The 3D Plaquette Ising Model, subsystem symmetries and fractons

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Plan of talk

Prompted by the talk “Fracton Gauge fields in Curved Space From Higher Dimensions” a few weeks back by Patricio Salgado-Rebolledo (and by making the mistake of emailing Denjoe about it....)

3D Plaquette Ising Model

Gauging and the X-Cube model

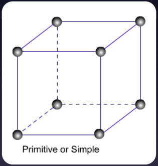
(Very little about) Continuum theories

3D Ising Model

Hamiltonian ($\beta = 1/k_b T$)

$$\mathcal{H} = -\beta \sum_{\langle i,j \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i$$

$\mathbb{Z}_2$ spins (± 1) on vertices of 3D cubic lattice



Objective - evaluate

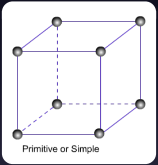
$$Z(\beta, h) = \sum_{\{\sigma\}} \exp(-\mathcal{H})$$

3D Plaquette Ising Model

Hamiltonian

$$\mathcal{H} = -\beta \sum_{\square} \sigma_i \sigma_j \sigma_k \sigma_l$$

$\mathbb{Z}_2$ spins (± 1) on vertices of 3D cubic lattice



Not 3D $\mathbb{Z}_2$ lattice gauge theory

Objective - evaluate

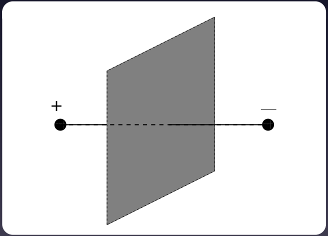
$$Z(\beta) = \sum_{\{\sigma\}} \exp(-\mathcal{H})$$

Spins Cluster Boundaries as Surface Models

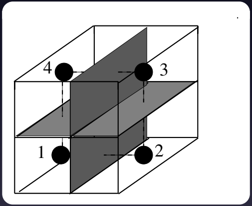
Spin cluster boundaries $\leftrightarrow$ surfaces

Edge spins: $U_{ij} = -1$

Vertex spins: $\sigma_i \sigma_j = -1$



Counting configurations with spins (areas and intersections)



Ising/Surface correspondence

Allow energy from areas, edges and intersections (*A. Cappi, P Colangelo, G. Gonella and A. Maritan*)

$$\beta\mathcal{H} = \sum (\beta_A n_A + \beta_E n_E + \beta_I n_I)$$

$$\beta_A = 2J_1 + 8J_2, \quad \beta_E = 2J_3 - 2J_2, \quad \beta_I = -4J_2 - 4J_3$$

$$\beta\mathcal{H} = -J_1 \sum_{\langle ij \rangle} \sigma_i \sigma_j - J_2 \sum_{\langle\langle ij \rangle\rangle} \sigma_i \sigma_j - J_3 \sum_{\Box} \sigma_i \sigma_j \sigma_k \sigma_l$$

Gonihedric $\rightarrow$ Tune Out Area Term

One parameter family of "Gonihedric" Ising models (Savvidy, Wegner)

$$\mathcal{H}^{\kappa} = -2\kappa \sum_{\langle i,j \rangle} \sigma_i \sigma_j + \frac{\kappa}{2} \sum_{\langle\langle i,j \rangle\rangle} \sigma_i \sigma_j - \frac{1-\kappa}{2} \sum_{\Box} \sigma_i \sigma_j \sigma_k \sigma_l$$

No area term

$$\beta_A = 2J_1 + 8J_2 = 4\kappa - 4\kappa = 0$$

$$\mathcal{H}^{\kappa=0} = -\frac{1}{2} \sum_{\Box} \sigma_i \sigma_j \sigma_k \sigma_l$$

Plaquette Ising/Gonihedric model

Hamiltonian

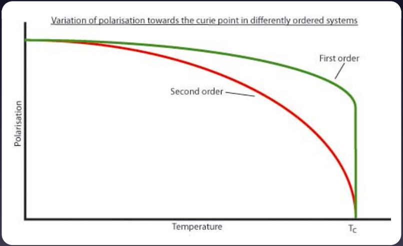
$$\mathcal{H} = -\frac{1}{2}\sum_{\square}\sigma_i\sigma_j\sigma_k\sigma_l$$

