

Hybrid Monte Carlo Simulation of the Bosonic BFSS Model

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Abstract

This report outlines the bosonic BFSS model which is conjectured to be equivalent to M-theory in the low-energy limit. The lattice discretised version of the model is obtained and a hybrid monte carlo algorithm is used to simulate the dynamics of the system. Our simulation result were compared to analytical predictions and were found to be in agreement.

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1 Introduction

M-theory aims to unify different consistent string theories [1]. The leading proposal for a non-perturbative formulation of this theory is expected to be the infinite matrix size limit of a matrix model [2], and one such model is the BFSS model. Named after its creators (T. Banks, W. Fischler, S. Shenker, and L. Susskind), this quantum mechanical matrix model was shown to be described by eleven-dimensional supergravity in the low-energy limit, and was proposed to be equivalent to M-theory [3]. Thus, it can be used to investigate M-theory in a simple setting.

This paper will investigate the dynamics of the 11-dimensional bosonic BFSS model in the context of D0-branes, which are objects on which the strings in string theory end [1]. A D0-brane refers to a point with 0 spatial dimensions and 1 time dimension. A lattice version of the model will be simulated using a Hybrid Monte Carlo simulation which, when given an initial starting point, will illustrate the dynamics of the system. The approach will follow that outlined by Filev and O'Connor [2].

This paper will be structured as follows:

1. An outline of the bosonic BFSS model on the lattice (including time discretisation and equations of motion).
2. An outline of the Hybrid Monte Carlo Algorithm.
3. A discussion of simulation results.

2 Bosonic BFSS Model

The BFSS matrix model is conjectured to correspond to M-theory compactified on a light-like circle [2]. Using dimensional reduction of ten-dimensional supersymmetric Yang-Mills theory to one dimension, the action is given by [4]:

$$S_M = \frac{1}{g^2} \int dt \text{Tr} \left\{ \frac{1}{2} (\mathcal{D}_0 X^i)^2 + \frac{1}{4} [X^i, X^j]^2 - \frac{i}{2} \Psi^T C_{10} \Gamma^0 D_0 \Psi + \frac{1}{2} \Psi^T C_{10} \Gamma^i [X^i, \Psi] \right\} ,$$

Here, X^i are $N \times N$ dimensional matrices with N degrees of freedom, and i, j running from 1 to 9 are the dimension. $\mathcal{D}_t = \partial_t + A(t)$ is the covariant derivative which allows us to account for changes in the gauge field $A(t)$. These gauge fields encode information about the dynamics of the D0-branes [3].

By performing a Wick rotation to Euclidean time and focusing on the bosonic part of the action [2], we obtain the Euclidean action:

$$S_b = \int_0^\beta \text{tr} \left\{ \frac{1}{2} (\mathcal{D}_t X^i)^2 - \frac{1}{4} [X^i, X^j]^2 \right\} dt \quad (1)$$

where $\beta = \frac{1}{k_b T}$ is the reciprocal of the thermodynamic temperature of the system.

2.1 Time Discretisation

In order to simulate this system, we discretise time to Λ sites with a lattice spacing $a = \beta/\Lambda$. We impose the periodic boundary condition $t_\Lambda = t_0$ whilst $t_n = an$, ($n = 0, 1, 2, \dots, \Lambda - 1$). The derivative part of \mathcal{D}_t on the lattice is given by:

$$\partial_t X^i(t_n) = \partial_t X_n^i \rightarrow \frac{X_{n+1}^i - X_n^i}{a} .$$

When considering the discretisation of $t \rightarrow t_n$, the covariant derivative $\mathcal{D}_t X^i$ becomes: [2]:

$$\mathcal{D}_t X^i \rightarrow \frac{U_{n,n+1} X_{n+1}^i U_{n+1,n} - X_n^i}{a} , \quad (2)$$

Where $U_{n,n+1}$ are unitary matrices ($UU^\dagger = \mathbb{1}$) that are transporter fields. By squaring the \mathcal{D}_t term and gathering boundary terms, we have the discrete bosonic action:

$$S_b = \sum_{n=0}^{\Lambda-1} \text{tr} \left\{ -\frac{1}{a} X_n^i U_{n,n+1} X_{n+1}^i U_{n,n+1}^\dagger + \frac{1}{a} (X_n^i)^2 - \frac{a}{4} [X_n^i, X_n^j]^2 \right\} , \quad (3)$$

