

Chaos in Classical D0-brane mechanics

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Abstract

We study classical chaos in a matrix system that can be thought of as describing N D0 branes. In particular, we compute Lyapunov exponents for the system. We define a classical analogue of the scrambling time and proceed to show that this system is a fast scrambler. We also investigate the energy dependence of Lyapunov exponents and we find that a bound on this energy dependence is satisfied.

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1 Introduction

In classical physics a black hole is a region of spacetime where the gravitational field is so strong that nothing can escape it. The geometry of a spherically symmetric black hole is described by the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2d\Omega^2 \quad (1)$$

written in natural units, where for $r > 2M$, t is the time coordinate, r is the radial coordinate, and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, where θ is the polar angle and ϕ is the azimuthal angle. The surface $r = 2M$ is known as the event horizon of the black hole. Anything that crosses the event horizon will never be able to get back outside it. The Schwarzschild metric describes black holes with no electric charge or angular momentum. In 1974, by studying quantum field theory in the background geometry of a black hole, Stephen Hawking was able to show that black holes are thermal systems that continuously radiate energy. Hawking produced a formula, now known as the Bekenstein-Hawking entropy, relating the entropy of a black to the area of its event horizon [1,2,3]

$$S_{BH} = \frac{k_B A}{4\hbar} \quad (2)$$

where k_B is Boltzmann's constant, \hbar is the reduced Planck constant, and A is the area of the event horizon. In most thermal systems, there is a microscopic interpretation of the thermodynamic entropy in terms of counting the number of microstates corresponding to the same macroscopic properties. However, the Bekenstein-Hawking entropy has no obvious microscopic origin. As a candidate theory of quantum gravity, string theory should be able to describe black holes. Indeed, string theory was able to make progress towards a microscopic description of black hole entropy when Strominger and Vafa [4] were able to produce the Bekenstein-Hawking entropy for a certain class of five dimensional extremal black holes in string theory. String theory contains D-brane solitons that are extended membranes of spacial dimensions [5,6]. Superimposing many of these objects gives rise to a soliton with properties of a black hole. There is a large degeneracy associated with this process that allows for a microscopic description of the black hole entropy.

There is a deep connection between thermal behaviour and chaos. Sta-

tistical descriptions of thermal systems depends upon these systems being ergodic. Ergodicity means that if we choose a point in the phase space of the system and consider its trajectory through phase space, then given a sufficient amount of time, the trajectory will eventually have passed through every point in the phase space. The ergodic nature of thermal systems arises from the exponential divergence of trajectories in phase space. Exponential divergence of nearby trajectories is also one of the defining characteristics of a chaotic system. Thus, we expect thermal systems to exhibit chaotic behaviour, and since black holes are thermal systems, it is natural to look for characteristics of chaos among black holes. Recent work in quantum information theory has conjectured an upper bound on the rate of growth of chaos in thermal systems with a large number of degrees of freedom. In [7], the authors conjecture that chaos can develop no faster than exponentially, with Lyapunov exponent

$$\lambda_L \leq \frac{2\pi k_B T}{\hbar} \quad (3)$$

Furthermore, black holes saturate this bound and so are not only chaotic, but are the most chaotic systems found in nature. Sekino and Susskind have studied the notion of a "scrambling time" for black hole horizons [8]. To define the scrambling time they consider a chaotic system with a large number of degrees of freedom that has been prepared in a pure state. Suppose the system has N degrees of freedom. Then they consider the density matrix of a subsystem of $m \ll N$ degrees of freedom. Over time, the entanglement entropy of the subsystem will tend towards its maximum value. They call the system "scrambled" when any subsystem with $m < N/2$ has achieved maximum entanglement entropy. Beginning with a scrambled system, they declare that one could add a single degree of freedom in a pure state and then the system would no longer be scrambled. But by waiting for the added bit of information to spread over the rest of the degrees of freedom, the system will once again become scrambled. The scrambling time is defined as the time one must wait before the system has re-scrambled after adding this small bit of information. They go on to conjecture that the fastest scramblers take a time logarithmic in the number of degrees of freedom, and systems whose degrees of freedom are $N \times N$ matrices saturate the bound. In this report we study the classical dynamics of such a system and so below we shall define a classical analogue to the scrambling time.

