Entropy of Supersymmetric Black Holes

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Abstract

We present our findings on black hole microstates in supersymmetric black holes. These results were obtained by calculating the Fourier coefficients of the inverse Siegel modular form of the twisted elliptic genus of $K3$. The mathematical background needed is discussed in the first three sections, with a historical introduction, followed by a review of the current literature of black holes and finally, an analysis of partition functions of bosons and fermions. We then use our results to discuss the elliptic genus of $K3$, a concept that we then used to calculate the Fourier coefficients of the inverse Siegel modular form, which ultimately granted us access to the black hole microstates, giving us the degeneracies.

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The purpose of life is the investigation of the sun, the moon, and the heavens.

- Anaxagoras

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Chapter 1

Introduction

One of the main goals in physics is to arrive at the *Grand Unified Theory* (GUT), commonly referred to as the “holy grail” of physics. A GUT would provide an explanation to phenomena at very large length scales, and very small length scales. A GUT is yet to be found due to the difficulty of *quantizing gravity*. There are several theories that currently address this issue, and in this report we will focus on one of these theories, *string theory*. We will use concepts in string theory to understand phenomena surrounding black holes. But before we do, it is worth discussing the physics that led us to string theory.

Classical and Quantum Mechanics

One of the first building blocks of the understanding of the universe was *classical mechanics*. This is the study of motion and interaction of everyday objects. Examples include; kinematics, momentum, statics, etc. This theory is in line with our intuition and has been studied since the time of the Greeks. One of the most famous examples of a classical mechanical theory is the *theory of gravity* by Sir. Issac Newton. He published this theory in *Philosophiæ Naturalis Principia Mathematica* [1] in 1687, where he argued that two masses attract each other at a rate inversely proportional to the square of the distance. It was in this theory where the equation

$$F = \frac{GMm}{r^2} \tag{1.1}$$

was discovered. This equation is known as *Newton's universal law of attraction*, and describes the force F between two bodies with masses M and m, that are separated by a distance r. G is known as the *gravitational constant*, with value $G = 6.6 \times 10^{-11} m^3 kg^{-1} s^{-2}$. These, as well as other theories related to gravity became known as *Newton's laws*. Although being a relatively easy theory and accurate in its measurements, Newton's laws are not the full story, and it took scientists many years to arrive to this conclusion. Classical mechanics fails when trying to explain relativistic situations (objects moving at an ambient percentage of the speed of light). Newton's theory also failed to explain phenomena such as the perihelion of Mercury, suggesting that the theory of gravity was incomplete.

By the twentieth century, it was believed that physics was completed, with Lord Kelvin claiming at the British Association for the Advancement of Science in 1900 that “there is nothing new to be discovered in physics now. All that remains is more and more precise measurement”. This bold claim was quickly challenged, when in that same year, Max Planck introduced the concept of a *photon*, a discrete or *quanta* of energy to explain blackbody radiation. This idea was then used by Albert Einstein to explain the photoelectric effect, which won him a Nobel Prize. The photon was also used in Niels Bohr's model of the atom, with the introduction of energy levels and orbitals. This seemed to explain the contradiction that had existed prior to this regarding the electron spiralling into the nucleus of the atom. Louis De Broglie introduces the concept of *wave-particle duality*, the idea that particles exhibit both particle-like properties and wave-like properties. This concept was received with backlash as it went against all our intuition and basic understanding of physics. Wave-particle duality was then reaffirmed by the Schoridinger equation, given by

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r},t) + V(\mathbf{r})\psi(\mathbf{r},t), \tag{1.2}$$

for a given *wavefunction* ψ, where $\mathbf{r}$ is the position vector, $V(\mathbf{r})$ is the potential and ∇ denotes the spatial derivatives.

Further studies continued to produce wilder and wilder ideas. Max Born showed that the square of the Schrödinger equation was a probability density of the location of a particle.

$$p(\mathbf{r},t) = |\psi(\mathbf{r},t)|^2. \tag{1.3}$$

All these theories were combined into a branch of physics that is now known as *quantum mechanics*. But why does quantum mechanics differ so much from classical mechanics? This can be explained by the mathematics of *commutator relations*. The commutator of two objects can be defined as

$$[A,B] = AB - BA. \tag{1.4}$$

If the commutator is equal to zero, then it is said that the two objects commute. Numbers are a prime example of objects that commute. In classical mechanics, observables such as position and momentum commute i.e.

$$[x,p] = xp - px = 0. \tag{1.5}$$

In quantum mechanics, the momentum observable becomes an operator $\hat{p} = -i\hbar\nabla$ and likewise position also becomes an operator $\hat{x}$. The commutator relation is then given by

$$[\hat{x},\hat{p}] = [\hat{x}, -i\hbar\nabla] = i\hbar. \tag{1.6}$$

A connection between the commutators between that of classical mechanics and quantum mechanics was found. Paul Dirac was the first to note that a connection between classical observables f, g and their quantum operators $\hat{f}, \hat{g}$ by

$$[\hat{f},\hat{g}] = i\hbar\widehat{\{f,g\}} \tag{1.7}$$

where $\{f,g\}$ is the *Poisson bracket* and is defined by

$$\{f,g\} = \sum_{i=1}^{N}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right) \tag{1.8}$$

for q_i, p_i holding the following relations, where $\mathcal{H}$ is a Hamiltonian of the system

$$\frac{dq}{dt} = \frac{\partial\mathcal{H}}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial\mathcal{H}}{\partial q}. \tag{1.9}$$

Classical and Quantum Field Theory

Up until the turn of the twentieth century, it was believed that Newton's point mass particle were fundamental in the understanding of nature. The overall behaviour of a larger system could in theory be predicted by an understanding of these point mass particles [2]. This way of thinking however, was not successful in explaining many concepts such as: electromagnetism, fluid dynamics and entropy. Ideas began to change when people began thinking of systems as *fields*. A field is a physical quantity at every point in space and time $(\mathbf{x},t)$. A field also has an infinite number of co-ordinates or *degrees of freedom* [3]. Theories which use the premise of fields are known as *field theories*. Field theories are the continuous version of a discrete theory. There are several ingredients one needs to establish for a fulfilling field theory. One must first find defining features of the system that will always remain true. These features are known as *constitutive equations*, and are usually a relation between two physical quantities. One then needs to describe some basic principles in integral form where the variable changes smoothly by the use of *field equations*. One must also define discontinuities at the boundary, which are known as *jump conditions*. Since jump conditions play a significant factor in reaching a field theory, one must also account for *boundary conditions*. These four ingredients provide the foundation needed to begin working on a field theory. The dynamics of a field are governed by the *Lagrangian*, which is a function of a scalar field $\phi(\mathbf{x}, t)$ and is given by [4]

$$L(t) = \int \mathcal{L}(\phi_a, \partial_\mu \phi_a). \tag{1.10}$$

The index a may label components of the same field, and $\mathcal{L}$ being the *Lagrangian density*. To introduce time, we must use the *action* of a system, which can tell us how the system has changed over time. The action is defined as

$$S(t) = \int_{t_1}^{t_2} L(t) dt. \tag{1.11}$$

In theory there is nothing preventing the Lagrangian from being dependent on higher derivatives $(\nabla^2\phi_a, \nabla^3\phi_a, \nabla^4\phi_a, ...)$. However with the introduction of *symmetries*, the Lagrangian is constrained to being dependent on $\nabla\phi_a$ [5]. Most of Newtonian point mass mechanics can be interpreted as a system of fields, a concept that we will explore later, with a more refined theory of gravity.

There is an obvious question arises from field theories: can we quantize field theories? Can we construct a theory whereby we can incorporate quantum mechanics, a theory which successfully tackled many problems in atomic physics, with field theory, a new and better approach to classical mechanics which also encapsulates many other physical theories, together to build a structure that describes all aspects of nature. Yes! These theories have been developed and have been elegantly called *quantum field theories* (QFT's). The earliest of these is *quantum electrodynamics* (QED) which is a field theory of the interaction of electrically charged particles by means of exchanging photons, and has been dubbed "the archetype of quantum field theory" [6]. But what are the main differences between QFT's and quantum mechanics?

A difference we see is in the commutation relations. If we take the QFT commutator of the position $\phi(\mathbf{x})$ and its momenta, known as the *canonical momenta* $\Pi(\mathbf{x}')$ we get [7]

$$[\phi(\mathbf{x}), \Pi(\mathbf{x}')] = \delta(\mathbf{x} - \mathbf{x}'), \tag{1.12}$$

where $\delta(x)$ represents the *Dirac-Delta function*. We must also find the analog of the Schrödinger equation. Paul Dirac was one of the first physicists to tackle the problem. He started by looking at the Klein-Gordon equation, given by [8]

$$\partial_\mu \partial^\mu \psi + m^2 \psi = 0. \tag{1.13}$$

We note that

$$\partial_\mu \partial^\mu \psi := \frac{\partial^2 \psi}{\partial t^2} - \nabla^2 \psi.$$

This however, has several issues since the solution yields negative energies, which imply negative probability densities, which don't exist. Dirac then turned his attention to the energy-momentum relation [9]. The energy-momentum relation in covariant form is given by

$$p_\mu p^\mu - m^2, \tag{1.14}$$

where p^μ is called the 4-*momentum* and is defined to be $p^\mu = (E, p_x, p_y, p_z)$, and

$$p_\mu p^\mu = p_0^2 - \mathbf{p} \cdot \mathbf{p}.$$

Dirac tried to rewrite the equation to give

$$p^\mu p_\mu - m^2 = (\beta^\kappa p_\kappa + m)(\gamma^\lambda p_\lambda - m), \tag{1.15}$$

where κ and λ run from 0 to 3. By choosing $\gamma^\kappa = \beta^\kappa$, we get

$$p^\mu p_\mu - m^2 = \gamma^\kappa \gamma^\lambda p_\kappa p_\lambda - m^2. \tag{1.16}$$

The only way to make the RHS equal the LHS is to have

$$(\gamma_0)^2 = 1, \quad (\gamma_1)^2 = (\gamma_2)^2 = (\gamma_3)^2 = -1.$$

Momentum also becomes an operator $p_\mu \to i\partial_\mu$, where

$$\partial_\mu = (\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}).$$

This leads us to the Dirac equation

$$(i\partial_\mu - m)\psi = 0. \tag{1.17}$$

This is the analog of the Schrödinger equation and allows for relativistic effects, such as high velocities or high energies. QFT is therefore a more generalised version of QM, and has proved its power in many subatomic theories.

General Relativity

In 1907, Einstein wrote a letter to a friend Conrad Habicht, talking about how he was working on a theory of gravity that could understand the issue of the ever changing perihelion of Mercury, a phenomenon that Newton's theory could not explain. It took Einstein another eight years before he fully developed his new theory of gravity. This theory would become to be known as *general relativity* (GR), and would become one of the biggest paradigm shifts in scientific history. According to GR, the universe is made of a curved 4-dimensional manifold known as *spacetime*. The four dimensions arise from three space dimensions and one time dimension. The bending and warping of said spacetime is what gives rise to gravity.

The underlying language of GR is known as *differential geometry*, and is the branch of mathematics that deals with these manifolds, or curved surfaces. In differential geometry, curved spaces can be defined by infinitesimal distance between two points $x + dx$. This distance is known as a *line element* and given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{1.18}$$

where $g_{\mu\nu}$ is called a *metric* and is a 4×4 symmetric matrix ($g_{\mu\nu} = g_{\nu\mu}$). In GR, the geometry of spacetime changes with position, and so the metric can be thought of as a tensor field. Einstein realised that the gravitational field could be embodied in the spacetime manifold M and the metric $g_{\mu\nu}$. This also implies that since M is a spacetime, the signature of the metric must be $(+, -, -, -)$.

In order to have a workable theory, one needs to establish a connection between tangent spaces on the manifold. This connection is known as the *Levi-Civita connection* which states that on any Riemannian manifold, there exists a unique affine connection $\Gamma^\rho_{\mu\nu}$ that is torsion-free and preserves the given metric. The Levi-Civita connection is given by

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right), \tag{1.19}$$

where $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$ and ∂_μ denotes $\partial/\partial x^\mu$. But what exactly is a connection?