Spins at vertices of 3D cubic lattice

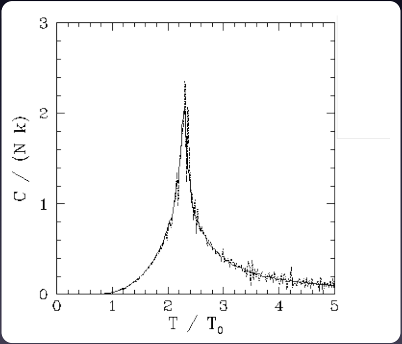
Strong first order phase transition (so no use for continuum limits)

First and Second Order Transitions - Characteristics

First order - discontinuities in magnetization, energy (latent heat)



Second order - divergences in specific heat, susceptibility



Second Order Transitions - Critical exponents

(Continuous) Phase transitions characterized by critical exponents

Define $t = |T - T_c|/T_c$

Then in general, $\xi \sim t^{-\nu}$, $M \sim t^{\beta}$, $C \sim t^{-\alpha}$, $\chi \sim t^{-\gamma}$

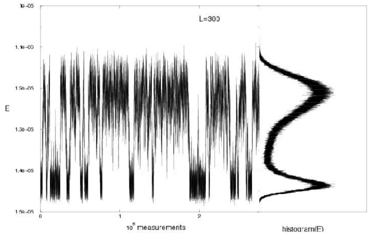
Can be rephrased in terms of the linear size of a system L

$\xi \sim L$, $M \sim L^{-\beta/\nu}$, $C \sim L^{\alpha/\nu}$, $\chi \sim L^{\gamma/\nu}$

First Order Transition: Heuristic two-phase model

A fraction W_{o} in q ordered phase(s), energy e_{o}

A fraction $W_{\mathrm{d}} = 1 - W_{\mathrm{o}}$ in disordered phase, energy e_{d}



1st Order FSS: Energy moments

Energy moments become

$$\langle e^n \rangle = W_\text{o} e_\text{o}^n + (1 - W_\text{o}) e_\text{d}^n$$

And the specific heat then reads:

$$C_V(\beta, L) = L^d \beta^2 \left(\left\langle e^2 \right\rangle - \langle e \rangle^2 \right) = L^d \beta^2 W_\text{o} (1 - W_\text{o}) \Delta e^2$$

Max of $C_V^\text{max} = L^d \left(\beta^\infty \Delta \hat{e}/2\right)^2$ at $W_\text{o} = W_\text{d} = 0.5$

where $\Delta e = e_d - e_o$, β^∞ "real" infinite volume transition point.

Volume scaling

1st Order FSS: Peak shifts

Probability of being in any of the states

$$p_{\mathrm{o}} \propto e^{-\beta L^d \hat{f}_{\mathrm{o}}} \text{ and } p_{\mathrm{d}} \propto e^{-\beta L^d \hat{f}_{\mathrm{d}}}$$

Time spent in the ordered states $\propto q p_{\mathrm{o}}$

$$W_{\mathrm{o}} / W_{\mathrm{d}} \simeq q e^{-L^d \beta \hat{f}_{\mathrm{o}}} / e^{-\beta L^d \hat{f}_{\mathrm{d}}}$$

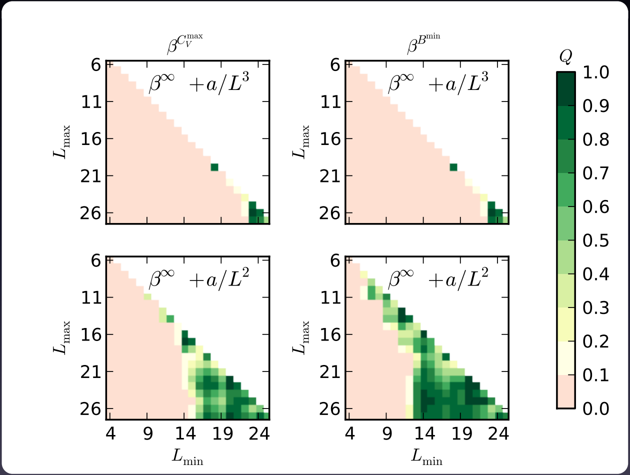
Take ln, expand around β^{∞}, $(W_o \sim W_d)$

$$0 = \ln q + L^d \Delta \hat{e} (\beta - \beta^{\infty}) + \dots$$

Solve for specific heat peak

$$\beta^{C_V^{\max}}(L) = \beta^{\infty} - \frac{\ln q}{L^d \Delta \hat{e}} + \dots$$