We will study chaos in the classical dynamics of a system whose degrees of freedom are $N \times N$ matrices and can be thought of as describing a system of N D0 branes. Moreover, we are interested in the large N limit of such a system. To quantify chaos in classical physics we can consider a dynamical system with a phase space \mathcal{M} of dimension n . Hamilton's equations describe the time evolution of a point x_0 to a point $x(t)$ in phase space. We may perturb the system to obtain a nearby point in phase space δx_0 and look at its time evolution to a final point $\delta x(t)$. The signature of a chaotic system is the exponential growth of these perturbations. We define a metric on phase space to discuss the growth of perturbations and for chaotic systems we expect to find

$$|\delta X(t)| \sim |\delta X(0)|e^{\lambda_L t} \quad (4)$$

where $|\delta X(t)| = |x(t) - \delta x(t)|$ is the distance between the phase space trajectories of the two points as defined by our metric on phase space. We can then define the Lyapunov exponent associated with the initial perturbation by

$$\lambda_L = \lim_{t \rightarrow \infty} \frac{1}{t} \log \left(\frac{|\delta x(t)|}{|\delta x_0|} \right) \quad (5)$$

For finite perturbations in compact phase space the behaviour described by Eq.(4) does not hold for all t . At early times, the growth will be exponential but the quantity $|\delta X(t)|$ cannot get any bigger than the size of the system, and so it will have to saturate to some value. The time it takes for the distance $|\delta X(t)|$ to saturate is what we will define as the classical scrambling time of the system, which we denote by t_* . We will show that this system is a fast scrambler which means that the scrambling time is logarithmic in the number of degrees of freedom. Since the number of degrees of freedom for a matrix system are the $N \times N$ components, the total number of degrees of freedom is proportional to N^2 and so we shall find that

$$t_* \sim \log N^2 \quad (6)$$

This report is structured as follows. In Section 2 we present the matrix model and describe the discretization that we use to simulate the system numerically before describing how we compute the largest Lyapunov exponent for the system. In Section 3 we discuss the results where we show

the computed Lyapunov exponents as well as the energy dependence of these exponents. We also discuss results regarding the scrambling time for this system. Then, in Section 4 we give a summary of the results obtained, followed by several appendices providing more mathematical details for the system.

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2 Method

2.1 The model

The model is a low-energy effective theory on a stack of D0-branes. The degrees of freedom are nine $N \times N$ Hermitian matrices $X_{ab}^i(t)$, where the index $i = 1, \dots, 9$ are the space coordinates, t is time and $a, b = 1, \dots, N$ index the N D0 branes. We work with the Lagrangian

$$L = \frac{1}{2g^2} \text{Tr} \left(\sum_i (D_t X^i)^2 + \frac{1}{2} \sum_{i \neq j} [X^i, X^j]^2 \right) \quad (7)$$

where $D_t = \partial_t + [A_t, \cdot]$ is the covariant derivative, and A_t is the $U(N)$ gauge field. We will take the large N limit, while keeping the 't Hooft coupling $\lambda = g^2 N$ fixed. From now on we set $\lambda = 1$, and so $g = 1/N^2$. Thus, in taking the large N limit, we are considering the weak coupling limit where classical dynamics provides a good approximation. We choose the gauge $A_t = 0$, and so the Lagrangian in this gauge is given by

$$L = \frac{1}{2g^2} \text{Tr} \left(\sum_i (\dot{X}^i)^2 + \frac{1}{2} \sum_{i \neq j} [X^i, X^j]^2 \right), \quad \dot{X}^i \equiv \partial_t X^i \quad (8)$$

Choosing $A_t = 0$ also gives rise to the Gauss law constraint

$$\sum_i [X^i, V^i] = 0, \quad V^i = \dot{X}^i \quad (9)$$

The full derivation of the Gauss law constraint is given in Appendix B. By taking derivatives of the Lagrangian with respect to X^i and \dot{X}^i we obtain two expressions

$$\frac{\partial L}{\partial X^k} = \frac{1}{g^2} \sum_i [X^i, [X^k, X^i]]^T, \quad \frac{\partial L}{\partial \dot{X}^K} = \frac{1}{g^2} \dot{X}^K T \quad (10)$$

Using these two expressions in the Euler-Lagrange equation gives the equations of motion for the system

$$\ddot{X}^i = \sum_j [X^j, [X^i, X^j]] \quad (11)$$

The derivation of the equations of motion is shown in detail in Appendix A. Equations (9) and (11) completely describe the system in the classical approximation. This system has been studied in [9] and we seek to reproduce some of their results here. They note that the equipartition theorem for this system relates the temperature, energy and number of degrees of freedom by

$$\langle K \rangle = 2\langle U \rangle = \frac{n_{dof}}{2} T \quad (12)$$

where T is the temperature and n_{dof} is the number of degrees of freedom. Since $E = K + U$, we have

$$E = \frac{3}{4} n_{dof} T \quad (13)$$

They further assert that after accounting for the Gauss law constraint, residual gauge symmetries, as well as conservation of angular momentum the number of degrees of freedom reduces to

$$n_{dof} = 8(N^2 - 1) - 36 \quad (14)$$

We take the temperature $T = 1$ and so the energy for each N is determined by

$$E = \frac{3}{4} \left(8(N^2 - 1) - 36 \right) \quad (15)$$

2.2 Discretization

For the purpose of simulating the system we discretize the equation of motion while conserving the Gauss law constraint. We write the equation of motion as

$$\begin{aligned} \dot{X}^K &= V^K \\ \dot{V}^K &= F^K = \sum_i [X^i, [X^K X^i]] \end{aligned}$$