A connection, is a mapping of the Cartesian product of all possible vector fields $\Omega(M)$, with manifold M, to itself $\Omega(M) \times (M) \to (M)$. The connection is defined by

$$\nabla_{e_\rho} e_\nu = \Gamma^{\rho}{}_{\mu\nu} e_\nu \tag{1.20}$$

with ∇_X being the covariant derivative. The connection also satisfies the following properties for vector fields X, Y and Z [10]:

$$\begin{aligned} \nabla_X(Y + Z) &= \nabla_X Y + \nabla_X Z, \\ \nabla_{(fX+gY)} Z &= f\nabla_X Z + g\nabla_Y Z, \\ \nabla_X(fY) &= f\nabla_X Y + (\nabla_X f)Y, \end{aligned}$$

for functions f, g.

From the Levi-Civita connection, we need both a torsion tensor and a curvature tensor to have a workable theory. Suppose ϕ is a smooth scalar field. One can show [11] that there exists a tensor $T^\rho_{\mu\nu}$ such that

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\,\phi = T^\rho_{\mu\nu} \nabla_\rho \phi, \tag{1.21}$$

where

$$\nabla_b V^a = \partial_b V^a + \Gamma^a_{bc} V^c, \tag{1.22}$$

and V^c is a covariant vector field. $T^{\rho}{}_{\mu\nu}$ is called the *torsion tensor* and if $T^{\rho}{}_{\mu\nu} = 0$, then it is said that the connection is *torsion-free*.

We now wish to introduce a tensor that accounts for curvature. Like we previously stated: curvature is analogous to gravity. It must then follow that the curvature tensor is analogous to some gravitational field. This tensor is called the *Riemann tensor* $R^{\sigma}{}_{\mu\nu\rho}$ and is defined as [11]

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\,V_\rho - T^{\sigma}{}_{\mu\nu} \nabla_\sigma V_\rho = -R^{\sigma}{}_{\mu\nu\rho} V_\sigma, \tag{1.23}$$

for a smooth vector field V_ρ. We can define other tensors from the Riemann tensor. One of these is called the *Ricci tensor* and is defined as

$$R_{\mu\nu} = R^\rho_{\mu\rho\nu}. \tag{1.24}$$

The Ricci tensor provides the measurement of a local curvature in a given direction and helps understand the changing of volume of a region of space. We also note that from the Bianchi identity, the Ricci tensor is symmetric:

$$R_{\mu\nu} = R_{\nu\mu}. \tag{1.25}$$

We can also construct a function R, known as the *Ricci scalar* defined as

$$R = g^{\mu\nu} R_{\mu\nu}, \tag{1.26}$$

which is a measure of curvature at a given point on the manifold.

GR, like most other physical theories is *co-ordinate invariant* [12], meaning that the laws of physics do not depend on a co-ordinate system so long as the reference frame is inertial. This implies that GR is also *diffeomorphism invariant*. A diffeomorphism is an isomorphism of a smooth manifold. If certain properties remain unchanged after the diffeomorphism, then those properties are said to be diffeomorphism invariant. A diffeomorphism is said to be a symmetry ϕ on a tensor T, if

$$\phi T = T. \tag{1.27}$$

In GR, the most important symmetries are those applied to the metric ($\phi g_{\mu\nu} = g_{\mu\nu}$). This can also be interpreted as preserving distance [13]. A diffeomorphism which holds this property is called an *isometry* [12]. A *Killing vector field* is defined to be a vector field V^μ, which generates one-parameter family isometries. Therefore it follows that

$$\nabla(_\mu V_\nu) = 0. \tag{1.28}$$

Killing vectors imply conserved quantities associated with the motion of free particles [12].

At the heart of GR lies one of the most important equations in all of science. Known as the *Einstein field equations*, they describe the interaction between matter and gravitation and are given by [11]

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{1.29}$$

where Λ is the *cosmological constant*. The nature of Λ is a subject of much debate in theoretical physics. Einstein initially introduced this term to allow the equations to satisfy a static universe. The static universe idea was disproved by Hubble in the 1920's and Einstein said it was his "biggest blunder". Current research suggests that $\Lambda > 0$, however it remains an open question in cosmology.

Black Holes

Karl Schwarzschild found a solution to Einsteins equations the following year. The solutions are constructed from the field equations in a vacuum, meaning that the gravitational field is outside a spherically symmetric static body (these properties will be discussed later). The solution yielded the following line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)^1, \tag{1.30}$$

where G is the universal gravitational constant, M is the mass of the body, and polar co-ordinates are used (r, θ, ϕ). He noticed however, that his solution became undefined at $r = 0$, and $r = 2M$, with the latter being later called the *Schwarzschild radius* and is denoted by r_s. The discovery of this radius lead to research being done on these celestial bodies. These bodies would later become to known as *black holes*. Many discoveries were made shortly after, like the *event horizon*: a boundary of region in spacetime whereby no signals can be received from an observer at infinity. In essence, the event horizon is an area whereby the gravitational effects of the black hole are so strong that light (signals) cannot escape.

Further work from Chandrasekhar and Oppenheimer showed that stars massive enough, will collapse under their own weight after the star stops producing energy. These were the first models of black holes, however they did not account for any angular momentum, or quantum effects. The limits that could be achieved from classical mechanics and GR of this model was finalised by some singularity theories from Penrose in the 1960's and some later work by Hawking in the 1970's [14].

Why String Theory?

String theory unifies GR with QFT. String theory might be the theory that *quantizes gravity*. The reason why GR and QFT are hard to unite is because gravity is *non-renormalizable* [8], meaning that the Feynman diagram expansion cannot be expressed in a finite number of terms. Another example of a non-renormalizable theory is the *Fermi weak interaction theory*. This becomes an issue as gravity, more specifically quantum gravity begins to play a bigger role at the *Planck scale* M_p. Therefore being able to quantize gravity is crucial in having a fundamental theory. One way around this is to treat gravity as a particle, which is known as a *graviton*. String theory then makes the graviton, and elementary particles as one-dimensional objects called *strings*, instead of points particles in a quantum field [15]. This not only allows gravity into the theory, but also any other non-renormalizable theory as well. Although this idea seems rather unusual, many branches of physics have appeared from this line of thinking, such as GR and electromagnetism. String theory does however have its issues. This theory introduces many more dimensions that do not appear to be observed in nature. But why study such a seemingly crazy theory?

Like we previously discussed, string theory is able to quantize gravity, which is no easy task. It also produces physics that coincides with the current understanding. String theory also seems to solve the issue regarding the cosmological constant, which we previously discussed [16], and provides answers to open questions for the unification of the Standard Model [17], explaining phenomena such as dark matter. There is also a special type of string theory called *superstring theory* [18], which incorporates aspects of *supersymmetry*, the prevalent framework in describing the Standard Model. The scales to which string theory is studied on are hard to reach experimentally, with some of the most advanced systems in the world, such as the LHC not even coming close to meeting the energy requirements [19]. But then how can we confirm this theory? Although at the moment the best engineering cannot reach these scales, historically many great scientific ideas have always been ahead of their time, so there is no reason to not explore this beautiful area of physics. It's worth noting that there are other experiments whereby string theory can prevail [8].

But how does this address black holes? We have seen that GR breaks down where there is infinite curvature (1/0). A place where this occurs in the universe is at the black hole singularity. String theory introduces *entropy* and *microstates*, which correspond to the overall macroscopic black hole.

[1] We note we will be using the units $G = c = \hbar = k_B = 1$ from herein.

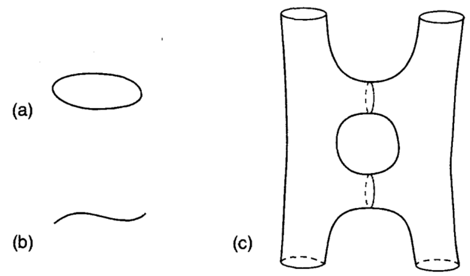

Figure 1.1: Representing the different type of stings. (a) is a closed string, (b) is an open string, (c) is the interaction of two point particles as strings. Source: [20].

These microstates can also be applied to the singularity, which allows us to introduce quantum gravity to the singularities, making the singularity finite in curvature, and providing us a of black holes . The purpose of this report is to study these microstates in the black hole. We will firstly look at *partition functions*, functions which help us count microstates.We will begin by discussing some gravitational phenomena observed in black holes. We will then look at bosonic strings and fermionic string partition functions and show a similarity between them. These similarities arise from superstring theory. We will then study the $K3$ *genus*, a genus used in black hole string theories. We will finally apply all of the acquired knowledge to study the string theory analog of these phenomena - black hole microstates. But why bother?

As we already mentioned, string theory is very hard to "prove" experimentally. Due to this caveat, any problem in physics that can be answered using string theory is deemed a success, and also as reasoning to continue pursuing this theory. In this report, we not only dive into the mathematics involved, but we also highlight the wonders of this theory. It is likely that we will not see any major leaps in experimentation throughout our lifetimes, but that is by no means a reason to avoid such an elegant theory.

Chapter 2

Semi-Classical Black Holes

2.1 Black Hole Thermodynamics

Black hole thermodynamics has been subject to much theoretical work over the span of several decades by some of the world's most brilliant minds and is one of the few theories with little to no experimentation to add to the theory. This theory of black hole mechanics was amplified by Jacob Bekenstein and Stephen Hawking. The two realised great analogies between the laws of thermodynamics and black hole mechanics. It was thought that black holes simply absorbed things and no information was released. Bekenstein argued that this thought process of classical black holes was illogical, as it violated laws of physics. If nothing can escape a black hole, then the black hole would be violating the second law of thermodynamics. Bekenstein realised that this paradox could be saved by proposing that a black hole *has* entropy. He thus began to work on a theory that would unite black hole mechanics and thermodynamics. Before we continue, it is worth reviewing the fundamental laws of thermodynamics:

- Zeroth law: If system A is in thermal equilibrium with system B, and system B is in thermal equilibrium with system C, then system A is in thermal equilibrium with system C.
- First law: The increment in the internal energy of a system is equal the increment of the energy supplied to the system (energy is conserved).
- Second law: The entropy S of a closed system increases or remains constant in time ($\Delta S \geq 0$).
- Third law: The entropy of a system approaches zero as the temperature of the system approaches absolute zero.

Our goal here is to find the black hole equivalences of these laws.

2.1.1 Zeroth Law

As we have just mentioned, the surface gravity on the event horizon is constant throughout. That means, particle A on the horizon will feel the same gravitational attraction as particle B and C. We would like to note that surface gravity is the analog of temperature. Surface gravity is essentially the strength of the gravitational field at the event horizon.

2.1.2 First Law

The first law states that Energy is conserved. This law can be written as

$$dE = TdS + \mu dQ + \Omega dJ, \tag{2.1}$$

for energy E, entropy S, temperature T, charge Q angular momentum Ω, and spin J. Bekenstein showed that for black holes, this relation was given by [21]

$$dM = \frac{\kappa}{8\pi} dA + \mu dQ + \Omega dJ, \tag{2.2}$$

where M is the mass of the black hole, and $\kappa = 1/GM$ is the surface gravity over the event horizon.

If we compare the two equations, we can already begin to see a connection between entropy and area, as well as energy and mass.

2.1.3 Second Law

Bekenstein was the first to propose a connection between entropy and the area of a black hole. The relationship was proved by Stephen Hawking to be

$$S = \frac{A}{4}. \tag{2.1}$$

Hawking also remarkably showed with the *area theorem*, that the area of a black hole never decreases $\Delta A \geq 0$ [22]. This is identical to the second law which states that $\Delta S \geq 0$.

There is also an associated temperature of the black hole, known as the Hawking temperature T_H. This is the temperature that a stationary observer measures from a black hole. The Hawking temperature is given by

$$T_H = \frac{\kappa}{2\pi}. \tag{2.2}$$

The Hawking temperature can be interpreted by the use of the *Rindler co-ordinates* and *Unruh temperature*. The Hawking temperature also appears by use of equations (2.1) and (2.2). For a Schwarzschild radius, $\Omega = \mu = 0$. The equations then become

$$dE = TdS \tag{2.3}$$

$$dM = \frac{\kappa}{8\pi}, \tag{2.4}$$

and since $E = Mc^2 = M$, And $S = A/4$, we find

$$T = \frac{\kappa}{2\pi} = T_H. \tag{2.5}$$

This again strengthens the bond between thermodynamics and black hole mechanics.