2.2 Transporter Fields

At each lattice site, there is a local $U(N)$ (unitary) symmetry. This fact can be used to write the action S_b in a more simple form as outlined by Filev and O'Connor [2], wherein they performed the transformation:

$$\begin{aligned} X_0^i &= X_0^i , \\ X_1^i &= U_{0,1} X_1^i U_{0,1}^\dagger , \\ &\dots \\ X_{\Lambda-1}^i &= (U_{0,1} U_{1,2} \dots U_{\Lambda-2, \Lambda-1}) X_{\Lambda-1}^i (U_{0,1} U_{1,2} \dots U_{\Lambda-2, \Lambda-1})^\dagger \end{aligned}$$

and introducing $\mathcal{W} = (U_{0,1} U_{1,2} \dots U_{\Lambda-2, \Lambda-1} U_{\Lambda-1,0})$, Eqn 3 becomes:

$$S_b = -\frac{1}{a} \text{tr} \left\{ \sum_{n=0}^{\Lambda-2} X_n^i X_{n+1}^i + X_{\Lambda-1}^i \mathcal{W} X_0^i \mathcal{W}^\dagger \right\} + \sum_{n=0}^{\Lambda-1} \text{tr} \left\{ \frac{1}{a} (X_n^i)^2 - \frac{a}{4} [X_n^i, X_n^j]^2 \right\}$$

Filev and O'Connor proceed by using the decomposition $\mathcal{W} = V D V^\dagger$ where $D = \text{diag}\{e^{i\theta_1}, \dots, e^{i\theta_N}\}$ (θ_N refer to angles associated with the rotation). However, it is possible to choose a gauge such that S_b becomes:

$$S_b[X, D] = \text{tr} \left\{ -\frac{1}{a} \sum_{n=0}^{\Lambda-2} X_n^i X_{n+1}^i - \frac{1}{a} X_{\Lambda-1}^i D X_0^i D^\dagger + \sum_{n=0}^{\Lambda-1} \left[\frac{1}{a} (X_n^i)^2 - \frac{a}{4} [X_n^i, X_n^j]^2 \right] \right\} \quad (4)$$

2.2.1 A note on transporter fields

It is necessary to account for the interaction between the transporter field and the gauge field (associated with the $U(N)$ symmetry) as well as the dynamics of the transporter field itself. This can be achieved using the covariant derivative of $U_{n,n+1}$ and yields a new phase term that must be accounted for in the Hamiltonian. A full treatment of this derivation is done by Filev and O'Connor [2].

$$S_{\text{FP}}[\theta] = - \sum_{l \neq m} \ln \left| \sin \frac{\theta_l - \theta_m}{2} \right|. \quad (5)$$

2.3 Hamiltonian Dynamics

The actions of this system, S_b and S_{FP} are given by Eqns 4 and 5, respectively. The corresponding Hamiltonian for this system is:

$$H = \frac{1}{2} \sum_{n=0}^{\Lambda-1} \text{tr} P_n^i \cdot P_n^i + \frac{1}{2} \sum_{l=0}^{N-1} P_d^l{}^2 + S_b[X, D(\theta)] + S_{\text{FP}}[\theta], \quad (6)$$

where P_n^i and P_d^l are the canonical momenta corresponding to the hermitian matrices X_n^i and the angles θ_l , respectively. For the Hybrid Monte Carlo algorithm, we are required to know the forces at play in this system. These can be found using Hamilton's equations:

$$\dot{q} = \partial H / \partial p \quad \dot{p} = -\partial H / \partial q = F(q)$$

where q and p are the generalised coordinates and momenta, respectively. In the context of this system, the forces are given by:

$$\begin{aligned} -\partial S_b / \partial X_{0,ml}^i &= \frac{1}{a} (X_1^i - 2X_0^i + D^\dagger X_{\Lambda-1}^i D)_{lm} + a [X_0^j, [X_0^i, X_0^j]]_{lm}, \\ -\partial S_b / \partial X_{n,ml}^i &= \frac{1}{a} (X_{n+1}^i - 2X_n^i + X_{n-1}^i)_{lm} + a [X_n^j, [X_n^i, X_n^j]]_{lm} \quad \text{for } n = 1, \dots, \Lambda - 2, \\ -\partial S_b / \partial X_{\Lambda-1,ml}^i &= \frac{1}{a} (D X_0^i D^\dagger - 2X_{\Lambda-1}^i + X_{\Lambda-2}^i)_{lm} + a [X_{\Lambda-1}^j, [X_{\Lambda-1}^i, X_{\Lambda-1}^j]]_{lm}, \\ -\partial S_b / \partial \theta_l &= \frac{2}{a} \sum_{m=0}^{N-1} \text{Re} (i X_{\Lambda-1,ml}^i X_{0,lm}^i e^{i(\theta_l - \theta_m)}) + \sum_{m, m \neq l} \cot \left(\frac{\theta_l - \theta_m}{2} \right) \end{aligned} \quad (7)$$

The following section will detail how these are implemented into the HMC algorithm.