We then use a Verlet integration algorithm to numerically simulate the dynamics of the system. The equations used in the Velocity Verlet algorithm can be obtained using Taylor expansions; see Appendix C. The discretized time evolution taken up to order δt^2 is

$$X^K(t + \delta t) = X^K(t) + V^K \delta t + F^K \frac{\delta t^2}{2} \quad (16)$$

$$V^K(t + \delta t) = V^K(t) + \left(F^K(t) + F^K(t + \delta t) \right) \frac{\delta t}{2} \quad (17)$$

It is easy to show that this time evolution ensures that the Gauss law constraint in Eq.(9) is preserved, namely if the constraint is satisfied at time t , then $\sum_i [X^i(t + \delta t), V^i(t + \delta t)] = 0$ for all later times. Thus, once we have chosen initial conditions satisfying the Gauss Law constraint, then the constraint will be satisfied for all later time. In order to observe the error caused by discretization, we chose a time step of $\delta t = 10^{-4}$ and $\delta t = 5 \times 10^{-4}$ and compared quantities $\text{Tr}((X_i)^2)$ and $\text{Tr}([X^i, X^j]^2)$. The results are shown in Appendix D.

2.3 Computing Lyapunov Exponents

In this section, the process of computing Lyapunov exponents is described. There are three steps involved. The first step is to thermalise the system by evolving it for a sufficiently long time. After obtaining a thermalised configuration (X, V) we perturb the system to obtain a nearby point (X', V') . We then evolve both configurations simultaneously, measuring the rate at which they diverge. Each step will be discussed below in more detail.

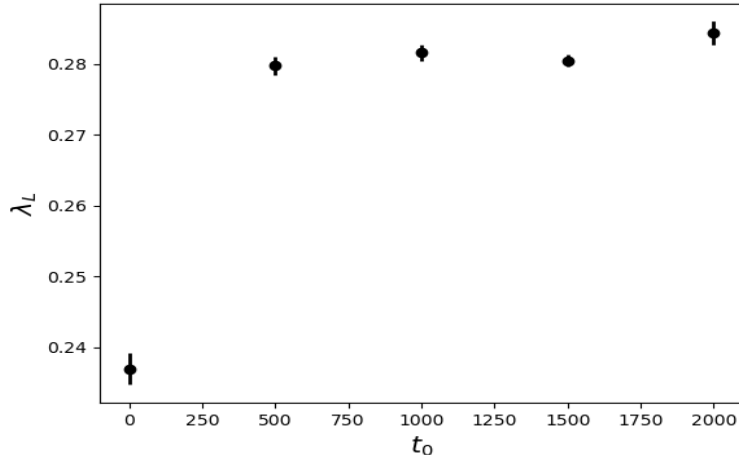


Figure 1: Lyapunov exponents as a function of the thermalisation time t_0 . The exponent can be seen to saturate for $t_0 \geq 500$

We begin by choosing our initial conditions satisfying the Gauss Law constraint. To do this we choose X^i to be random Hermitian matrices with zero trace, and initially set $V^i = 0$. We then evolve the system for a sufficiently long time to obtain a configuration corresponding to a typical state that is uncorrelated with our initial conditions. The time taken for a system to reach a typical state is what we call the thermalisation time t_0 . In order to determine how long we needed to wait before the system was thermalised, we investigated the effect of t_0 on the Lyapunov exponents. The results are shown in Fig.(1). In what follows we set $t_0 = 1000$.

Once we have obtained a thermalised configuration (X, V) , we must perturb it slightly in order to obtain the perturbed configuration (X', V') . To do this we deform the potential by adding the following interaction terms to the potential

$$\sum_{k=1}^{k_0} c_k \text{Tr} \left[\left(\sum_i X_i^2 \right)^k \right] \quad (18)$$

where we have chosen $k_0 = 2$ and the coefficients c_k are chosen from a normal distribution $\mathcal{N}(0, 10^{-8})$. The details of the deformation do not affect

the results, the only limitation on choosing the deformation is that it must obey the Gauss Law constraint. Addition of the interactions modifies the force from F^i to

$$\tilde{F}^i(t) = F^i(t) + \sum_k^{k_0} k c_k \left\{ X^i(t), \left(\sum_j X_j^2(t) \right)^{k-1} \right\} \quad (19)$$

We evolve our thermalised configuration (X, V) for $t = 1$ under this modified force to obtain our perturbed configuration (X', V') . Once obtained, we switch off the deformation, restoring the original potential.

Having obtained a reference configuration (X, V) and a perturbed configuration (X', V') , we evolve them both simultaneously and measure the distance between them as a function of time. In order to measure the distance we must introduce a metric on phase space. The distance function we use is

$$|\delta X(t)| = \sqrt{\sum_{i=1}^9 \text{Tr}(\delta X_i^2(t))}, \quad \delta X \equiv |X' - X| \quad (20)$$

It is proven in [9] that if the phase space is compact then the Lyapunov exponents are independent of the chosen metric. We find that the distance grows exponentially

$$|\delta X(t)| = |\delta X(0)| e^{\lambda_L t} \quad (21)$$

where λ_L is the Lyapunov exponent that quantifies the rate at which chaos develops within the system.