2.1.4 Third Law

The third law in essence tells us, that we can never reach absolute zero ($0K$). One would require infinite energy to reach such temperatures. This seems to match our ideas in our theory of black holes. If we recall the equation κ

$$\kappa = \frac{1}{4M}, \tag{2.1}$$

we can see that $\kappa \to 0$, when $M \to \infty$. One therefore requires infinite mass to reach zero surface gravity.

Equations (2.2) and (2.5) hold true for black hole with any mass, charge, and in any number of dimensions. There is a fascination and excitement about the introduction of entropy into black hole theory as it can give us access to the *degeneracy*, d, or total number of *microstates*. A microstate is a possible configuration of the thermal system at a given instant, and the degeneracy is the number of all the possible microstates. Entropy and degeneracy are related by

$$S = \ln(d). \tag{2.2}$$

The degeneracy can provide us information regarding the degrees of freedom of the system. The degrees of freedom are a crucial ingredient for any physical system, and is a good first step in developing a quantum theory of gravity.

2.2 Extremal Black Holes

We now turn our attention to a new type of black hole which resembles the black hole that we will calculate the microstates from. An *extremal* black hole, is a black hole whereby the magnitude of the electric charges Q, and the magnetic charges P is equal to the mass of the black hole ($\sqrt{Q^2+P^2}=M$) [23]. Extremal black holes arise from *Reissner-Nordström* (RN) black holes. These black holes arise as a general solution to the Einstein equations, and have a line element given by [23]

$$ds^2 = -\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}+\frac{P^2}{r^2}\right)dt^2+\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}+\frac{P^2}{r^2}\right)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta\, d\phi^2). \quad (2.1)$$

We would like to highlight that if $Q=P=0$, we recover the Schwarzschild metric (1.30). Similarly to argument made in (1), the solution becomes singular at

$$1-\frac{2M}{r}+\frac{Q^2}{r^2}+\frac{P^2}{r^2}=0. \quad (2.2)$$

This leads to

$$r_\pm = M \pm \sqrt{M^2-Q^2-P^2}. \quad (2.3)$$

These can be interpreted as the inner horizon radius r_- and the outer horizon radius r_+, with the area being dependent on r_+. The surface gravity is defined to be [24]

$$\kappa_\pm = \frac{r_\pm - r_\mp}{2r_\pm^2}. \quad (2.4)$$

We will use κ_+ since we want the work with the surface of the black hole. By using (2.3) we get

$$\begin{aligned}\kappa &= \frac{M+\sqrt{2(M^2-Q^2-P^2)}-(M-\sqrt{M^2-Q^2-P^2})}{2M(M+2\sqrt{M^2-Q^2-P^2}+M^2+Q^2+P^2)} && (2.5)\\ &= \frac{\sqrt{M^2-Q^2}}{2M(M+\sqrt{M^2-Q^2})-Q^2-P^2}. && (2.6)\end{aligned}$$

This means that the Hawking temperature is

$$T_H = \frac{\kappa}{2\pi} = \frac{\sqrt{M^2-Q^2-P^2}}{2\pi(2M(M+\sqrt{M^2-Q^2-P^2})-Q^2-P^2)}. \quad (2.7)$$

The entropy is defined as

$$S = \frac{A}{4} = \frac{4\pi r_+^2}{4} = \pi(M+\sqrt{M^2+Q^2+P^2}). \quad (2.8)$$

We can easily see that at ($\sqrt{Q^2+P^2}=M$), we have no Hawking temperature. Extremal black holes are interesting as they have no Hawking temperature, yet remarkably still carry entropy! We would like to explain this phenomena using string theory and microstates, and we will see in order to carry this calculation out, we will need some supersymmetry.

Chapter 3

Partition Functions in String Theory

3.1 What are Partition Functions?

Partition functions are a concept used to understand thermodynamic systems. In string theory, partition functions give us much information regarding the string, such as the tension, actions, open strings and much more [25]. It is therefore essential to understand partition functions in order to fully understand string theory. The *canonical* partition function Z is given by

$$Z = \Sigma_i e^{\beta E_i}, \tag{3.1}$$

where β is the thermodynamic beta $\beta = 1/T$, for given temperature T, and E_i is the energy for each *microstate*. A microstate is a specific configuration of the states (atoms) involved. The partition function of the microstates can explain the macroscopic properties of the system, such as pressure for a thermodynamic system. We will see in this report that the string theory analog microstates can provide us with much information regarding the macroscopic properties of black holes.

3.2 Bosonic Partition Functions on a Torus

Bosonic strings are a toy model in string theory. Bosonic strings give way to tachyons (particles moving faster than light), which prevents it from being a physical theory [26]. It is nevertheless, a good place to begin working up a theory for the purpose of this report. We begin by analysing bosonic partition functions on a torus. We will use the co-ordinates to be $X = X(\tau, \sigma)$, where τ is the timelike co-ordinate and σ is the spacelike co-ordinate. Before we continue, it is worth discussing the co-ordinate system. (τ, σ) is the parameterization of a *worldsheet*. A worldsheet is a sheet that is traced out by a string in Minkowski space [8]. Similarly to a particle tracing a worldine, a string will trace out a sheet since it has extra dimensions, contrary to a particle being defined by a point.

The free Lagrangian can be written as

$$\mathcal{L} = \frac{1}{4\pi\alpha'} \int \partial_\alpha X \partial^\alpha X d\sigma, \tag{3.1}$$

where α' is associated with the tension of the string. We will assume to be in Minkowski space, and as such, we will use the Lorentzian metric given by [11]

$$ds^2 = d\tau^2 - d\sigma^2. \tag{3.2}$$

By use of this metric, the Lagrangian can be rewritten as

$$L = \frac{1}{4\pi\alpha'} \int (\dot{X}^2 - X'^2) d\sigma. \tag{3.3}$$

where

$$\dot{X} = \frac{\partial X}{\partial \tau}$$
$$X' = \frac{\partial X}{\partial \sigma} \tag{3.4}$$

To calculate the equation of motion, we must use the Euler-Lagrange equation, given by

$$\frac{\partial \mathcal{L}}{\partial X} - \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{X}}\right) - \frac{d}{d\sigma}\left(\frac{\partial \mathcal{L}}{\partial X'}\right) = 0, \tag{3.5}$$

for Lagrangian density

$$\mathcal{L} = \frac{1}{4\pi\alpha'}\left(\dot{X}^2 - X'^2\right). \tag{3.6}$$

We find that

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{X}}\right) = \frac{1}{2\pi\alpha'}\ddot{X} \tag{3.7}$$
$$\frac{d}{d\sigma}\left(\frac{\partial \mathcal{L}}{\partial X'}\right) = -\frac{1}{2\pi\alpha'}X'' \tag{3.8}$$
$$\frac{\partial \mathcal{L}}{\partial X} = 0. \tag{3.9}$$

We then conclude that the equation of motion (EOM) is

$$\frac{1}{2\pi\alpha'}\left(\ddot{X} - X''\right) = \ddot{X} - X'' = 0. \tag{3.10}$$

This can be written in the Einstein convention as

$$\partial_\alpha \partial^\alpha X = 0. \tag{3.11}$$

If we assume a solution of the form

$$X = a(\tau)b(\sigma), \tag{3.12}$$

by using (3.11), we get

$$\ddot{a}(\tau)b(\sigma) = -a(\tau)b''(\sigma) \tag{3.13}$$
$$\frac{\ddot{a}}{a} = -\frac{b''}{b}. \tag{3.14}$$

We can see that these must be equal to a constant k^2 since two functions of independent variables are only equal to one another if they are constant functions. We note that the $-$ sign here is dependent on the a choice of metric.

$$\frac{\ddot{a}}{a} = -\frac{b''}{b} = k^2. \tag{3.15}$$

From here we can construct and solve the two equations

$$\frac{\ddot{a}}{a} = k^2, \quad -\frac{b''}{b} = k^2. \tag{3.16}$$

We can see that these equations have solution

$$a = e^{-k\tau} \quad b = e^{\pm ik\sigma}. \tag{3.17}$$

The solution is therefore

$$X = \alpha_k e^{-k\tau + ik\sigma} + \tilde{\alpha}_k e^{-k\tau + ik\sigma}, \tag{3.18}$$

where α_k and $\tilde{\alpha}_k$ represent the left moving and right moving modes respectively. We must remember that these modes are summed over all k, which gives us

$$X = \sum_{k\neq 0} \alpha_k e^{-k\tau + ik\sigma} + \sum_{k\neq 0} \tilde{\alpha}_k e^{-k\tau + ik\sigma}. \tag{3.19}$$

We also need to account for the special case $k = 0$, the *zero modes.* We see that when $k = 0$, a becomes a function of σ and b becomes a function of τ

$$a = r\sigma + y, \tag{3.20}$$
$$b = p\tau + x, \tag{3.21}$$

with r, y, p and x are constants. After imposing the boundary condition for a closed string $X(\tau, \sigma) = X(\tau, \sigma + 2\pi)$, we see that

$$(r\sigma + y)(p\tau + x) = (r(\sigma + 2\pi))(p\tau + x). \tag{3.22}$$

We can see that this solution works only when $q = 0$. For simplicity, we set $y = 1$. The solution now reads

$$X = i\sqrt{\frac{\alpha'}{2}} \left(\sum_{k\neq 0} \frac{1}{k}\alpha_k e^{-k\tau + ik\sigma} + \sum_{k\neq 0} \frac{1}{k}\tilde{\alpha}_k e^{-k\tau - ik\sigma} \right) + \alpha' p\tau + x, \tag{3.23}$$

where we choose a convenient renormalization factor. The $1/k$ factor is introduced to accommodate for derivatives of X, which we will encounter soon.

We will now use the following commutator relation $[X(\sigma), \Pi(\sigma')] = \delta(\sigma - \sigma')$ [27] which will help us in deriving another useful relation. We will assume equal times and we will set $\tau = 0$. Π is the canonical momentum and is defined as

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{X}}. \tag{3.24}$$

When applied to (3.6), we get

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{X}} = \frac{1}{4\pi\alpha'}(2\dot{X}) = \frac{1}{2\pi\alpha'}\dot{X}. \tag{3.25}$$

From this we get

$$[X(\sigma), \Pi(\sigma')] = \left[X(\sigma),\ \frac{1}{2\pi\alpha'}\dot{X}(\sigma')\right] = \frac{1}{2\pi\alpha'}[X(\sigma), \dot{X}(\sigma')]. \tag{3.26}$$

It then follows that

$$[X(\sigma), \dot{X}(\sigma')] = 2\pi\alpha'\delta(\sigma - \sigma'). \tag{3.27}$$

We note that at $\tau = 0$, we get

$$X = i\sqrt{\frac{\alpha'}{2}} \left(\sum_{k\neq 0} \frac{1}{k}\alpha_k e^{ik\sigma} + \sum_{k\neq 0} \frac{1}{k}\tilde{\alpha}_k e^{-ik\sigma} \right) + x, \tag{3.28}$$

$$\dot{X} = i\sqrt{\frac{\alpha'}{2}} \left(\sum_{k\neq 0} \alpha_k e^{ik\sigma} + \sum_{k\neq 0} \tilde{\alpha}_k e^{-ik\sigma} \right). \tag{3.29}$$

$$\tag{3.30}$$

We now wish to isolate the modes α_k. To do this, we will carry out a Fourier transform on (3.28) and (3.29). This is done by multiplying both sides by $e^{-ik'\sigma}$ and then integrating [28]. From this we get

$$\int Xe^{-ik'\sigma}d\sigma = i\sqrt{\frac{\alpha'}{2}}\int\left(\sum_{k\neq 0}\frac{1}{k}\alpha_k e^{ik\sigma}+\sum_{k\neq 0}\frac{1}{k}\tilde{\alpha}_k e^{-ik\sigma}\right)e^{-ik'\sigma}d\sigma \tag{3.31}$$

$$= i\sqrt{\frac{\alpha'}{2}}\sum_{k\neq 0}\frac{1}{k}\int\left(\alpha_k e^{ik\sigma}+\tilde{\alpha}_k e^{-ik\sigma}\right)e^{-ik'\sigma}d\sigma \tag{3.32}$$

$$= i\sqrt{\frac{\alpha'}{2}}\sum_{k\neq 0}\frac{1}{k}\int\left(\alpha_k e^{i(k-k')\sigma}+\tilde{\alpha}_k e^{-i(k+k')\sigma}\right)d\sigma. \tag{3.33}$$

$$\tag{3.34}$$

We must remember that we are in a closed string and therefore the integrals are bounded from $[0, 2\pi)$, leading us to

$$\int_0^{2\pi} Xe^{-ik'\sigma}d\sigma = i\sqrt{\frac{\alpha'}{2}}\sum_{k\neq 0}\frac{1}{k}\int_0^{2\pi}\left(\alpha_k e^{i(k-k')\sigma}+\tilde{\alpha}_k e^{-i(k+k')\sigma}\right)d\sigma. \tag{3.35}$$

