algebra: 2 real parameter

FEM: 3 real parameter

Euclide 2 real parameter

Affine Geometry 0 real parameter

All primitive Simplexes are Affine Equivalent

Poicare: $d(d+1)/2$ Affine: $d(d+1)$

SVD form, $x = Ax + b \equiv U\Sigma V + b$,

where U, V are rotations and the d diagonal singular values (or shearing parameters) transform circles into ellipsoids.

Finite Elements (FEM) — keeping score! Goggle FEM about 696,000,000 results Google String Theory about 198,000,000 results