3 Discussion

3.1 Lyapunov exponents

The time evolution of $|\delta X(t)|$ is shown in Fig(2). It is clear that exponential growth sets in quickly. Growth continues until the size of the perturbation becomes of the order of the size of the system. Since phase space is compact, the distance will not be able to grow larger than the total size of the system and so it must saturate at some value. As previously mentioned, the time it

takes for the growth to saturate is what we will define to be the scrambling time of this classical system. We can compute Lyapunov exponents directly from this growth by plotting the logarithm of $|\delta X(t)|/|\delta X(0)|$. Then from Eq.(21) it is obvious that the Lyapunov exponent can be computed from the slope of the line in the region where the distance is growing exponentially. The resulting Lyapunov exponents for various values of N are shown in Fig(3).

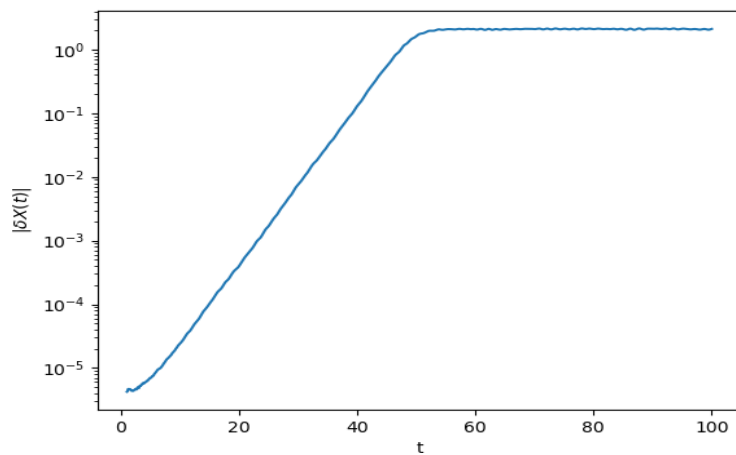


Figure 2: Growth of perturbations for $N = 16$, $t = 0$ is the time of the initial perturbation

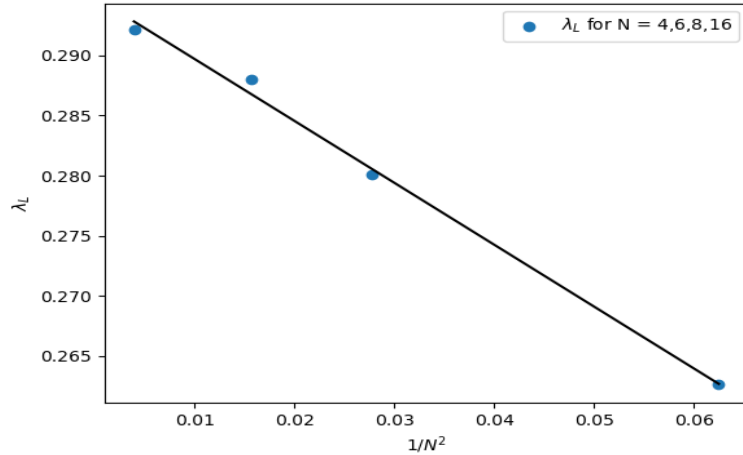


Figure 3: Lyapunov exponents for $N = 4, 6, 8, 16$. The line shows Eq.(x) with these exponents.

We determined the large N behaviour to be governed by

$$\lambda_L = \left[0.294 - \frac{0.515}{N^2} \right] \quad (22)$$

which is in good agreement with those results found in [9].

3.2 Scrambling time of the system

Since we have defined the scrambling time to be the time taken for the infinitesimal perturbation $|\delta X(t)|$ to grow to be of the order of the size of the system, the scrambling time will depend on the size of the initial perturbation $|\delta X(0)|$. Thus, in order to determine how the scrambling time scales with N , we must increase N while keeping the size of the initial perturbation fixed. A fast scrambler is a system whose scrambling time scales as $t_* \sim \log N^2$. We compute the value that $|\delta X(t)|$ saturates to by considering a reference configuration X and a perturbed configuration X' , and use our metric defined in Eq.(20) to compute the distance between them at late time. Fig(4) shows the late time behaviour of $|\delta X(t)|$ for various values of N .