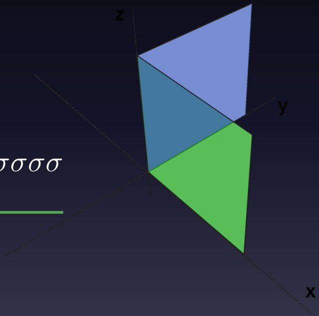
Scaling oddity (Janke/Mueller)



Standard $1/L^3$ gives much poorer quality

Anisotropic plaquette model - "Fuki-Nuke"

$$H_{\text{fuki-nuke}}(\{\sigma\}) \;=\; -J_x \sum_{\Box_{yz}} \sigma\sigma\sigma\sigma \;\; -J_y \sum_{\Box_{zx}} \sigma\sigma\sigma\sigma$$



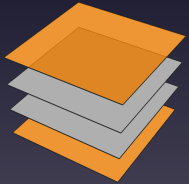
Three dimensional plaquette model: free boundaries in z-direction

Spin-bond-transformation in z-direction

$$\tau_{x,y,z} = \sigma_{x,y,z}\sigma_{x,y,z+1}$$

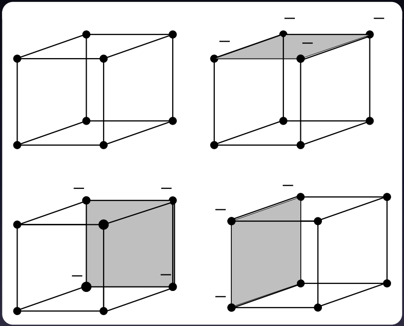
partition function factorises:

$$H_{\text{fuki-nuke}}(\{\tau\}) = -\sum_{x=1}^{L}\sum_{y=1}^{L}\sum_{z=1}^{L_z-1}\left(\tau_{x,y,z}\tau_{x+1,y,z} + \tau_{x,y,z}\tau_{x,y+1,z}\right)$$



$$\begin{aligned} Z_{\text{fuki-nuke}} &= \sum_{\{\tau\}} \exp\left(-\beta H_{\text{fuki-nuke}}(\{\tau\})\right) \\ &= 2^{L^2}\left(Z_{2d\ \text{Ising}}\right)^{L_z-1} \end{aligned}$$

Groundstate



Persists into low temperature phase: degeneracy 2^{3L}

Aside: Duality

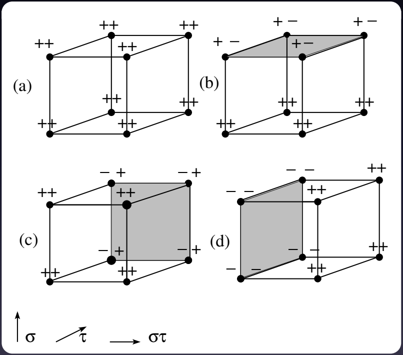
Look at dual of 3D plaquette Ising

$$
\begin{aligned}
Z(\beta) &= \sum_{\{\sigma\}} \exp(-\mathcal{H}) \\
&= \sum_{\{\sigma\}} \prod_{\Box} \cosh(\beta)\left[1 + \tanh(\beta)\left(\sigma_i\sigma_j\sigma_k\sigma_l\right)\right] \\
&= \left[2\cosh(\beta)\right]^{3L^3} \sum_{\{S\}} \left[\tanh(\beta)\right]^{n(S)}
\end{aligned}
$$

(sum runs over closed surfaces with an even number of plaquettes at any vertex)

$$
H_{dual} = -\beta^* \left[\sum_{\langle ij \rangle} \sigma_i\sigma_j - \sum_{\langle ik \rangle} \tau_i\tau_k - \sum_{\langle jk \rangle} \sigma_j\sigma_k\tau_j\tau_k \right]
$$

