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Black Hole Entropy in String Theory

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A report submitted upon completion of the
Dublin Institute for Advanced Studies
Summer Studentship Programme

31 July 2023

Contents

Notation and Conventions

- We use natural units, in which $c, \hbar, k_{\mathrm{B}} = 1$. Unless otherwise specified, we also use units in which $G = 1$.
- We choose the branch of the square root function $\sqrt{z}$ which is positive when $z > 0$.

Acknowledgments

I'd like to thank my supervisor Dr. Aradhita Chattopadhyaya for her guidance throughout this project, and her helpful feedback on the draft of this report. I'm also very grateful to my collaborators, Rachel, Jack, and Anito, for their contributions to the project.

The figures in this report were generated using Mathematica [44], Ti*k*Z [45], Adobe Illustrator [46], and the Python packages `mpmath` [47] and `cplot` [48].

Abstract

The study of black hole entropy provides one of the most useful guides in the development of string theory, particularly in the absence of direct experimental tests. A condition for the consistency of a theory of quantum gravity is that it successfully explain the macroscopic thermodynamic entropy of a black hole as a logarithm of the number of microstates, in accordance with the Boltzmann relation of statistical mechanics. One of the most significant successes of string theory has been its prediction that the entropy of a class of extremal supersymmetric black holes, computed in terms of the number of microstates, is equal to that predicted by the Bekenstein–Hawking formula for black hole entropy. In this report, we compute microstate degeneracies of quarter-BPS extremal black holes in $\mathcal{N} = 4$ supersymmetric type II string theory compactified on $\mathrm{K3} \times T^2$. This computation involves using the Fourier coefficients of the elliptic genus of K3 to calculate the Igusa cusp form of weight ten. The microstate numbers are obtained, up to a sign, from the Fourier coefficients of the reciprocal of this cusp form. The report concludes with a discussion of the decomposition of the Fourier–Jacobi coefficients of meromorphic Jacobi forms, and the application of this procedure to the computation of microstate degeneracies of single-centred black holes.

Chapter 1

Introduction

Our current understanding of the laws of physics is built upon two vastly different theories. The theory of general relativity describes the large-scale behaviour of our universe, while the theory of quantum mechanics describes the behaviour of microscopic systems under the framework of the Standard Model of particle physics. One of the major aims of modern physics is to unify these two theories into a single fundamental theory that fully describes the laws of nature [1].

One of the prime candidates for this unifying theory is string theory, in which point particles are replaced by one-dimensional strings. String theory has many features that would be desirable in a unified theory, foremost of which is its ability to provide a consistent theory of quantum gravity; other attractions include its uniqueness, its natural incorporation of spacetime supersymmetry, and its ability to describe the Standard Model. However, a significant problem that plagues research into string theory is the lack of direct experimental evidence for its claims. The effects of quantum gravity only become appreciable at the Planck scale, which corresponds to an energy on the order of 10^{18} GeV. The Large Hadron Collider only probes the electroweak scale, on the order of 10^{3} GeV, so there does not seem to be much hope of finding direct experimental evidence of string theory in the foreseeable future [2, 3].

The difficulty of testing string theory experimentally means that any ways of extracting quantitative information about the fundamental degrees of freedom of quantum gravity are highly significant. One such source of information arises in the study of the entropy of black holes. A major success of string theory is that it can successfully explain the macroscopic thermodynamic entropy of a certain class of black holes as a statistical entropy that takes the form of a logarithm of the microscopic degeneracy of states [4].

In this report, we will introduce the main concepts of string theory and quantum field theory, which will allow us to derive the partition functions for bosonic and fermionic strings. By introducing some mathematical concepts from number theory, we will compute the elliptic genus of the manifold K3. The work will culminate in the computation of the microstate degeneracies for a class of extremal supersymmetric black holes in string theory.