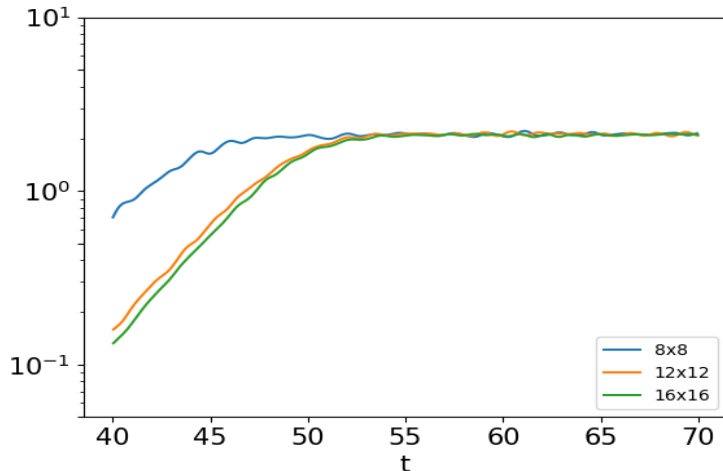


Figure 4: The late time behaviour of $|\delta X(t)|/\sqrt{N}$ for various values of N . At late times we see convergence to a value of order 1

We can see that distance between the two configurations is of the order of $|X - X'| \sim \sqrt{N}$. We can thus expect that the scrambling time is given by

$$e^{\lambda_L t_*} \sim \sqrt{N} \implies t_* \sim \frac{1}{4\lambda_L} \log N^2 \quad (23)$$

Since we have already found that λ_L is independent of N to leading order, and we can verify numerically that at late time $|\delta X(t)| \sim \sqrt{N}$, this establishes the system as a "fast scrambler".

3.3 Energy dependence of lyapunov exponents

It was conjectured in [10] that there is an upper bound on the energy dependence of Lyapunov exponents for classical and quantum Hamiltonian systems. The conjecture is that the Lyapunov exponent $\lambda(E)$ cannot grow faster than linearly in the Energy E . This results it consistent with the chaos bound in [7]. The energy dependence of the exponents is given by

$$\lambda(E) \propto E^c, \quad c \leq 1 \quad (24)$$

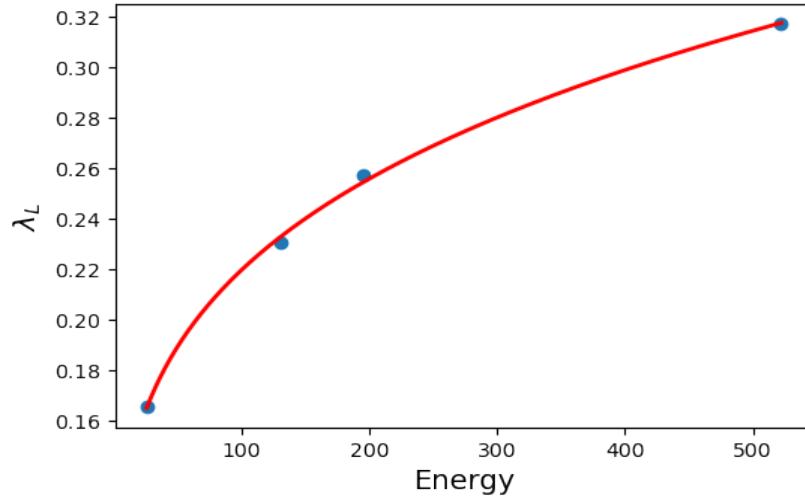


Figure 6: Lyapunov exponent as a function of energy for $N = 7$

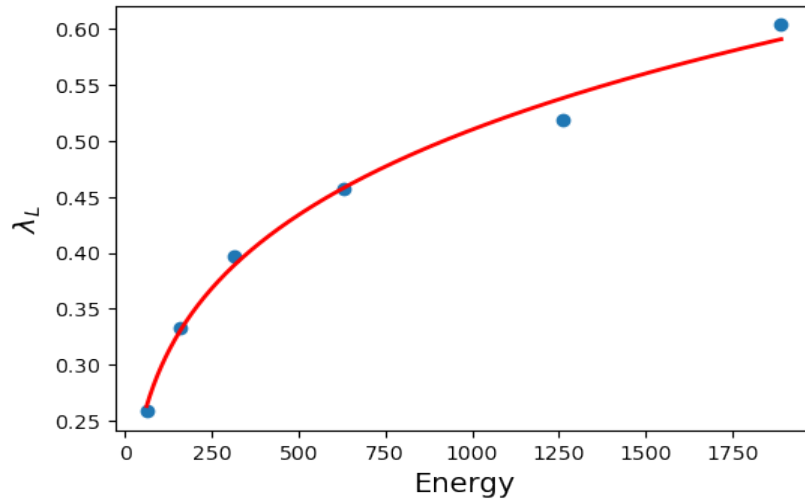


Figure 5: Lyapunov exponents as a function of energy for $N = 4$

We studied the dependence of λ_L on the total energy for the case of $N = 4$ and $N = 7$. The results are shown in Fig.(5) and Fig.(6) respectively.

For the case of $N = 4$ we found that the Lyapunov exponent depends on energy according to

$$\lambda_L(E) \sim E^{0.24} \quad (25)$$

while for $N = 7$ we found the dependence

$$\lambda_L(E) \sim E^{0.22} \quad (26)$$

and thus, we can conclude that this system satisfies the energy bound conjectured in [10].