3 Hybrid Monte Carlo

The idea behind the HMC algorithm is that we begin with a random initial configuration of positions, add random momentum kicks, and let the system evolve according to Hamiltonian dynamics [5]. The steps implemented were as follows:

1. Initialise random X^i , and P^i which are lists of Λ matrices. Also, initialise random θ_l and P^l .
2. Use a leapfrog algorithm to make a half-step momentum update, full-step position update, and another half-step momentum update. The equations used are [6]:

$$\begin{aligned}
 P_{i+1/2} &= P_i + \frac{\Delta t}{2} F(X_i) \\
 X_{i+1} &= X_i + \Delta t P_{i+1/2} \\
 P_{i+1} &= P_{i+1/2} + \frac{\Delta t}{2} F(X_{i+1})
 \end{aligned}$$

where $F(X_i)$ are the forces given by Eqn 7. Δt is a time-step that is given a small value.

3. Perform a Metropolis Acceptance Check [7] to see if the new configurations are probable enough to be accepted. If $dS = S_{i+1} - S_i < 0$ i.e. the action has decreased, the new configuration is accepted. If $dS > 0$, then the configuration is accepted with a probability $\exp(-dS)$. In practise, the configuration is accepted if $\exp(-dS) > \text{rand}[0, 1]$

Steps 2 and 3 are repeated until a sufficiently long chain of configurations has been generated.

4 Discussion

In order to test the validity of our algorithm, we compared our results to analytical estimates and expected behaviours of these systems. One check that we used was $\langle S \rangle$, the expectation value of the action. The second check was examining whether the eigenvalues of our configurations were distributed according to the Wigner Semicircle Law.

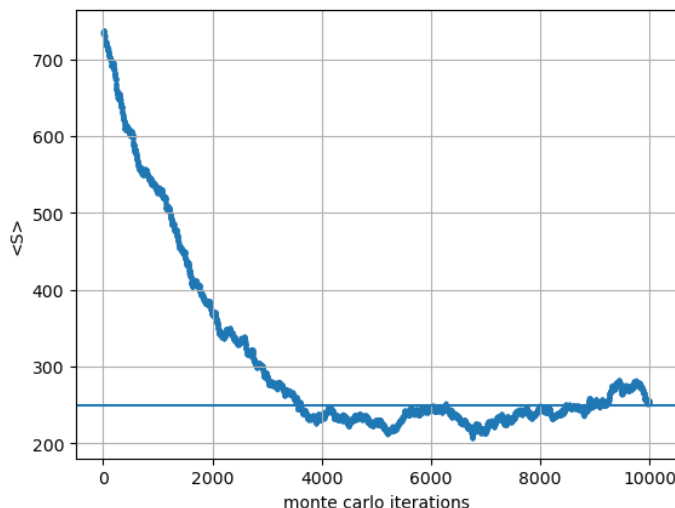


Figure 1: Plot of $\langle S \rangle$ vs monte carlo iterations for $N = 10, \Lambda = 5, \beta = 1$ in the one dimensional case. The system can clearly be seen to approach an equilibrium around the analytically derived result, given by the horizontal line.

4.1 Expectation value of the action

The partition function $Z = \int [dx] e^{-(S_2[x] + S_3[x] + \dots + S_k[x])}$, where $S_k[x]$ is the action (a polynomial of order k), encodes the statistical properties of a system in thermodynamic equilibrium [8].

We can define $Z_B = B^{-N^2\Lambda} Z$ where B is a dummy variable that will be set to 1 later. Introducing $\tilde{x} = x/B$, we have:

$$Z_B = \int d^{N^2\Lambda} \tilde{x} e^{-(B^2 S_2[\tilde{x}] + B^3 S_3[\tilde{x}] + \dots + B^k S_k[\tilde{x}])}$$

So, we then have:

$$\frac{B}{Z_B} \frac{dZ_B}{dB} = -N^2\Lambda$$

$$\frac{B}{Z_B} \frac{dZ_B}{dB} = - \int d^{N^2\Lambda} \tilde{x} B \frac{d}{dB} (B^2 S_2[\tilde{x}] + B^3 S_3[\tilde{x}] + \dots + B^k S_k[\tilde{x}]) e^{-(B^2 S_2[\tilde{x}] + B^3 S_3[\tilde{x}] + \dots + B^k S_k[\tilde{x}])}$$

However, by definition, the terms on the right are just the expectation values.