We will use the property of the Dirac-delta function, which tells us that

$$\int_{-\infty}^{\infty} e^{i(a-a')x}dx = 2\pi\delta(a-a'). \tag{3.36}$$

When applied to (3.35), $(-\infty, \infty)$ becomes $[0, 2\pi)$ since we are on a closed string, and the integral comes out to be

$$\int_0^{2\pi} Xe^{-ik'\sigma}d\sigma = i\sqrt{\frac{\alpha'}{2}}\sum_{k\neq 0}\frac{2\pi}{k}\left(\alpha_k\delta(k-k')-\tilde{\alpha}_k\delta(k+k')\right). \tag{3.37}$$

When we sum over the modes we get

$$\int_0^{2\pi} Xe^{-ik'\sigma}d\sigma = i\sqrt{\frac{\alpha'}{2}}\frac{2\pi}{k}\left(\alpha_k-\tilde{\alpha}_k\right). \tag{3.38}$$

A similar argument can be made for $\dot{X}$, leading to

$$\int_0^{2\pi} \dot{X}e^{-ik'\sigma}d\sigma = i\sqrt{\frac{\alpha'}{2}}2\pi\left(\alpha_+\tilde{\alpha}_k\right). \tag{3.39}$$

We can see here that the modes α_k and $\tilde{\alpha}_k$ can be solved to give

$$\alpha_k = \frac{1}{2i\sqrt{\frac{\alpha'}{2}}2\pi}\left(k\int Xe^{-ik\sigma}+\dot{X}\int e^{-ik\sigma}\right)d\sigma \tag{3.40}$$

$$\tilde{\alpha}_k = -\frac{1}{2i\sqrt{\frac{\alpha'}{2}}2\pi}\left(k\int Xe^{-ik\sigma}-\dot{X}\int e^{-ik\sigma}\right)d\sigma \tag{3.41}$$

The Hamiltonian can be constructed by using the following equation:

$$H = \dot{X}\frac{\partial L}{\partial \dot{X}} - L \tag{3.42}$$

The Hamiltonian, by use of (3.3) then becomes

$$\mathcal{H} = \frac{1}{4\pi\alpha'}\int\left(\dot{X}(2\dot{X}) - (\dot{X}^2 - X'^2)\right)d\sigma \tag{3.43}$$

$$\tag{3.44}$$

yielding

$$\mathcal{H} = \frac{1}{4\pi\alpha'}\int\left(\dot{X}^2 - X'^2\right)d\sigma \tag{3.45}$$

We will now find the commutator relation $[\alpha_k, \alpha_{-k}]$, which is given by

$$[\alpha_k, \alpha_{-k}] = \alpha_k\alpha_{-k} - \alpha_{-k}\alpha_k \tag{3.46}$$

We find that

$$\begin{aligned}[\alpha_k, \alpha_{-k}] = &-\frac{1}{8\pi^2\alpha'}\left(k\int X(\sigma)e^{-ik\sigma} + \int \dot{X}(\sigma)e^{-ik\sigma}\right)\left(k\int X(\sigma')e^{-ik\sigma'} + \int \dot{X}(\sigma')e^{-ik\sigma'}\right)\\ &-\left(-\frac{1}{8\pi^2\alpha'}\right)\left(k\int X(\sigma)e^{-ik\sigma'} + \int \dot{X}(\sigma)e^{-ik\sigma'}\right)\left(k\int X(\sigma')e^{-ik\sigma} + \int \dot{X}(\sigma')e^{-ik\sigma}\right)d\sigma d\sigma'\end{aligned} \tag{3.47}$$

This factorizes to

$$\begin{aligned}[\alpha_k, \alpha_{-k}] = -\frac{k}{8\pi^2\alpha'}&\left(k\int\int X(\sigma)X(\sigma') + \int\int X(\sigma)\dot{X}(\sigma') + \int\int \dot{X}(\sigma)X(\sigma') + \int\int \dot{X}(\sigma)\dot{X}(\sigma')\right.\\ &\left.-k\int\int X(\sigma)X(\sigma') + \int\int X(\sigma)\dot{X}(\sigma') - \int\int X(\sigma)\dot{X}(\sigma') - \int\int \dot{X}(\sigma)\dot{X}(\sigma')\right)e^{ik(\sigma'-\sigma)}d\sigma d\sigma'\end{aligned} \tag{3.48}$$

The integrals can be factored out and we recover some commutator relations, as well as some terms cancelling out

$$[\alpha_k, \alpha_{-k}] = -\frac{1}{8\pi^2\alpha'}\int\int\left(k[X(\sigma), \dot{X}(\sigma')] + k[X(\sigma), \dot{X}(\sigma')]\right)e^{ik(\sigma'-\sigma)}d\sigma d\sigma' \tag{3.49}$$

We assume the string to be closed, and therefore the integrals become bounded.

$$[\alpha_k, \alpha_{-k}] = -\frac{1}{8\pi^2\alpha'}\int_0^{2\pi}\int_0^{2\pi} 4\pi k\alpha'\delta(\sigma - \sigma')e^{ik(\sigma'-\sigma)}d\sigma d\sigma' = k \tag{3.50}$$

3.2.1 Dedekind η Function Derivation for a Bosonic String

We begin with the following commutation relations:

$$[\alpha_k, \alpha_{-k}] = k,\ \ [\tilde{\alpha}_k, \tilde{\alpha}_{-k}] = k \tag{3.1}$$

The Hamiltonian $\mathcal{H}$ after restrictions on the modes is defined by [27]

$$\mathcal{H} = \frac{1}{2}\left(\sum(\alpha_k\alpha_{-k}) + \sum(\tilde{\alpha}_k\tilde{\alpha}_{-k}) + \sum(\alpha_{-k}\alpha_k) + \sum(\tilde{\alpha}_{-k}\tilde{\alpha}_k)\right). \tag{3.2}$$

By the use of (3.1), we arrive at

$$\mathcal{H} = \frac{1}{2}\left(\sum([\alpha_k, \alpha_{-k}] + [\tilde{\alpha}_k, \tilde{\alpha}_{-k}] + (\alpha_{-k}\alpha_k) + (\tilde{\alpha}_{-k}\tilde{\alpha}_k)\right). \tag{3.3}$$

The difference between (3.2) and (3.3) is constant. This constant can be understood as a changes in zero energy standards.

We can split $\mathcal{H}$ into it's left and right mover energy levels $\mathcal{H}_{\mathcal{L}}$ and $\mathcal{H}_{\mathcal{R}}$ ($\mathcal{H} = \mathcal{H}_{\mathcal{L}} + \mathcal{H}_{\mathcal{R}}$) by separating α_k and $\tilde{\alpha}_k$ respectively. Therefore the Hamiltonian can be split as follows

$$\mathcal{H}_L = \sum \left(\frac{1}{2}([\alpha_k, \alpha_{-k}]) + \alpha_k \alpha_{-k} \right), \quad \mathcal{H}_R = \sum \left(\frac{1}{2}([\tilde{\alpha}_k, \tilde{\alpha}_{-k}]) + \tilde{\alpha}_k \tilde{\alpha}_{-k} \right). \tag{3.4}$$

We wish to find the partition function $\mathrm{Tr}[e^{-\beta \mathcal{H}}]$, and as such we can split $\mathcal{H}$ into its $\mathcal{H}_{\mathcal{L}}$ and $\mathcal{H}_{\mathcal{R}}$ components

$$\mathrm{Tr}[e^{-\beta \mathcal{H}}] = \mathrm{Tr}[e^{-\beta(\mathcal{H}_{\mathcal{L}} + \mathcal{H}_{\mathcal{R}})}] = \mathrm{Tr}[e^{-\beta_L(\mathcal{H}_{\mathcal{L}})} e^{-\beta_R(\mathcal{H}_{\mathcal{R}})}]. \tag{3.5}$$

We will first look at the $\mathrm{Tr}[e^{-\beta_L \mathcal{H}_{\mathcal{L}}}]$ case.

$$\mathrm{Tr}[e^{-\beta_L \mathcal{H}_{\mathcal{L}}}] = \mathrm{Tr}[\exp(-\beta_L \left(\sum \left(\frac{1}{2}([\alpha_k, \alpha_{-k}]) + \alpha_k \alpha_{-k} \right) \right)]. \tag{3.6}$$

Applying the commutator relation and the rule of indices used in (3.5), we find

$$\mathrm{Tr}[e^{-\beta_L \mathcal{H}_{\mathcal{L}}}] = \mathrm{Tr}[\exp(-\beta_L \sum \frac{k}{2}) \exp(-\beta_L \sum \alpha_k \alpha_{-k})] \tag{3.7}$$

Using the zeta function regularization

$$\sum_{n=1}^{\infty} n = -\frac{1}{12}, \tag{3.8}$$

we get

$$\mathrm{Tr}[e^{-\beta_L \mathcal{H}_{\mathcal{L}}}] = e^{\frac{\beta_L}{24}} \mathrm{Tr}[\exp(-\beta_L \sum \alpha_k \alpha_{-k})] \tag{3.9}$$

We can now see that the second exponential turns into a product of an infinite geometric series

$$\mathrm{Tr}[e^{-\beta_L \mathcal{H}_{\mathcal{L}}}] = e^{\frac{\beta_L}{24}} \prod_k \left(1 + e^{-\beta_L k} + e^{-2\beta_L k} + ...\right). \tag{3.10}$$

This can be calculated using

$$S_\infty = a + ar + ar^2 + ar^3 + ... = \frac{a}{1-r}, \tag{3.11}$$

leading to

$$\mathrm{Tr}[e^{-\beta_L \mathcal{H}_{\mathcal{L}}}] = e^{\frac{\beta_L}{24}} \prod_k \left(\frac{1}{1 - e^{\beta_L k}} \right). \tag{3.12}$$

We can now see that the RHS of this equation, when we let $q = e^{\beta_L}$, we recover the inverse Dedekind η function

$$\mathrm{Tr}[e^{\beta_L} \mathcal{H}_{\mathcal{L}}] = q^{\frac{1}{24}} \prod_k \left(1 - q^k\right)^{-1} = \eta(q)^{-1}. \tag{3.13}$$

The same argument can be made for $\mathrm{Tr}[e^{\beta_R \mathcal{H}_{\mathcal{R}}}]$ giving

$$\mathrm{Tr}[e^{\beta_R} \mathcal{H}_{\mathcal{R}}] = \bar{q}^{\frac{1}{24}} \prod_k \left(1 - \bar{q}^k\right)^{-1} = \eta(\bar{q})^{-1}. \tag{3.14}$$

When combined to form the overall partition function (3.5) we conclude that a bosonic partition function is in the form of a Dedekind η function. We note that since $\bar{q} = e^{\beta_R}$, we get

$$\mathrm{Tr}[e^{\beta \mathcal{H}}] = |\eta(q)|^{-2}. \tag{3.15}$$

As of now, we have only explored periodic boundary condition, namely $X(\tau,\sigma) = X(\tau,\sigma+2\pi)$, however, we can also consider *anti-periodic* boundary conditions: $X(\tau,\sigma) = -X(\tau,\sigma+2\pi)$. If we sub this condition into (3.19) , we find that

$$e^{ik\sigma} = e^{ik(\sigma+2\pi)} = -e^{ik\sigma}e^{ik2\pi} \tag{3.16}$$

$$-1 = e^{2\pi ik}. \tag{3.17}$$

It must follow that k is half integer. For convenience, in the commutator relations we let $k \to k+1/2$ and $k \to k-1/2$, since k is an integer, $k \pm 1/2$ is a half integer. The commutation relations then become

$$[\alpha_{k+1/2}, \alpha_{-k-1/2}] = k + 1/2 \tag{3.18}$$

$$[\alpha_{k-1/2}, \alpha_{-k-+1/2}] = k - 1/2. \tag{3.19}$$

.