4 Summary

We studied classical chaos in a system of N D0 branes by numerically computing Lyapunov exponents for various values of N . The Lyapunov behaviour was found to be governed by

$$\lambda_L = \left[0.294 - \frac{0.515}{N^2} \right] \quad (27)$$

We then proceeded to define a classical analogue of the scrambling time as the time taken for the small perturbations to grow to be of the order of the size of the system. Saturation of the perturbations occurred in a time logarithmic in the number of degrees of freedom $t_* \sim \log N^2$. Since "fast scramblers" have been defined as systems whose scrambling time is logarithmic in the number of degrees of freedom, we conclude that this system is indeed a fast scrambler. Lyapunov exponents were then considered as a function of energy and the case for $N = 4$ and $N = 7$ were found to satisfy the bound conjectured in [10]. This work has been a precursor to studying classical chaos in pure D-brane black holes in $\mathcal{N} = 8$ String Theory. Such a system is similar, yet more complicated than the system studied here. We intend on using similar methods as those described here to study Lyapunov behaviour within this system, as well as to study the scrambling time for this system. We hope to find a scrambling time logarithmic in the black hole entropy. If we could find such behaviour, this would imply that classical chaos is somehow aware of a highly quantum phenomenon.

A Equations of motion

After choosing the gauge $A_t = 0$, the Lagrangian for the system is

$$L = \frac{1}{2g^2} \text{Tr} \left(\sum_i (\dot{X}^i)^2 + \frac{1}{2} \sum_{i \neq j} [X^i, X^j]^2 \right) \quad (28)$$

We obtain the equations of motion by using the Euler-Lagrange equations

$$\frac{\partial L}{\partial X^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{X}^i} = 0 \quad (29)$$

The kinetic term in the Lagrangian is independent of the generalised and so differentiating the Lagrangian with respect to the generalised coordinates amounts to differentiating the potential term

$$\begin{aligned} \frac{\partial L}{\partial X_{ab}^K} &= \frac{1}{4g^2} \frac{\partial}{\partial X_{ab}^K} \sum_{i \neq j} \text{Tr} \left([X^i, X^j]^2 \right) \\ &= \frac{1}{4g^2} \frac{\partial}{\partial X_{ab}^K} \sum_i \left(\text{Tr} ([X^K, X^i]^2) + \text{Tr} ([X^i, X^K]^2) \right) \\ &= \frac{1}{2g^2} \frac{\partial}{\partial X_{ab}^K} \sum_i \text{Tr} ([X^i, X^K]^2) \end{aligned}$$

the second equality follows from the fact that the derivative of the trace will be only non-zero when $i = K$ or $j = K$ and the last equality from the fact that $[X^i, X^K]^2 = [X^K, X^i]^2$. Expanding the commutator and using the cyclic property of the trace gives

$$\begin{aligned} \frac{\partial L}{\partial X_{ab}^K} &= \frac{1}{2g^2} \frac{\partial}{\partial X_{ab}^K} \sum_i \text{Tr} \left(X^i X^K X^i X^K - X^i X^K X^K X^i - X^K X^i X^i X^K + X^K X^i X^K X^i \right) \\ &= \frac{1}{2g^2} \frac{\partial}{\partial X_{ab}^K} \sum_i \left(2\text{Tr}(X^i X^K X^i X^K) - 2\text{Tr}(X^i X^K X^K X^i) \right) \\ &= \frac{1}{g^2} \frac{\partial}{\partial X_{ab}^K} \sum_{i,j,n,m,p} \left(X_{jn}^i X_{nm}^K X_{mp}^i X_{pj}^K - X_{jn}^i X_{nm}^K X_{mp}^K X_{pj}^i \right) \end{aligned}$$

Taking the derivative gives

$$\begin{aligned}
\frac{\partial L}{\partial X_{ab}^K} &= \frac{1}{g^2} \sum_{i,j,p,n,m} \left(X_{ja}^i X_{bp}^i X_{pj}^K + X_{bn}^i X_{nm}^K X_{ma}^i - X_{jn}^i X_{na}^K X_{bj}^i - X_{bp}^K X_{pj}^i X_{ja}^i \right) \\
&= \frac{1}{g^2} \sum_i \left((X^i X^K X^i)_{ba} + (X^i X^K X^i)_{ba} - (X^i X^i X^K)_{ba} - (X^K X^i X^i)_{ba} \right) \\
&= \frac{1}{g^2} \sum_i \left([X^i, X^K X^i]_{ba} - [X^i, X^i X^K]_{ba} \right) \\
&= \frac{1}{g^2} \sum_i [X^i, [X^K, X^i]]_{ba}
\end{aligned}$$

Differentiating the Lagrangian with respect to the generalised velocities gives

$$\begin{aligned}
\frac{\partial L}{\partial \dot{X}_{ab}^K} &= \frac{1}{2g^2} \frac{\partial}{\partial \dot{X}_{ab}^K} \text{Tr} \sum_i (\dot{X}^i)^2 \\
&= \frac{1}{2g^2} \frac{\partial}{\partial X_{ab}^K} \sum_{i,j,k} \dot{X}_{jk}^i \dot{X}_{kj}^i \\
&= \frac{1}{g^2} \dot{X}_{ba}^K
\end{aligned}$$