Dual model Groundstate



Aside^2: Duals Galore

There is a thicket of related dual spin models, such as......

$$H_{dual2} = -\sum_{\langle ij \rangle} \sigma_i \sigma_j \mu_i \mu_j - \sum_{\langle ik \rangle} \tau_i \tau_k \mu_i \mu_k - \sum_{\langle jk \rangle} \sigma_j \sigma_k \tau_j \tau_k$$

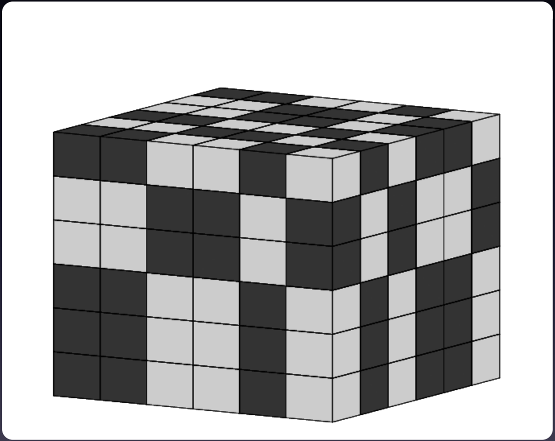
Or

$$\begin{aligned} H_{dual3} &= -\sum_{\langle ij \rangle} \left(\sigma_i U^1_{ij} \sigma_j + \mu_i U^1_{ij} \mu_j \right) - \sum_{\langle ik \rangle} \left(\tau_i U^2_{ik} \tau_k + \mu_i U^2_{ik} \mu_k \right) \\ &- \sum_{\langle jk \rangle} \left(\sigma_j U^3_{jk} \sigma_k + \tau_j U^3_{jk} \tau_k \right) \end{aligned}$$

A decoration/iteration transformation gets you from *dual*3 to *dual*2

$$\sum_{\{U^1_{12}\}} \exp \left[\tilde{\beta} \left(\sigma_1 U^1_{12} \sigma_2 + \mu_1 U^1_{12} \mu_2 \right) \right] = A \exp(\beta \sigma_1 \sigma_2 \mu_1 \mu_2)$$

Groundstate in the large



1st Order FSS with Exponential Degeneracy

Normally q is constant

If instead $q \propto e^L$ ($q = e^{(3\ln 2)L}$), as in Gonihedric model

$$\beta^{C_V^{\max}}(L) = \beta^{\infty} - \frac{\ln q}{L^d \Delta e} + \dots$$

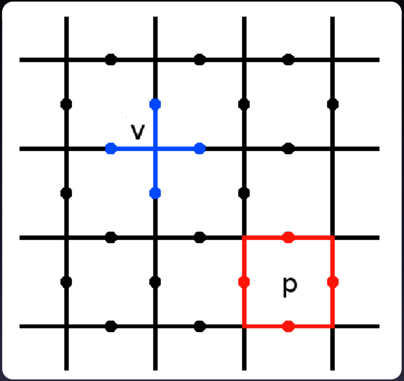
becomes

$$\beta^{C_V^{\max}}(L) = \beta^{\infty} - \frac{3\ln 2}{L^{d-1} \Delta e} + \dots$$

First observation

The 3D plaquette Ising and its dual/duals model has/have a (planar) subsystem symmetry

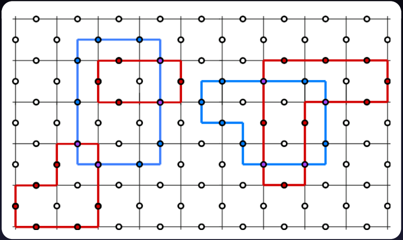
Toric Code (Kitaev)



$$A_v = \prod_{i \in v} \tau_i^x, \quad B_p = \prod_{i \in p} \tau_i^z$$

$$H = -J_v \sum_v A_v - J_p \sum_p B_p$$

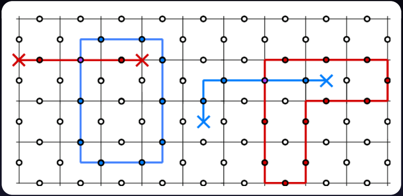
Toric Code: Ground State



$$|\xi_0\rangle = \prod_v \frac{1}{\sqrt{2}} (\mathbb{1}_v + A_v) \underbrace{|0\rangle \otimes \ldots \otimes |0\rangle}_{N_e \text{times}}$$