1.1 Introduction to String Theory

As mentioned above, string theory is one of the most promising candidates for a theory that unifies all four fundamental forces of nature. In string theory, all elementary particles are postulated to be one-dimensional objects called *strings*, rather than the point particles that form the basis of quantum field theory. These strings may be open or closed. One of the major attractions of string theory is that it incorporates general relativity within the framework of a consistent quantum theory. String theory also leads to gauge groups large enough to contain the Standard Model [1, 2].

1.1.1 Supersymmetry

The consistency of string theory requires *supersymmetry*, which is a symmetry that relates bosons and fermions. Representations of the supersymmetry algebra pair every bosonic state with an equal-energy fermionic state, and vice versa. This means that each particle from one group is paired with another particle in the other group, called its *superpartner*. Supersymmetry transformations are generated by *supercharges*, which are operators that transform bosons into fermions, and vice versa [1, 5].

An important concept related to supersymmetry in string theory is that of *Bogomolnyi–Prasad–Sommerfeld (BPS) states*. These are states that are invariant under a nontrivial subalgebra of the full supersymmetry algebra. All BPS states carry conserved charges, which exactly determine the mass of the state in a given moduli space [2]. If such a state preserves half of the total supersymmetries, it is known as a half-BPS state, while if it preserves a quarter of the original supersymmetries, it is a quarter-BPS state [4].

1.1.2 Worldsheets

Whereas a particle sweeps out a worldline in Minkowski space as time progresses, a string sweeps out a *worldsheet*. This worldsheet can be parameterised by one timelike coordinate, τ, and one spacelike coordinate, σ. For a closed string, σ is periodic and we take it to have range $\sigma \in [0, 2\pi)$. A string therefore sweeps out a surface in spacetime which defines a map $X^\mu(\tau, \sigma)$ from the worldsheet to Minkowski space, where $\mu = 0, \ldots, D-1$ for a spacetime of dimension D. Sometimes the two worldsheet coordinates are combined into one entity $\sigma^\alpha = (\tau, \sigma)$, with $\alpha = 0, 1$. The worldsheets of typical open and closed strings are illustrated in Figure 1.1. In most cases, the solution to the equation of motion for a string has left-moving (holomorphic) and right-moving (anti-holomorphic) components [3]. In this work, we follow the notation of [2] and distinguish between the two types by placing a tilde ˜ above anti-holomorphic operators.

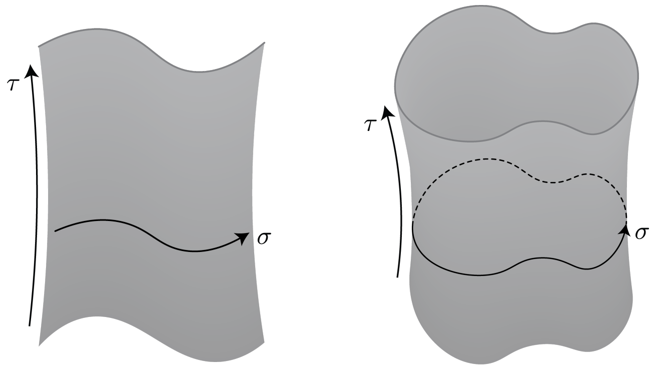

Figure 1.1: Typical worldsheets swept out by an open string (shown on the left) and closed string (shown on the right) in spacetime.

The original version of string theory is *bosonic string theory*, in which string worldsheets have only bosonic modes. In this case, the required *critical dimension* of spacetime is found to be $D = 26$. Since this theory does not predict the existence of fermions, it is not a valid candidate for a true physical model. In contrast, *superstring theory* also incorporates fermionic modes, and the corresponding critical dimension is $D = 10$. We will denote the bosonic string worldsheet by $X(\tau, \sigma)$, and the fermionic string worldsheet by $\psi^\mu(\tau, \sigma)$. Depending on the theory, it is possible for strings to obey periodic or anti-periodic boundary conditions, or even more general twisted boundary conditions. In the basic unorbifolded case, for a closed bosonic string, we require that

$$X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi), \tag{1.1}$$

while for a closed fermionic string, we require that

$$\psi^\mu(\tau, \sigma) = \pm\psi^\mu(\tau, \sigma + 2\pi), \tag{1.2}$$

where the plus sign corresponds to periodic boundary conditions (also known as the *Ramond* sector or R sector), and the minus sign corresponds to anti-periodic boundary conditions (also known as the *Neveu–Schwarz* sector or NS sector).