Now using the Euler-Lagrange equations gives the equations of motion for the system

$$\frac{\partial L}{\partial X^K} - \frac{d}{dt} \frac{\partial L}{\partial \dot{X}^K} = \frac{1}{g^2} \sum_i [X^i, [X^K, X^i]]^T - \frac{1}{g^2} \ddot{X}^K = 0$$

Thus we have arrived at the following equations of motion for the system

$$\ddot{X}^K = \sum_i [X^i, [X^K, X^i]]$$

B Gauss law constraint

The Gauss law constraint is obtained by computing the equations of motion for the gauge field and then choosing the gauge to be $A_t = 0$. The equations of

motion for the gauge field can be obtained using the Euler-Lagrange equation. Before choosing the gauge $A_t = 0$, the total Lagrangian for the system is

$$L = \frac{1}{2g^2} \text{Tr} \left(\sum_i (D_t X^i)^2 + \frac{1}{2} \sum_{i \neq j} [X^i, X^j]^2 \right) \quad (30)$$

where the covariant derivative is defined as $D_t X^i = \partial_t X^i + [A_t, X^i]$. The Lagrangian is independent of \dot{A}_t so immediately we have

$$\frac{\partial L}{\partial \dot{A}_t} = 0 \quad (31)$$

Thus, the Euler-Lagrange equation reduces to

$$\frac{\partial L}{\partial A_t} = 0 \quad (32)$$

The potential term in the Lagrangian is independent of A_t and so the calculation amounts to differentiating the kinetic term with respect to A_t

$$\begin{aligned} \left(\frac{\partial L}{\partial A_t} \right)_{ab} &= \frac{1}{2g^2} \frac{\partial}{\partial (A_t)_{ab}} \text{Tr} \sum_i (D_t X^i)^2 \\ &= \frac{1}{2g^2} \frac{\partial}{\partial (A_t)_{ab}} \text{Tr} \sum_i (\partial_t X^i + [A_t, X^i])^2 \end{aligned}$$

Expanding the term in brackets and using the cyclic property of the trace gives

$$\begin{aligned}
\frac{\partial L}{\partial(A_t)_{ab}} &= \frac{1}{2g^2} \frac{\partial}{\partial(A_t)_{ab}} \text{Tr} \sum_i \left((\dot{X}^i)^2 - \dot{X}^i A_t X^i + \dot{X}^i X^i A_t - A_t X^i \dot{X}^i + X^i A_t \dot{X}^i \right. \\
&\quad \left. + A_t X^i A_t X^i - A_t X^i X^i A_t - X^i A_t A_t X^i + X^i A_t X^i A_t \right) \\
&= \frac{1}{2g^2} \frac{\partial}{\partial(A_t)_{ab}} \sum_i \left(\text{Tr}(\dot{X}^i)^2 - 2\text{Tr}(\dot{X}^i A_t X^i) + 2\text{Tr}(\dot{X}^i X^i A_t) \right. \\
&\quad \left. + 2\text{Tr}(A_t X^i A_t X^i) - 2\text{Tr}(A_t X^i X^i A_t) \right) \\
&= \frac{1}{g^2} \frac{\partial}{\partial(A_t)_{ab}} \left(\sum_{i,j,k,n} X_{jk}^i X_{kn}^i (A_t)_{nj} - \sum_{i,j,k,n} X_{jk}^i X_{kn}^i (A_t)_{nj} \right. \\
&\quad \left. + \sum_{i,j,k,n,m} (A_t)_{jk} X_{kn}^i (A_t)_{nm} X_{mj}^i - \sum_{i,j,k,n,m} (A_t)_{jk} X_{kn}^i X_{nm}^i (A_t)_{mj} \right) \\
&= \frac{1}{g^2} \left(\sum_{i,j} \dot{X}_{bj}^i X_{ja}^i - \sum_{i,n} \dot{X}_{na}^i X_{bn}^i + \sum_{i,n,m} (A_t)_{nm} X_{ma}^i X_{bn}^i + \sum_{i,j,k} X_{bj}^i (A_t)_{jk} X_{ka}^i \right. \\
&\quad \left. - \sum_{i,m,j} (A_t)_{bm} X_{mj}^i X_{ja}^i - \sum_{i,j,k} X_{bj}^i X_{jk}^i (A_t)_{ka} \right) \\
&= \frac{1}{g^2} \left([\dot{X}^i, X^i]_{ba} + 2(X^i A_t X^i)_{ba} + [X^i X^i, A_t]_{ba} \right)
\end{aligned}$$