A “Loop Soup”

Toric Code: Excitations



Defects (i.e. quasiparticles) appear on the end of strings

$$W_e = \prod \tau_z. \quad W_m = \prod \tau_x$$

Braiding excitations reveals anyonic behavior

Toric Code: Excitations (Anyons)

e, m bosonic w.r.t. themselves

Take e for a walk around m, gives -1 phase $\implies$ anyons

Other interesting properties, topological degeneracy of ground state etc

Gauging: From here to there:

Here - (2D) Quantum Transverse Ising:

$$\mathcal{H} = -\beta \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z - h \sum_i \sigma_i^x$$

Gauge the global $\mathbb{Z}_2$ symmetry

$$\mathcal{H} = -\beta \sum_{\langle i,j \rangle} \sigma_i^z \tau_{ij}^z \sigma_j^z - h \sum_i \sigma_i^x - J_p \sum_{\square} \tau_i^z \tau_j^z \tau_k^z \tau_l^z$$

$\beta \to 0$, gauge invariance: $\sigma_i^x \prod_{i \in v} \tau_i^x = 1$

There - Toric Code:

$$\mathcal{H} = -h \sum_v A_v - J_p \sum_p B_p$$

Sketch/caricature of a continuum limit

$$W_e = \prod \tau_z . \quad W_m = \prod \tau_x$$

With $[E, A] = -i$ set

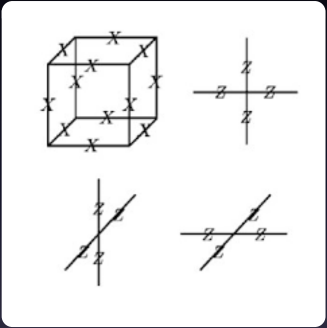
$$W_e \sim \exp(i \int_C A) \ , \quad W_m \sim \exp(i\pi \int_{\bar{C}} E)$$

For $\mathbb{Z}_2$, A " $=$ " $0, \pi$ and E " $=$ " $0, 1$

Then

$$L \sim \int dt\, d^2x\, \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$$

The X-cube Model (Vijay, Haah and Fu)



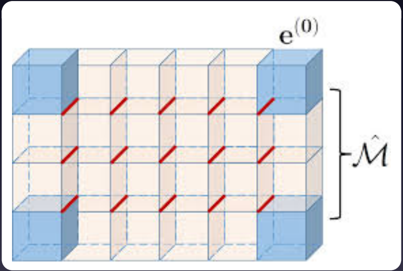
$$A_c = \prod_{i \in \Box} \tau_i^x, \quad B_i^{xy,yz,xz} = \prod_{j \in +,i} \tau_j^z$$

$$H = -J_{\Box} \sum A_{\Box} - J_{xy} \sum_i B_i^{xy} - J_{yz} \sum_i B_i^{yz} - J_{xz} \sum_i B_i^{xz}$$

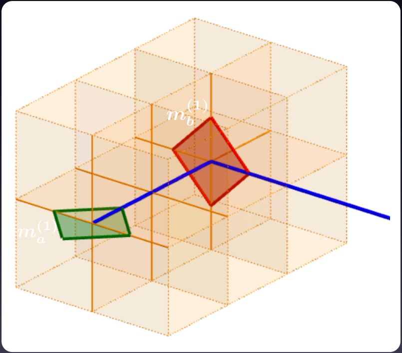
X-Cube: Excitations (Fractons)

“Fracton: Not to be confused with a fracton, the fractal analog of a phonon.”

Electric excitations τ_z



Magnetic excitations τ_x



Pics c/o Vijay, Haah and Fu.