The main difference here is the zeta function regularization (3.8). This time the sum will have an extra 1/2 term, which changes the sum [27]

$$\sum_{n=1}^{\infty} n - \frac{1}{2} = \frac{1}{24}. \tag{3.20}$$

This gives

$$\mathrm{Tr}\left[e^{-\beta_L \mathcal{H}_L}\right] = e^{-\frac{\beta_L}{48}} \prod_{k=1}^{\infty} \frac{1}{1-e^{-\beta(k-1/2)}} \tag{3.21}$$

$$\mathrm{Tr}\left[e^{-\beta_R \mathcal{H}_R}\right] = e^{-\frac{\beta_R}{48}} \prod_{k=1}^{\infty} \frac{1}{1-e^{-\beta(k-1/2)}}, \tag{3.22}$$

and when combined together gives

$$\mathrm{Tr}\left[e^{-\beta \mathcal{H}}\right] = e^{-\frac{\beta}{48}} \prod_{k=1}^{\infty} \frac{1}{1-e^{-\beta_L(k-1/2)}} \frac{1}{1-e^{-\beta_R(k-1/2)}}. \tag{3.23}$$

3.3 Fermionic Partition Functions on a Torus

In superstring theory, we wish to describe the dynamics of bosons and fermions under supersymmetry [18]. It is then a natural step to try and derive a partition function for fermionic strings. We begin with the Lagrangian density for a single fermion, with co-ordinates $\psi = \psi(\tau, \sigma)$ and is given by

$$\mathcal{L} = \int \frac{1}{4\pi} \left(\bar{\psi} \partial_\omega \psi + \tilde{\psi} \partial_{\bar{\omega}} \bar{\tilde{\psi}} \right) d\sigma. \tag{3.1}$$

where $\bar{\psi} = \psi^\dagger \gamma^0$ [29] and γ are the *Dirac matrices* which are 4×4 matrices. In two dimensions however, they break down to give the regular 2×2 Pauli matrices and 2×2 identity matrix. We also note that $w = \tau + i\sigma$ and $\bar{w} = \tau - i\sigma$.

Using the two identities for w and $\bar{w}$, we will now solve for ∂_w and $\partial_{\bar{w}}$. By combining the two, we can solve for τ and σ.

$$\tau = \frac{1}{2}(w + \bar{w}) \tag{3.2}$$

$$i\sigma = \frac{1}{2}(w - \bar{w}), \tag{3.3}$$

which gives

$$\partial_\tau = \frac{1}{2}(\partial_w + \partial_{\bar{w}}) \tag{3.4}$$

$$i\partial_\sigma = \frac{1}{2}(\partial_w - \partial_{\bar{w}}). \tag{3.5}$$

Solving for ∂_w and $\partial_{\bar{w}}$, and by use of the chain rule

$$\frac{df}{dw} = \frac{df}{dg}\frac{dg}{dw} + \frac{df}{dh}\frac{dh}{dw}$$

for some $f = f(g, h)$, we get

$$\partial_w = \frac{1}{2}(\partial_\tau - i\partial_\sigma) \tag{3.6}$$

$$\partial_{\bar{w}} = \frac{1}{2}(\partial_\tau + i\partial_\sigma). \tag{3.7}$$

The action is given by

$$S = \frac{1}{8\pi} \int \left(\bar{\psi}\partial_\tau \psi + \bar{\tilde{\psi}} \partial_\tau \tilde{\psi} - i\bar{\psi}\partial_\sigma \psi + i\bar{\tilde{\psi}}\partial_\sigma \tilde{\psi} \right) d\sigma d\tau J, \tag{3.8}$$

where J is a Jacobi transformation and is defined by

$$J = det \begin{bmatrix} \frac{\partial \omega}{\partial \tau} & \frac{\partial \omega}{\partial \sigma} \\ \frac{\partial \bar{\omega}}{\partial \tau} & \frac{\partial \bar{\omega}}{\partial \sigma} \end{bmatrix} = -2i.$$

The equations of motion can be solved by use of the Euler-Lagrange equations. The Lagrangian density is given by

$$\mathcal{L} = \frac{1}{4\pi i} \int \left(\bar{\psi}\partial_\tau \psi + \bar{\tilde{\psi}} \partial_{\tilde{\tau}} \tilde{\psi} - i\bar{\psi}\partial_\sigma \psi + i\bar{\tilde{\psi}}\partial_{\tilde{\sigma}} \tilde{\psi} \right) d\sigma. \tag{3.9}$$

The canonical momentum is defined as [30]

$$\Pi(\psi) = \left(\frac{\partial \mathcal{L}}{\partial(\partial_\tau \bar{\psi})} \right). \tag{3.10}$$

We can see that the canonical momenta for ψ and $\tilde{\psi}$ are

$$\Pi(\psi) = \left(\frac{\partial \mathcal{L}}{\partial(\partial_\tau \bar{\psi})} \right) = \frac{1}{4\pi i} \bar{\psi} \tag{3.11}$$

$$\Pi(\tilde{\psi}) = \left(\frac{\partial \mathcal{L}}{\partial(\partial_\tau \bar{\psi})} \right) = \frac{1}{4\pi i} \bar{\tilde{\psi}}. \tag{3.12}$$

The equations of motion can again be obtained from the Euler-Lagrange equations. The Euler-Lagrange equation of a field is given by [31]

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\phi})} \right) - \frac{\partial \mathcal{L}}{\partial \bar{\phi}} = 0. \tag{3.13}$$

For our Lagrangian we get

$$\partial_w \left(\frac{\partial \mathcal{L}}{\partial(\partial_w \bar{\psi})} \right) + \partial_{\bar{w}} \left(\frac{\partial \mathcal{L}}{\partial(\partial_{\bar{w}} \bar{\psi})} \right) - \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = 0 \tag{3.14}$$

$$\partial_w \left(\frac{\partial \mathcal{L}}{\partial(\partial_w \bar{\tilde{\psi}})} \right) + \partial_{\bar{w}} \left(\frac{\partial \mathcal{L}}{\partial(\partial_{\bar{w}} \bar{\tilde{\psi}})} \right) - \frac{\partial \mathcal{L}}{\partial \bar{\tilde{\psi}}} = 0 \tag{3.15}$$

From this we get

$$\frac{\partial \mathcal{L}}{\partial(\partial_w \bar{\psi})} = \frac{\partial \mathcal{L}}{\partial(\partial_{\bar{w}} \bar{\psi})} = \frac{\partial \mathcal{L}}{\partial(\partial_w \bar{\tilde{\psi}})} = \frac{\partial \mathcal{L}}{\partial(\partial_{\bar{w}} \bar{\tilde{\psi}})} = 0. \tag{3.16}$$

This leads to

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = 0 \tag{3.17}$$

$$\frac{\partial \mathcal{L}}{\partial \bar{\tilde{\psi}}} = 0. \tag{3.18}$$

It therefore follows that

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = \frac{1}{8\pi i} (\partial_\tau - i\partial_\sigma)\psi = 0 \tag{3.19}$$

$$\frac{\partial \mathcal{L}}{\partial \bar{\tilde{\psi}}} = \frac{1}{8\pi i} (\partial_\tau + i\partial_\sigma)\tilde{\psi} = 0. \tag{3.20}$$

The equations of motions are then

$$(\partial_\tau - i\partial_\sigma)\psi = 0 \tag{3.21}$$

$$(\partial_\tau + i\partial_\sigma)\tilde{\psi} = 0 \tag{3.22}$$

The mode expansions can be calculated using the equation of motion.

$$(\partial_\tau - i\partial_\sigma)\psi = 0 \tag{3.23}$$

$$(\partial_\tau + i\partial_\sigma)\tilde{\psi} = 0. \tag{3.24}$$

If we assume that $\psi = X(\sigma)T(\tau)$, from the method of separation of variables, (3.23) becomes

$$X\dot{T} = iTX', \tag{3.25}$$

where $X' = \frac{dX}{d\sigma}$ and $\dot{T} = \frac{dT}{d\tau}$.

This leads to

$$\frac{X}{X'} = i\frac{T}{\dot{T}} = k, \tag{3.26}$$

with k being constant.

This has solutions

$$\psi = Ae^{k\tau + ik\sigma}, \tag{3.27}$$

with A being a constant. More information of k can be extracted. If we assume periodic boundary conditions $\psi(\tau, \sigma) = \psi(\tau, \sigma + 2\pi)$, we get

$$\psi = Ae^{k\tau + ik(\sigma + 2\pi)} = Ae^{2\pi ik}e^{k\tau + ik\sigma}. \tag{3.28}$$

We can see that for the periodic conditions to occur, k must be integer valued. If we assume anti-periodic boundary conditions $\psi(\tau, \sigma) = -\psi(\tau, \sigma + 2\pi)$ we get that

$$Ae^{k\tau + ik(\sigma + 2\pi)} = Ae^{2\pi ik}e^{k\tau + ik\sigma} = -Ae^{k\tau + ik\sigma}. \tag{3.29}$$

It must follow that

$$e^{2\pi ik} = -1, \tag{3.30}$$

meaning that for anti-periodic boundary conditions to hold k must be half integer valued. We conclude that k can either be integer or half integer valued.

We must remember that we are summing over all modes to get the mode expansion, which therefore leads to

$$\psi = \sum_k \psi_k e^{k\tau + ik\sigma}. \tag{3.31}$$

We must account for normalization, and therefore we must introduce a factor of $\sqrt{-i}$. The mode expansions then becomes

$$\psi = \sqrt{-i}\sum_k \psi_k A e^{k\tau + ik\sigma}. \tag{3.32}$$

The same process can be applied to $\tilde{\psi}$ to get

$$\tilde{\psi} = \sqrt{-i}\sum_k \psi_k A e^{k\tau - ik\sigma}. \tag{3.33}$$

The Hamiltonian density is defined $\mathcal{H}$ [3] to be

$$\mathcal{H} = \sum_i \frac{\partial\mathcal{L}}{\partial\dot{\psi}_i}\dot{\psi}_i - \mathcal{L} \tag{3.34}$$

$$= \frac{\partial\mathcal{L}}{\partial\dot{\tilde{\psi}}}\dot{\tilde{\psi}} + \frac{\partial\mathcal{L}}{\partial\dot{\psi}}\dot{\psi} - \mathcal{L} \tag{3.35}$$

$$= \frac{\partial\mathcal{L}}{\partial(\partial_\tau\psi)}\partial_\tau\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\tau\tilde{\psi})}\partial_\tau\tilde{\psi} - \mathcal{L}. \tag{3.36}$$

We find that

$$\frac{\partial\mathcal{L}}{\partial(\partial_\tau\psi)} = \frac{1}{4\pi i}\bar{\psi}d\sigma \tag{3.37}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\tau\tilde{\psi})} = \frac{1}{4\pi i}\bar{\tilde{\psi}}d\sigma. \tag{3.38}$$

The Hamiltonian then becomes

$$H = \frac{1}{4\pi i}\int \left(\bar{\psi}\partial_\tau\psi + \bar{\tilde{\psi}}\partial_\tau\tilde{\psi} - (\bar{\psi}\partial_\tau\psi + \bar{\tilde{\psi}}\partial_{\tilde{\tau}}\tilde{\psi} - i\bar{\psi}\partial_\sigma\psi + i\bar{\tilde{\psi}}\partial_{\tilde{\sigma}}\tilde{\psi})\right) d\sigma \tag{3.39}$$

$$= \frac{1}{4\pi i}\int \left(i\bar{\psi}\partial_\sigma\psi - i\bar{\tilde{\psi}}\partial_{\tilde{\sigma}}\tilde{\psi}\right) d\sigma \tag{3.40}$$

$$= \frac{1}{4\pi}\int \left(\bar{\psi}\partial_\sigma\psi - \bar{\tilde{\psi}}\partial_{\tilde{\sigma}}\tilde{\psi}\right) d\sigma. \tag{3.41}$$