$$\frac{B}{Z_B} \frac{dZ_B}{dB} = 2B^2 \langle S_2 \rangle + 3B^3 \langle S_3 \rangle + \dots + kB^k \langle S_k \rangle$$

$$\therefore N^2\Lambda = 2 \langle S_2 \rangle + 3 \langle S_3 \rangle + \dots + k \langle S_k \rangle$$

after setting $B = 1$. Plotting $\langle S[X^i] \rangle$ for the one-dimensional case vs the Monte Carlo time t , as displayed in Fig 1 for $N = 10, \Lambda = 5$, the system quickly approaches and oscillates around an equilibrium value. The horizontal blue line is

the analytically derived expectation value, so it is clear that our algorithm returns the expected results. Fig 2 displays the five-dimensional case for $N = 10$, $\Lambda = 4$ with similar behaviour.

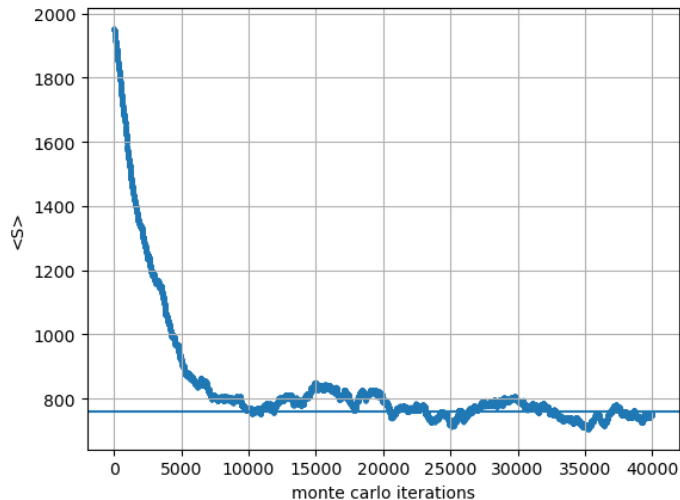


Figure 2: Plot of $\langle S \rangle$ vs monte carlo iterations for $N = 10$, $\Lambda = 4$, $\beta = 1$ in the five-dimensional case. The system can clearly be seen to approach an equilibrium around the analytically derived result, given by the horizontal line.

4.2 Eigenvalue distribution

Wigner's semicircle law can be used as an additional test of our algorithm. If we let X be a symmetric $N \times N$ matrix where the entries are independent and identically distributed random variables with bounded moments, then Wigner's semicircle law states that the eigenvalue distribution of X converges to a distribution in the shape of a semicircle as N goes to infinity [9].

The simulations carried out were limited by our available computational power, so very high values of N could not be used. However, the expected trend was still visible for $N = 150$, as shown in Fig 3

Increasing values of N would display a more apparent semicircle. The infinite N limit is conjectured to be a formulation of a non-perturbative model of M-theory [2]. Hence with enough computational power, this algorithm could be used to study dynamics of M-theory. The low-energy limit ($\beta \ll 1$) BFSS model algorithm could be used to study supergravity, for example, or known results could be compared to this model to further investigate the proposal that the BFSS model is equivalent to M-theory [3].

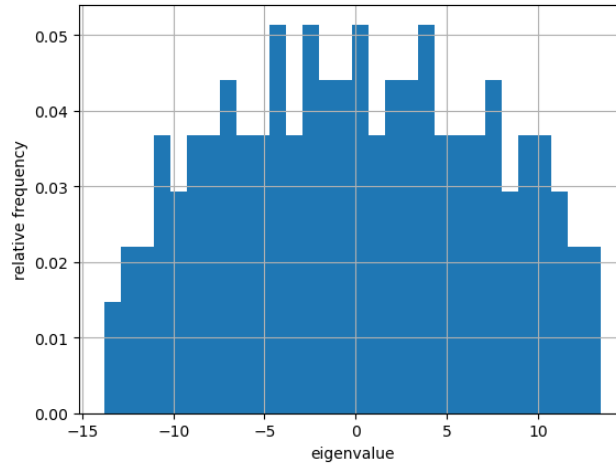


Figure 3: Eigenvalue distribution for $N = 150$. A semi-circular distribution is apparent, which agrees with the expected distribution according to the Wigner semicircle law.

5 Conclusion

In this paper, we outlined the bosonic BFSS model and how a Hybrid Monte Carlo algorithm can be used to simulate the dynamics of an 11-dimensional system. Our results agree with expected behaviour from random matrix theory and analytical derivations from the BFSS model. However, our simulations were not extensive as we were severely limited by computational capabilities.

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