From here to there: Subsystem Symmetry Gauging

Here - Quantum Transverse Plaquette Ising:

$$\mathcal{H} = -\beta \sum_{\Box} \sigma_i^z \sigma_j^z \sigma_k^z \sigma_l^z - h \sum_i \sigma_i^x$$

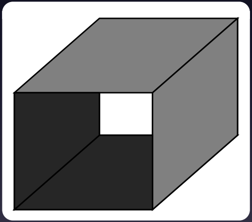
Gauge the $\mathbb{Z}_2$ subsystem symmetry

$$\mathcal{H} \;=\; -\beta \sum_{\Box} \tau_{\Box}^z \sigma_i^z \sigma_j^z \sigma_k^z \sigma_l^z - h \sum_i \sigma_i^x \;+\; \ldots$$

A picture, or two, is worth a thousand words

From here to there: Subsystem Symmetry Gauging II

Equivalent of plaquette flux term in 2D is matchbox (not cube)



Gives $B_i^{xy,yz,xz} = \prod \tau_j^z$ flux terms

From here to there: Subsystem Symmetry Gauging

There (almost)

$$
\begin{aligned}
\mathcal{H} &= -\beta \sum_{\square} \tau^z_{\square} \sigma^z_i \sigma^z_j \sigma^z_k \sigma^z_l - h \sum_i \sigma^x_i \\
&\quad - J_{xy} \sum_i B^{xy}_i - J_{yz} \sum_i B^{yz}_i - J_{xz} \sum_i B^{xz}_i
\end{aligned}
$$

$\beta \to 0$, gauge invariance: $\sigma^x_i \prod_i \tau^x_i = 1$

There

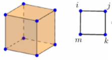
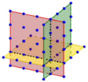
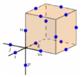
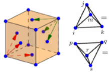
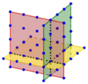
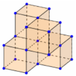
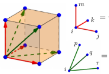
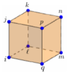
$$
H = -h \sum A_{\cube} - J_{xy} \sum_i B^{xy}_i - J_{yz} \sum_i B^{yz}_i - J_{xz} \sum_i B^{xz}_i
$$

Second (and third) Observation(s)

Gauge global symmetry, get Toric Code, anyons etc

Gauge (this particular) subsystem symmetry, get fractons (reduced mobility quasiparticles)

Fracton zoo (Vijay, Haah and Fu)

Classical Spin System	Subsystem Symmetry	Fracton Topological Phase
 $=\tau_i^z\tau_j^z\tau_k^z\tau_m^z$ **Plaquette Ising Model**	 Planar	 $A_c=\prod_{n\in\partial c}\sigma_n^x$ $B_s^{(xz)}=\sigma_i^z\sigma_p^z\sigma_n^z\sigma_k^z$ $B_s^{(xy)}=\sigma_i^z\sigma_j^z\sigma_k^z\sigma_m^z$ $B_s^{(yz)}=\sigma_j^z\sigma_n^z\sigma_m^z\sigma_p^z$ **X-Cube Model** $\left[\text{Type I}:\ e_a^{(0)},\ m_a^{(1)},\ m_b^{(1)}\right]$
 $=\tau_i^z\tau_j^z\tau_k^z\tau_m^z$ $=\tau_p^z\tau_q^z\tau_r^z\tau_s^z$ **Tetrahedral Ising Model**	 Planar	 $A_c=\prod_{n\in\partial c}\sigma_n^x$ $B_c=\prod_{n\in\partial c}\sigma_n^z$ **Checkerboard Model** $\left[\text{Type I}:\ e_a^{(0)},\ m_a^{(0)}\right]$
 $=\tau_i^z\tau_j^z\tau_k^z\tau_m^z$ $=\tau_i^z\tau_p^z\tau_q^z\tau_r^z$ **Fractal Ising Model**	 Fractal	 $A_c=\mu_i^z\sigma_j^z\mu_k^z\mu_m^z\sigma_n^z\mu_p^z\sigma_p^z\sigma_q^z$ $B_c=\mu_i^x\sigma_j^x\mu_k^x\sigma_\ell^x\mu_\ell^x\mu_m^x\sigma_n^x\mu_p^x\sigma_q^x$ **Haah's Code** $\left[\text{Type II}:\ e_a^{(0)},\ m_a^{(0)}\right]$