3.3.1 Dedekind η Function Derivation for a Fermionic String

We wish to find the partition function $\mathrm{Tr}[e^{-\beta\mathcal{H}}]$. We begin with the following anti-commutator relations

$$\{\psi_k, \psi_{-k}\} = 1 \tag{3.1}$$

$$\{\tilde{\psi}_k, \tilde{\psi}_{-k}\} = 1 \tag{3.2}$$

$$\{\tilde{\psi}_k, \psi_{k'}\} = 0 \tag{3.3}$$

The Hamiltonian for a fermionic string is defined to be

$$\mathcal{H} = \frac{1}{2}\sum \left(k\psi_{-k}\psi_k + k\tilde{\psi}_{-k}\tilde{\psi}_k\right). \tag{3.4}$$

Similarly to the bosonic case, we introduce 2 new terms and their inverses

$$\mathcal{H} = \frac{1}{2}\sum \left(k\psi_{-k}\psi_k + k\tilde{\psi}_{-k}\tilde{\psi}_k + k\psi_k\psi_{-k} - k\psi_k\psi_{-k} + k\tilde{\psi}_k\tilde{\psi}_{-k} - k\tilde{\psi}_k\tilde{\psi}_{-k}\right). \tag{3.5}$$

This can be rearranged in terms of anti-commutators to give

$$\mathcal{H} = \frac{k}{2}\sum \left(\{\psi_k, \psi_{-k}\} + \{\tilde{\psi}_k, \tilde{\psi}_{-k}\} - k\psi_k\psi_{-k} - k\tilde{\psi}_k\tilde{\psi}_{-k}\right). \tag{3.6}$$

We will now apply the same process as in 3.2.1

$$\mathcal{H} = \mathcal{H}_{\mathcal{L}} + \mathcal{H}_{\mathcal{R}}, \tag{3.7}$$

and

$$\mathcal{H}_{\mathcal{L}} = \frac{k}{2}\sum \left(\{\psi_k, \psi_{-k}\} - k\psi_k\psi_{-k}\right) \tag{3.8}$$

$$\mathcal{H}_{\mathcal{R}} = \frac{k}{2}\sum(\{\tilde{\psi}_k, \tilde{\psi}_{-k}\} - k\tilde{\psi}_k\tilde{\psi}_{-k}). \tag{3.9}$$

We can similarly split β into $\beta = \beta_L + \beta_R$. The partition function now becomes

$$\mathrm{Tr}[e^{-\beta\mathcal{H}}] = \mathrm{Tr}[e^{-(\beta_L\mathcal{H}_L + \beta_R\mathcal{H}_R)}]. \tag{3.10}$$

Taking $[e^{+\beta_R\mathcal{H}_R)}]$, the same argument from 3.2.1 can be used to show

$$\mathrm{Tr}[e^{-\beta_L\mathcal{H}_L}] = e^{-\frac{\beta_L}{24}}\prod_k(1 + e^{\beta_L k}). \tag{3.11}$$

Likewise

$$\mathrm{Tr}[e^{-\beta_R\mathcal{H}_R}] = e^{-\frac{\beta_R}{24}}\prod_k(1 + e^{\beta_R k}). \tag{3.12}$$

Combining the equations will give to overall partition function

$$\mathrm{Tr}[e^{-\beta\mathcal{H}}] = \mathrm{Tr}[e^{-\beta_L\mathcal{H}_L}]\mathrm{Tr}[e^{-\beta_R\mathcal{H}_R}] \tag{3.13}$$

$$= e^{-\frac{\beta_L}{24}}\prod_k(1 + e^{\beta_L k})e^{-\frac{\beta_R}{24}}\prod_k(1 + e^{\beta_R k}). \tag{3.14}$$

We will now write $\beta_L = 2i\pi\tau$ and $\beta_R = 2i\pi\bar{\tau}$.

This gives

$$e^{-\frac{2i\pi\tau}{24}}\prod_k(1+e^{2i\pi\tau k})e^{-\frac{i\pi\bar{\tau}}{24}}\prod_k(1+e^{i\pi\bar{\tau}k}). \tag{3.15}$$

Finally, letting $q=e^{2i\pi\tau}$ and $\bar{q}=e^{2i\pi\bar{\tau}}$ we get

$$\mathrm{Tr}[e^{-\beta\mathcal{H}}] = q^{\frac{1}{2}}\prod_k(1+q^k)\bar{q}^{\frac{1}{2}}\prod_k(1+\bar{q}^k) \tag{3.16}$$

$$= \eta(q)\eta(\bar{q}) \tag{3.17}$$

$$= |\eta(q)|^2. \tag{3.18}$$

We have now shown that a fermionic partition functions on a torus can also be written as eta functions, similarly to the bosonic case, further strengthening ties between the two.

We will also look at the anti-periodic boundary condition, $\psi(\tau,\sigma)=-\psi(\tau,\sigma+2\pi)$. A similar argument can be made to that of (3.2.1), in letting $k\to k+1/2$ and $k\to k-1/2$. We find from an identical argument that the anti-peridic partition function is given by

$$\mathrm{Tr}[e^{\beta\mathcal{H}}] = e^{\frac{\beta_L}{48}}\prod_{k=1}^{\infty}\left(1+e^{-\beta(k-1/2)}\right)+e^{\frac{\beta_R}{48}}\prod_{k=1}^{\infty}\left(1+e^{-\beta_R(k-1/2)}\right). \tag{3.19}$$

Chapter 4

Elliptic Genus of $K3$

We wish to further enhance our theory of partition functions from the last section. We will now add a chemical potential (to introduce thermodynamics), to these functions. It can be shown that with this addition, the functions will give rise to Jacobi forms. Jacobi forms are power series which behave similarly to to modular forms.

4.1 What is a $K3$ Surface?

There are many ways in which one can define a $K3$ surface. In this report, we will think of a $K3$ surface as a four dimensional manifold where $SU(2)$ *holonomy* holds [23]. This means that if one *parallel transports* a vector V_i on a tangent space at a point P along a closed loop, and returns to the starting point, the vector will have have rotated by a rotation U_{ij}.

$$V_i(P) \to U_{ij} V_i(P), \quad U_{ij} \in SU(2). \tag{4.1}$$

To further illustrate this point, it is best to discuss more about parallel transport. Recall the idea of a connection from Chapter 1. We note that for this purpose, the connection connects tangent spaces at different points on the manifold. Such a connection (mapping) is said to be a parallel transport. Mathematically, this can be interpreted as a vector field X, on a smooth curve C, with co-ordinates $x^\mu(\tau)$ so that [10]

$$X^\mu \bigg|_C = \frac{dx^\mu}{d\tau}. \tag{4.2}$$

A vector field T, is parallel transported along C if

$$\nabla_X T = 0, \tag{4.3}$$

for every point on C.

$K3$ surfaces are very important in string theory as they allow for string compactification, a concept we will discuss later. Because of this, $K3$ surfaces act as the arena for many concepts in string theory, such as black hole microstates [32]. $K3$ can also be thought of as an *orbifold*, manifolds which contain singularities of T^4, a torus in 4 dimensions.

4.2 Elliptic Genus

To allow for extra compactification, we use *elliptic genera*, which are a topological invariant of $K3$. They can also be interpreted as extensions of the Witten index [33], a partition function of the form [34]

$$\mathrm{Tr}[(-1)^F e^{-\beta H}],$$

for some fermionic number F. F changes depending on the particle in question. If we are considering a boson, then $F = 1$. If we are considering a fermion, then $F = 0$. The choice of using the elliptic genus will prove vital in later sections, as the elliptic genus will allow us to count types of states that are present in black hole string theory.

We will consider 2 complex (4 real) bosons and 2 complex (4 real) fermions compactified on T^4 [27]. This allows us to use partition functions from the previous section. The elliptic genus of $K3$ can be defined as [27]

$$Z_{K3} = \frac{1}{2}\sum_{a,b=0}^{1} \mathrm{Tr}_{RRg^a}\left((-1)^{F_{K3}+\bar{F_{K3}}} g^b e^{2\pi i\nu J_{K3}} q^{L_0-\frac{c}{24}} \bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\right), \tag{4.1}$$

for a genus g, where $L_0, \bar{L}_0$ are the left moving and right moving Hamiltonians without the zero point energy, $c, \bar{c} = 6$, the central charge number ($c_b = 1$, the bosonic charge number and $c_f = \frac{1}{2}$, the fermionic charge number), J_{K3} is an angular momentum component, with $J_{K3} = \pm 1$ for fermions. This makes J_{K3} form a doublet under a $SU(2)$ transformation, with $z = e^{2\pi i\nu}$. We note that RR here denotes the *Ramond-Ramond sector*, meaning that we will be dealing with periodic boundary conditions.

We note that $(a, b) = (0, 0)$ yields a trivial solution, and therefore is excluded. The following values appear for the different (a, b) values:

$$(0,1) : \mathrm{Tr}_{RR_{g^0}}\left[(-1)^{F_{K3}+\overline{F}_{K3}}\ g\, e^{2\pi i\nu J_{K3}}\ q^{L_0-\frac{c}{24}} \bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\right] = 16\frac{\theta_2^2(q,z)}{\theta_2^2(q,1)} \tag{4.2}$$

$$(1,0) : \mathrm{Tr}_{RR_{g^1}}\left[(-1)^{F_{K3}+\overline{F}_{K3}}\ e^{2\pi i\nu J_{K3}}\ q^{L_0-\frac{c}{24}} \bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\right] = 16\frac{\theta_2^4(q,z)}{\theta_2^4(q,1)} \tag{4.3}$$

$$(1,1) : \mathrm{Tr}_{RR_{g^1}}\left[(-1)^{F_{K3}+\overline{F}_{K3}}\ e^{2\pi i\nu J_{K3}}\ q^{L_0-\frac{c}{24}} \bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\right] = 16\frac{\theta_2^4(q,z)}{\theta_2^4(q,1)}. \tag{4.4}$$

The elliptic genus Z_{K3} can then be written as

$$Z_{K3}(\tau, z) = 8\left[\frac{\theta_2^2(q,z)}{\theta_2^2(q,1)} + \frac{\theta_3^2(q,z)}{\theta_3^2(q,1)} + \frac{\theta_4^2(q,z)}{\theta_4^2(q,1)}\right], \tag{4.5}$$

which we will denote as $A(\tau, z)$. Conformal field theory (CFT) implies [35] that the elliptic genus is a *weak Jacobi form* (), which behave similarly to modular forms. The Fourier expansion [36] of the elliptic genus can also be defined as

$$A(\tau, z) = \sum_{n\geq 0, \ell\in\mathbb{Z}} c(n,\ell) q^n y^\ell \tag{4.6}$$

where c are the Fourier coefficients, $q = e^{2\pi i\tau}$ and $y = e^{2\pi iz}$. We note that we computed $A(\tau, z)$ with the first terms being

$$A(\tau, z) = 20 + 2/z + 2z + q(216 + 20/z^2 - 128/z - 128z + 20z^2) + \mathcal{O}(q^2). \tag{4.7}$$

The *twisted elliptic genus*, is an automorphism of the elliptic genus and is defined as [27]

$$F^{(r,s)}(\tau, z) = \frac{1}{N}\mathrm{Tr}_{RRg^r}\left((-1)^{F_{K3}+\overline{F}_{K3}}\ {g'}^s\ e^{2\pi izF_{K3}}\ q^{L_0-\frac{c}{24}} \bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\right), \tag{4.8}$$

with automorphism g'.

The twisted elliptic genus can also be written as a Fourier expansion:

$$F^{(r,s)}(\tau, z) = \sum_{r,s,n,l} c^{(r,s)}(4n - l^2) q^n z^l, \tag{4.9}$$

where $c^{(r,s)}$ are the Fourier coefficients. In later sections we will explore properties of the Fourier coefficients which will give us access to information regarding black holes.