Finally, choosing the gauge $A_t = 0$ and using Eq.(23) leaves the Gauss law constraint

$$\sum_i [X^i, \dot{X}^i] = 0 \tag{33}$$

C Velocity Verlet algorithm

To discretize the equations of motion we write

$$\dot{X}^K = V^K \tag{34}$$

$$\dot{V}^K = F^K = \sum_i [X^i, [X^K, X^i]] \tag{35}$$

The discretized time evolution can be obtained by using Taylor expansions and terminating at the order of δt^2 . Expanding gives

$$X^K(t + \delta t) = X^K(t) + \dot{X}^K(t)\delta t + \ddot{X}^K(t)\frac{\delta t^2}{2} \quad (36)$$

$$V^K(t + \delta t) = V^K(t) + \dot{V}^K(t)\delta t + \ddot{V}^K(t)\frac{\delta t^2}{2} \quad (37)$$

We must develop an expression for \ddot{V}^K in terms of known quantities. To do this we can expand \dot{V}^K and terminate the expansion at the order δt^2 .

$$\dot{V}^K(t + \delta t) = \dot{V}^K(t) + \ddot{V}^K(t)\delta t \quad (38)$$

We can take the expansion up to order δt since we are looking for an approximation of the quantity $\frac{\delta t^2}{2}\ddot{V}^K$ that holds to order δt^2 . Rearranging Eq.(29) for \ddot{V}^K and substituting the result into Eq.(28) gives

$$V^K(t + \delta t) = V^K(t) + \left(\dot{V}^K(t) + \dot{V}^K(t + \delta t) \right) \frac{\delta t}{2} \quad (39)$$

Using the definitions in Eq.(25) and Eq.(26) gives the discretized time evolution

$$X^K(t + \delta t) = X^K(t) + V^K\delta t + F^K\frac{\delta t^2}{2} \quad (40)$$

$$V^K(t + \delta t) = V^K(t) + \left(F^K(t) + F^K(t + \delta t) \right) \frac{\delta t}{2} \quad (41)$$

D Numerical error

We examined two quantities, $\text{Tr}((X^i)^2)$ and $\text{Tr}([X^i, X^j]^2)$ where summation over repeated indices is implied. We computed the time evolution of these quantities using two different time steps $\delta t = 1 \times 10^{-4}$ and $\delta t = 5 \times 10^{-4}$ in order to observe the numerical errors. The results are shown below. We find good agreement for $t \lesssim 50$.

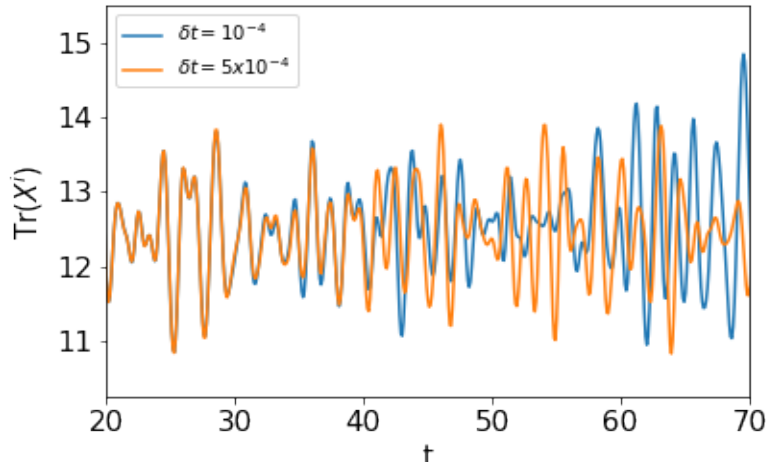


Figure 7: Time evolution of $\text{Tr}((X^i)^2)$ for a time step $\delta t = 1 \times 10^{-4}$ and $\delta t = 5 \times 10^{-4}$

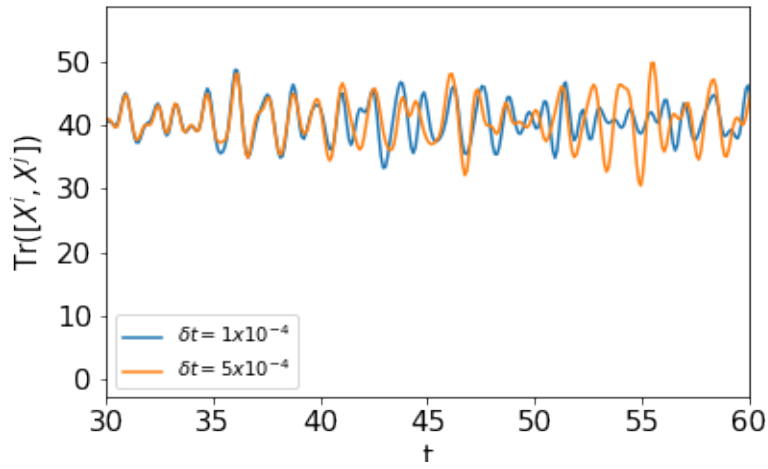


Figure 8: Time evolution of $\text{Tr}([X^i, X^j]^2)$ for a time step $\delta t = 1 \times 10^{-4}$ and $\delta t = 5 \times 10^{-4}$

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