Sketch/caricature of a continuum limit

Continuum limit of the X-cube model is a BF theory

$$\mathcal{L} = i\frac{N}{4\pi}\left[A_0\hat{B} + A_{ij}\hat{E}^{ij}\right].$$

where

$$\hat{B} = \partial_i\partial_j\hat{A}^{ij}\ ,\quad \hat{E}^{ij} = \partial_0\hat{A}^{ij} - \partial_k\hat{A}_0^{k(ij)}$$

Defects

$$W = \exp\left[i\int_{-\infty}^{\infty} dt\ A_0(t,x,y,z)\right]$$

$$\hat{W}^z(x,y,\hat{\mathcal{C}}_z) = \exp\left[i\int_{\hat{\mathcal{C}}_z}(dt\hat{A}_0^{z(xy)} + dz\hat{A}^{xy})\right],$$

References

G.K. Savvidy and F.J. Wegner: *Geometrical String and Spin Systems*, Nucl. Phys. B **413**, 605 (1994).

S. Vijay, J. Haah and L. Fu: *Fracton Topological Order, Generalized Lattice Gauge Theory and Duality*, Phys. Rev. **B94** (2016) 235157

R. M. Nandkishore and M. Hermele: *Fractons*, Annual Review of Condensed Matter Physics, 10, 295-313 (2019) [arXiv:1803.11196]

Xie Chen, Han Ma, Michael Pretko, Kevin Slagle, Nathan Seiberg, Shu-Heng Shao........

Gerbes?

(Abelian) Gauge theory, one forms:

$$S = \frac{1}{4g^2} \int d^4 x F_{\mu\nu} F^{\mu\nu}$$

$$F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$$

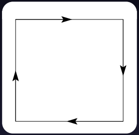
(Abelian) Gerbe theory, two forms:

$$S = \frac{1}{4g^2} \int d^d x \; H_{\mu\nu\lambda} H^{\mu\nu\lambda}$$

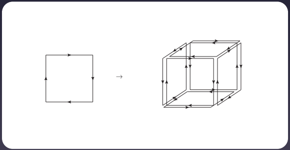
$$H_{\mu\nu\lambda} = \partial_{[\mu} B_{\nu\lambda]}$$

Lattice Gauge/Gerbe Theory

Gauge - Hamiltonian defined on *plaquettes*



Gerbe - Hamiltonian defined on *cubets*



Higher Abelian Gauge theory

General Framework

$$H = -\sum_{C_{n+1}} \left(\prod_{C_n \in \partial C_{n+1}} U(C_n) + c.c. \right)$$

$U(C_n) = \exp(iA(C_n))$ live on the boundaries C_n of cells C_{n+1}

Hamiltonian given by the sum of products of the $U(C_n)$ around the boundary of a C_{n+1}

$\mathbb{Z}_2$ Gauge theory

Many of the properties are visible already in simplest $\mathbb{Z}_2$ case

Symmetries, observables (loops)

$$\Gamma(L) = \left\langle \prod_{C_1 \in L} U(C_1) \right\rangle$$

If a confining transition exists

$$\Gamma(L) \sim \begin{cases} \exp(-A(L)) & \beta < \beta_c \\ \exp(-P(L)) & \beta > \beta_c \end{cases}$$

$\mathbb{Z}_2$ Gerbe theory

Play same game with Gerbe theory

Symmetries, observables (surfaces)

$$\Gamma(S) = \left\langle \prod_{C_2 \in S} U(C_2) \right\rangle$$

If a confining transition exists

$$\Gamma(S) \sim \begin{cases} \exp(-V(S)) & \beta < \beta_c \\ \exp(-A(S)) & \beta > \beta_c \end{cases}$$

So Far, so general

We can use Wegner's results on duality for generalized Ising models (1971) to say more

Lattice N d-dimensional hypercubes

$M_{d,n}$ model, $N_s = \binom{d}{n-1} N$ spins sited at the centres of the $(n-1)$-dimensional hypercubes

Hamiltonians H_{dn}, product of $2n$ spins on the $(n-1)$-dimensional faces of the $N_b = \binom{d}{n} N$ n-dimensional hypercubes.

$M_{d,3}$ are lattice Gerbe theories