Chapter 5

Supersymmetric Black Holes

Superstring theory naturally arises in 10 dimensional Loreznțian spacetime $\mathcal{M}_{10}$ [23]. In order to account for these extra dimensions, one uses *compactification*, the lowering of dimensions in a theory by assuming the extra dimensions are wrapped up on themselves. Mathematically, this can be achieved by taking the Cartesian product of Minkowski spacetime and the extra dimensions. Mathematically this can be written as $\mathbb{R}^{1,3} \times X_6$. The "extra dimensions", X_6, is known as a *Calabi-Yau threefold* [23], and for our report we will take $X_6 = K3 \times T^2$, which corresponds to Type-II string theory [23]. This compactification corresponds to a $\mathcal{N} = 4$ supersymmetry. We will work with *heterotic* strings, a hybrid of a bosonic and superstring.

In superstring theory, there exists certain states, called BPS states, which preserve a fraction of supersymmetry. *Supersymmetric* black holes are extremal black holes that are governed by BPS states. In this chapter, we compute the microstates of supersymmetric black holes, which will give us the degeneracy. Lastly, to fully understand this theory, we need to define *dyons*. Dyons are particles in string theory which have both electric and magnetic charges. These charges are denoted by Q and P for the electric and magnetic charges respectively. We will be computing the degeneracy of these dyons to find the degeneracy of the black hole.

Dijkgraaf et al. [37] were the first to propose a formula for BPS black hole states. From this Jatkar et al. [38] proposed a formula for the degeneracy of 1/4 BPS dyons in CHL models (models which work with heterotic strings) in terms of Siegel modular forms. David et al. [39] further enhanced this by introducing the product formulae of these modular forms. We will begin with the twisted elliptic genus. The twisted elliptic genus, when expanded is given as

$$F^{(r,s)}(\tau, z) = \sum_{b=0} \sum_{\substack{j \in 2\mathbb{Z}+b \\ n \in \mathbb{Z}/N \\ 4n-j^2 \geq -b^2}} c^{(r,s)}(4n - j^2) e^{2\pi i n\tau + 2\pi i j z}. \tag{5.1}$$

This can also be written in terms of q and z

$$F^{(r,s)}(q, z) = \sum_{r \in \mathbb{Z}} \sum_{n} c_b^{(r,s)}(4n - r^2) q^n z^r. \tag{5.2}$$

The Siegel modular form of this twisted elliptic genus is given by

$$\Phi(\rho, \sigma, v) = e^{2\pi i(\tilde{\alpha}\rho + \tilde{\beta}\sigma + v)} \prod_{r=0}^{N-1} \prod \left(1 - e^{2\pi i(k\sigma + l\rho + jv)}\right)^{\sum_{s=0}^{N-1} e^{2\pi i s l/N} c^{(r,s)}(4kl - j^2)}. \tag{5.3}$$

with $c^{(r,s)}$ being the Fourier coefficients [39].

$$F^{(0,0)}(\tau, z) = \frac{8}{N} A(\tau, z) \tag{5.4}$$

$$F^{(0,s)}(\tau, z) = \frac{8}{N(N+1)} A(\tau, z) - \frac{2}{N+1} B(\tau, z) E_N(\tau) \quad \text{for } 1 \leq s \leq (N-1) \tag{5.5}$$

$$F^{(r,rk)}(\tau, z) = \frac{8}{N(N+1)} A(\tau, z) + \frac{2}{N(N+1)} E_N\left(\tau + \frac{k}{N}\right) B(\tau, z) \tag{5.6}$$

for

$$1 \leq r \leq (N-1), \quad 0 \leq k \leq (N-1).$$

We note that [40]

$$B_N(\tau,z) = \eta(\tau)^{-6}\theta_1(\tau,z)^2 \tag{5.7}$$

$$E_N(\tau,z) = \frac{12i}{\pi(N-1)}\partial_\tau\left(\ln(\eta(\tau) - \ln(\eta(N\tau))\right). \tag{5.8}$$

We note that these coefficients were already calculated (4.7).
For the purpose of this report, we will use the following values.

$$\tilde{\alpha} = 1 \tag{5.9}$$

$$\tilde{\beta} = 1/N \tag{5.10}$$

For $K3$, we have $N = 1$ and also need not concern ourselves with $b = 0, 1$. We note that with these conditions, (5.2) turn becomes (4.9), and more importantly, (5.3) becomes

$$\Phi(\rho,\sigma,v) = e^{2\pi i(\rho+\sigma+v)} \prod_{\substack{k,l,j\in\mathbb{Z} \\ k,l\geq 0,\ j<0,\ k=l=0}} \left(1 - e^{2\pi i(k\sigma+l\rho+jv)}\right)^{c^{(r,s)}(4kl-j^2)}. \tag{5.11}$$

In string theory, there exists particles which have both electric and magnetic charges. These charges are denoted by Q and P for the electric and magnetic charges respectively. The degeneracy is given by [39]

$$d(Q,P) = (-1)^{Q\cdot P+1}\int_C d\rho dv d\sigma e^{-\pi i(\rho Q^2+\sigma P^2 m+2vQ\cdot P)}\frac{1}{\Phi(\rho,\sigma,v)}, \tag{5.12}$$

where C is a real three dimensional subspace over complex space. We note that (ρ,σ,v) and Φ is a modular form under a subgroup of $SL_2(\mathbb{Z})$ [40]. We note that Φ is sometimes referred to as the *Igusa cusp form*, and has a weight 10 [33]. The Igusa cusp form is the *multiplicative lift* of the elliptic genus of $K3$ [27].

The reciprocal of the Igusa cusp form can be rewritten as

$$\frac{1}{\Phi(\rho,\sigma,v)} = \sum_{m,n,p} g(m,n,p)e^{2\pi i(m\rho+n\sigma+pv)}. \tag{5.13}$$

.

Introducing (5.13) into (5.12), we get

$$d(Q,P) = (1)^{Q\cdot P+1}\sum_{m,n,p} g(m,n,p)\int_C d\rho dv d\sigma e^{-\pi i(\rho Q^2+\sigma P^2 m+2vQ\cdot P)}e^{2\pi i(m\rho+n\sigma+pv)}. \tag{5.14}$$

If we focus the exponential we find that

$$\exp\left(-\pi i(\rho Q^2+\sigma Q^2 m+2vQ\cdot P)\right)\exp\left(2\pi i(m\rho+n\sigma+pv)\right) \tag{5.15}$$

$$= \exp\left(2\pi i((-Q^2/2+m)\rho+(-P^2/2+n)\sigma+(-Q\cdot P+p)v)\right) \tag{5.16}$$

$$\tag{5.17}$$

By linearity of the integral ,we can integrate $d\rho, d\sigma$ and dv respectively. We can again implement the Dirac-Delta function to give

$$\int_C e^{-\pi i(m-Q^2/2)\rho} d\rho = \delta(Q^2/2 - m) \tag{5.18}$$

$$\int_C e^{-\pi i(n-P^2/2)} d\sigma = \delta(P^2/2 - n) \tag{5.19}$$

$$\int_C e^{-\pi i(2Q\cdot P+p} dv = \delta(Q \cdot P - p). \tag{5.20}$$

We see that from the delta functions

$$m = Q^2/2 \quad n = P^2/2 \quad p = Q \cdot P. \tag{5.21}$$

From this we get

$$d(Q,P) = (-1)^{Q\cdot P} g\left(\frac{Q^2}{2}, \frac{P^2}{2}, Q \cdot P\right). \tag{5.22}$$

This means that if we can find the coefficients g, then we have found our degeneracies! To do this we must make some substitutions. Using $x = e^{2\pi i\rho}$, $y = e^{2\pi i\sigma}$ and $z = e^{2\pi iv}$, (5.13) becomes

$$\frac{1}{\Phi(\rho,\sigma,v)} = \sum_{m,n,p} g(m,n,p)x^m y^n z^p, \tag{5.23}$$

and (5.11) becomes

$$\Phi(\rho,\sigma,v) = xyz \prod_{k,l\geq 0,\ j<0,\ k=l=0} \left(1 - x^l y^k z^j\right)^{c^{(r,s)}(4kl-j^2)}, \tag{5.24}$$

which leads to

$$\frac{1}{\Phi(\rho,\sigma,v)} = \frac{1}{xyz} \prod_{k,l\geq 0,\ j<0,\ k=l=0} \left(1 - x^l y^k z^j\right)^{-c^{(r,s)}(4kl-j^2)}. \tag{5.25}$$

(5.25) can be computed numerically which can then be used to find the coefficients g of the Fourier expansion (5.13), which then yields the degeneracies d (5.12). There are several constraints that one must introduce to the 1/4 BPS states in order to have a single-centered black hole [27]. The constraints are given by

$$Q \cdot P \geq 0, \quad (Q\cdot P)^2 < Q^2P^2, \quad Q^2, P^2 > 0. \tag{5.26}$$

The following table gives results for different combinations of Q^2, P^2, and $Q \cdot P$ from (5.12). To calculate the entropy, one needs to take the natural log of the degeneracies (2.2). We note that some values will give an undefined answer. This is due to the aforementioned constraints.

The importance of these findings cannot be overstated. These results are seen by many physicists as a “success” of string theory because not only does it add validity to the theory, it also provides us with an insight into quantum gravity!

(Q^2, P^2)	$Q \cdot P = -2$	$Q \cdot P = 0$	$Q \cdot P = 1$	$Q \cdot P = 2$	$Q \cdot P = 3$	$Q \cdot P = 4$
$(2, 2)$	-209304	50064	25353	648	327	0
$(2, 4)$	-2023536	1127472	561576	50064	8376	-648
$(4, 4)$	-16620544	32861184	18458000	3859456	561576	12800
$(2, 6)$	-15493728	16491600	8533821	1127472	130329	-15600
$(4, 6)$	-53249700	632078672	392427528	110910300	18458000	1127472
$(6, 6)$	2857656828	16193130552	11232685725	4173501828	920577636	110910300

Table 5.1: The results of the degeneracy values for given Q^2,P^2 and $Q \cdot P$ values. Taking the natural logarithm of any of these values will yield the black hole entropy. The negative signs in some of the values arise from nonphysical circumstances and therefore do not give any solutions. These results also match with the current literature [40].

Chapter 6

Conclusion

We have been able to count the BPS microstates on a supersymmetric black hole. We began with a brief introduction which guided us to read into the relevant literature of black holes. We then turned our attention to string theory, primarily into partition functions, which led to the $K3$ elliptic genus. This was the key that we used in our final section to compute the microstates by using Siegel modular forms.

Although we have done much calculations, this report only scratches the surface of black hole string theory. The areas I enjoyed working on the most was learning about black hole dynamics, and the connection between black holes and string theory. In the future, I would like to explore more of this connection between string theory and black holes, such as the *Hawking information paradox.* Another area I would like to study, and appeared throughout the literature review was *anti de-Sitter* space (AdS) and *Conformal Field Theory* (CFT), topics used when studying quantum gravity.

Throughout these last few months I learned more than I could have imagined, and this internship has given me a good understanding of how research in theoretical physics is carried out, which is something I hope to do in the future.

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Appendix

Rindler Co-ordinates and the Unruh effect

Like we previously discussed, for Schwarzschild black holes the black hole is singular at $r = 0$ and $r = r_s$. We would like to make r_s continuous. This can be done by introducing a new co-ordinate system, which is used to study accelerated observers [41]. We begin with the Minkowski metric

$$ds^2 = dt^2 - dx^2.$$

The other spacial dimensions become irrelevant and as such can be omitted.

We define some co-ordinates $\bar{u}$, $\bar{v}$ [42]

$$\begin{aligned}\bar{u} &= t - v,\\ \bar{v} &= t + v.\end{aligned}$$

() can be written as

$$ds^2 = d\bar{u}d\bar{v}. \tag{.1}$$

We then make the following transformation

$$\begin{aligned}x &= \frac{1}{a}e^{a\xi}\cosh(a\eta),\\ t &= \frac{1}{a}e^{a\xi}\sinh(a\eta),\end{aligned}$$

where a is the norm of the acceleration $a := \sqrt{a_\mu a^\mu}$ [43]. These co-ordinate systems (ξ, η) are known as *Rindler co-ordinates*. From this we can construct hyperbolae from ()

$$\begin{aligned}x^2 - t^2 =& \frac{1}{a^2}e^{2a\xi}(\cosh^2(a\eta) - \sinh^2(a\eta))\\ =& \frac{1}{a^2}e^{2a\xi}\end{aligned}$$

Since a is already constant, for constant ξ, we get lines of hyperbolae, whilst lines of constant η are straight lines [42]. We note that the Rindler co-ordinates only take up a segment of Minkowski space. This section is known as a *Rindler wedge* for $x > |t|$. A second Rindler wedge $x < -|t|$ can be constructed by means of reflection of the first wedge [42]. We will denote the wedge on the LHS as “L” and the wedge on the RHS as “R”. It is important to note that observers in R are causally separated from L. *Rindler observers* moving along with constant ξ, or more simply, an observer moving with uniform acceleration.

Calculating dx and dt we get,

$$\begin{aligned}dx &= e^{a\xi}\left(\cosh(a\eta)d\xi + \xi\sinh(a\eta)d\eta\right),\\ dt &= e^{a\xi}\left(\sinh(a\eta)d\eta + \eta\cosh(a\eta)d\eta\right).\end{aligned}$$

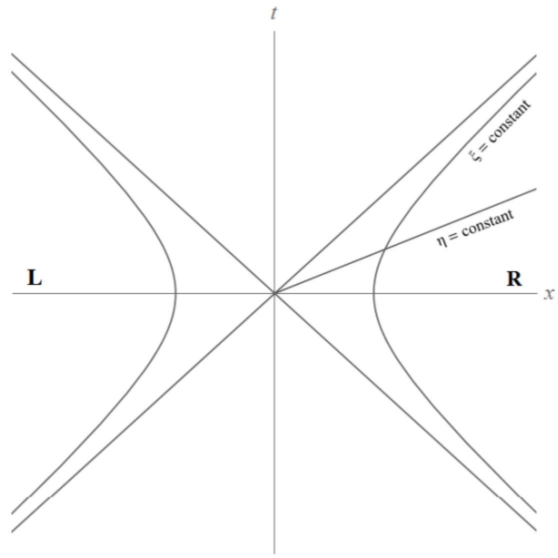

Figure .1: Diagram explaining the Rindler co-ordinates. We can the Rindler wedge in L and R [44]

This leads to

$$ds^2 = e^{2a\xi}\left(\xi^2 d\eta^2 - d\xi^2\right).$$

We can see from here that ds^2 is independent of η. Therefore ∂_η is a Killing vector field in (ξ, η) [45]. We note that in this context $\partial_\eta = \frac{\partial}{\partial\eta}$. By use of the chain rule, we can define ∂_η to be

$$\begin{aligned}\partial_\eta =& \frac{\partial t}{\partial\eta}\partial_t + \frac{\partial x}{\partial\eta}\partial_x \\ =& e^{a\xi}(\cosh(a\eta)\partial_t + \sinh(a\eta)\partial_x) \\ =& a(x\partial_t + t\partial_x).\end{aligned}$$

∂_η is therefore a timelike Killing vector, which generates the boost Lorentz symmetry. We note that the norm of ∂_η on the orbit $\xi = 1/a$ is $1/a$.

With these co-ordinate systems, it can be shown that the accelerated observer will experience the *Unruh effect*, vacuum fluctuations that appear to be a thermal bath with a temperature

$$T = \frac{a}{2\pi}.$$

We note that an observer at rest will not observe this temperature. One way to get to this is to use density matrices, which allow for the different particle states that the observer sees.

A density matrix ρ in the Minkowski vacuum is defined to be [46]

$$\rho = Z^{-1}e^{-\beta H},$$

for some partition function Z and Hamiltonian H. This can also be interpreted as a thermal state with respect to boosts. Density matrcies use *vacuum states* $|0\rangle$, defined as a state where particles cannot be annihilated [47]. In mathematical form, it can be expressed as

$$\mathbf{a}_i\,|0\rangle = 0, \quad \forall\, i$$

where $\mathbf{a}_i$ is the annihilation operator.

ρ when traced over $x < 0$, gives a *Gibbs state*, an equilibrium state in a quantum system for a boost Hamiltonian H_B, given by [46]

$$\mathrm{tr}_{\mathrm{x}<0} \left|0\right\rangle \left\langle 0\right| = \mathrm{Z}^{-1} \exp(-2\pi \mathrm{H_B})$$
$$H_B = \int_\Sigma T_{ab} (\partial/\partial\eta)^a d\Sigma^b,$$

for some Cauchy surface Σ, and energy-stress tensor T_{ab}. We can now see that $T = 1/2\pi$, since $\beta = 1/T$. We note that the temperature is dimensionless. This is due to the simple fact that boost energy is also dimensionless. However, to determine the temperature seen by an observer on a Killing field following an orbit ξ, we must re-scale by a. If we scale the boost Hamiltonian by a to match this, we see that the temperature observed is indeed

$$\frac{a}{2\pi}.$$

We can also arrive to this temperature by the use of the Rindler time co-ordinate $\eta\ \eta + 2\pi i$, which is periodic under η. This is at the heart of the work done by Hawking. Let us consider the Unruh effect just outside the black hole event horizon.

We note that the horizons $x = \pm t$ are *Killing horizons*, horizons associated with Killing vector fields. The analog of the Unruh effect was first introduced by Hawking (although the Unruh effect was discovered after Hawking's findings) when he showed that the temperature T_H experienced by an observer very far away from black hole horizon is related to the acceleration of the surface gravity κ by

$$T_H = \frac{\kappa}{2\pi}.$$

$SL_2(\mathbb{Z})$

$SL_2(\mathbb{Z})$, the *special linear group* is a group whereby the following properties hold [48]:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a, b, c, d \in \mathbb{Z}$ and $det(A) = 1$.

$Sp(2, \mathbb{Z})$

$Sp(2, \mathbb{Z})$, the *symplectic group* is a group whereby the following properties hold:

For a matrix M

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with A, B, C, and D are 2×2 matrices satisfying:

$$MJ^TM = J, \quad M \in Sp(2, \mathbb{Z}),$$

for

$$J = \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$$

.

We also that the following conditions must hold:

$$-C^{\mathrm{T}}A + A^{\mathrm{T}}C = 0, \quad -C^{\mathrm{T}}B + A^{\mathrm{T}}D = \mathbb{I}, \quad -D^{\mathrm{T}}B + B^{\mathrm{T}}D = 0.$$

Weak Jacobi Forms

Weak Jacobi forms $\phi(\tau,z)$ are defined to have the following properties [36]

$$\phi\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)=(c\tau+d)^w\exp\left(2\pi im\frac{cz^2}{c\tau+d}\right)\phi(\tau,z),\quad \begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})$$

$$\phi\left(\tau,z+\ell\tau+\ell'\right)=\exp\left(-2\pi im(\ell^2\tau+2\ell z)\right)\phi(\tau,z),\quad (\ell,\ell'\in\mathbb{Z})\,.$$

We note that for this report, $m=1$.
Finally, they present the Fourier expansion of the elliptic genus:

Modular Forms

We let $f(z)$ be a holomorphic function on $\mathbb{H}$, where

$$\mathbb{H}=\{z\in\mathbb{C},\mathrm{Im}(z)>0\}.$$

$f(z)$ is said to be a *modular form* of $SL_2(\mathbb{Z})$ if

$$f\left(\frac{az+b}{cz+d}\right)=(cz+d)^kf(z),$$

where k is the *weight* of the modular form. The generators of this group are given by the S and T matrices, defined by

$$S=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$$

$$T=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

We can see that these matrices, when applied to the transformation yield

$$f\left(\frac{az+b}{cz+d}\right)_S=f(z+1)=f(z)$$
$$f\left(\frac{az+b}{cz+d}\right)_T=f\left(\frac{-1}{z}\right)=z^kf(z)$$

We will define a *modular function* to be a meromorphic function that satisfies () with a phase factor $\zeta=\zeta(a,b,c,d)$.

Dedekind η function

The Dedekind η function

$$\eta(q)=q^{\frac{1}{24}}\prod_{k=0}^{\infty}(1-q^k),$$

where $q=e^{2i\pi\tau}$ and $\tau=\tau_1+i\tau_2$, red and is invariant on

$$\mathbb{H}:=\{z\in\mathbb{C}\mid\mathrm{Im}(z)>0\}.$$

The η function can also be rewritten [49] as

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right)=\exp\left(\frac{\pi i}{12c}(a+d)+s(-d,c)\right)(-i(c\tau+d))^{1/2}\eta(\tau)$$

where $s(d,c)$ is known as the Dedekind sum and is defined as [50]

$$s(d,c)=\sum_{x\ mod\ c}\frac{x}{c}\frac{dx}{c}$$

.

When we apply the generator matrices S and T to (), it can be shown that the following transformations will occur (see Apostol [49] for full derivation)

$$\eta(z+1)=e^{\frac{i\pi}{12}}\eta(\tau)$$
$$\eta(\frac{-1}{z})=\sqrt{(-i\tau)}\ \eta(\tau)$$

We will show for context that () also yields the same transformation for $\eta(\tau+1)$.

$$\begin{aligned}\eta(\tau+1)&=e^{\frac{2i\pi(\tau+1)}{24}}\prod(1-e^{2i\pi(\tau+1)})\\&=e^{\frac{i\pi}{12}}e^{\frac{i\pi\tau}{12}}\prod(1-e^{2i\pi}e^{2i\pi\tau})\\&=e^{\frac{i\pi}{12}}e^{\frac{i\pi\tau}{12}}\prod(1-e^{2i\pi\tau})\\&=e^{\frac{i\pi}{12}}\eta(\tau).\end{aligned}$$

The η function is thus a modular form of $SL_2(\mathbb{Z})$, of weight 1/2.

Jacobi Theta Functions

This classical theta function is defined [20] by

$$\theta(\nu,\tau)=\sum_{n=-\infty}^{\infty}\exp(\pi in^2\tau+2\pi in\nu).$$

This can also be rewritten as a product

$$\theta(\nu,\tau)=\prod_{m=1}^{\infty}(1-q^m)(1+q^{\frac{m-1}{2}}z)(1+q^{\frac{m-1}{2}}z^{-1}),$$

where $q=exp(2\pi i\tau)$ and $z=exp(2\pi i\nu)$.

There are several theta functions for different input values. We will use the following theta functions throughout.

$$\theta_1(z,q)=\sum_{n=-\infty}^{\infty}(-1)^{n-\frac{1}{2}}q^{\frac{1}{2}(n-\frac{1}{2})^2}z^{(n-\frac{1}{2})}$$

$$\theta_2(z,q)=\sum_{n=-\infty}^{\infty}q^{\frac{1}{2}(n+\frac{1}{2})^2}e^{(n+\frac{1}{2})}$$

$$\theta_3(z,q)=\sum_{n=-\infty}^{\infty}q^{\frac{1}{2}n^2}z^n$$

$$\theta_4(z,q)=\sum_{n=-\infty}^{\infty}(-1)^nq^{\frac{1}{2}n^2}z^n$$

These functions have the following transformations;

$$\theta_2(\tau+1,\nu) = e^{\frac{\pi i}{4}}\theta_2(\tau)$$
$$\theta_3(\tau+1,\nu) = \theta_4(\tau)$$
$$\theta_2\left(-\frac{1}{\tau},-\frac{\nu}{\tau}\right) = (-i\tau)^{\frac{1}{2}}\, e^{\frac{\pi i\nu^2}{\tau}}\theta_4(\tau,\nu)$$
$$\theta_3\left(-\frac{1}{\tau},-\frac{\nu}{\tau}\right) = (-i\tau)^{\frac{1}{2}}\, e^{\frac{\pi i\nu^2}{\tau}}\theta_3(\tau,\nu)$$

Siegel Modular Forms

A Siegel modular form $F(\Omega)$, is modular forms defined by

$$F\left[(A\Omega+B)(C\Omega+D)^{-1}\right] = \{\det(C\Omega+D)\}^k F(\Omega)$$

for a given matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where A, B, C, and D are (2×2) matrices defined by

$$AB^T = BA^T, \quad CD^T = DC^T, \quad AD^T - BC^T = \mathbb{I}.$$

We note that the the determinant cannot be equal to zero.