When fermionic modes are added to the bosonic modes on the worldsheet, the resulting worldsheet theory is supersymmetric, so the strings in this theory are called *superstrings*. Depending on whether fermionic modes are included in both the left- and right-moving sectors or just in the right-moving sector, there are two different classes of superstring theory, which are further subdivided into five distinct theories:

1. **Type II** strings have both left- and right-moving worldsheet fermions. The resulting spacetime theory in ten dimensions has $\mathcal{N} = 2$ supersymmetry, meaning that there are 32 (real) supercharges. This class is further subdivided into *type IIA*, where the supersymmetries associated with the left-moving and right-moving modes have opposite handedness, and *type IIB*, where the left-movers and right-movers have the same handedness. [1, 3]. A third theory called *type I* can be derived from type IIB using a procedure called orientifold projection, which is similar to the concept of orbifolding discussed in Section 1.1.3, but results in a theory with unoriented strings [6].

2. **Heterotic** strings have just right-moving worldsheet fermions. The resulting spacetime theory in ten dimensions has $\mathcal{N} = 1$ supersymmetry, meaning 16 (real) supercharges. Since this theory corresponds to the use of the $D = 26$ bosonic formalism for the left-moving modes and the $D = 10$ superstring formalism for the right-moving modes, there are 16 extra left-moving dimensions. These extra dimensions must take the form of a torus, with a corresponding Lie algebra of $SO(32)$ or $E_8 \times E_8$. For this reason, the heterotic class is further subdivided into $SO(32)$ *heterotic* and $E_8 \times E_8$ *heterotic* [1, 3].

In the late 1980s, the concept of *string duality* was discovered. It was found that the two type II string theories can be related by an equivalence of compactification geometries (see the discussion of compactification below) known as *T-duality*, as can the two heterotic theories, meaning that these are not truly distinct string theories. In another major development, the discovery of *S-duality* in the 1990s showed that strongly-coupled type I theory is equivalent to weakly-coupled $SO(32)$ heterotic theory, and that the type IIB theory is self-dual. Finally, the type IIA and $E_8 \times E_8$ heterotic string theories were found to give rise to an eleventh dimension at strong coupling, leading to the emergence of a new 11-dimensional theory called M-theory. In fact, M-theory allows all five superstring theories to be unified into a single theory [1, 7].

In fact, string theory exhibits even more dualities, since the S-duality and T-duality groups can be combined to produce the more general *U-duality* group. In Chapter 6, we will consider type II string theory compactified on the internal manifold K3 $\times T^2$. This theory has been found to exhibit a highly nontrivial duality with heterotic string theory compactified on T^6 [8, 9].

1.1.3 Compactification and Orbifolds

String theory is only consistent in a ten-dimensional (or in some cases eleven-dimensional) spacetime. The concept of *compactification* attempts to reconcile this higher-dimensional spacetime with the four-dimensional spacetime that we observe. The idea is that the extra dimensions of string theory may be *compactified* on an internal manifold, or *target space*, of a very small size. In other words, six (or seven) of the spacetime dimensions are 'curled up' so as to be too small for us to detect, and we observe only the remaining four [1]. Superstring theories are generally formulated in ten-dimensional Lorentzian spacetime $\mathcal{M}_{10}$, and a compactification to four dimensions is achieved by taking $\mathcal{M}_{10} = \mathbb{R}^{1,3} \times X_6$, where $\mathbb{R}^{1,3}$ is the familiar non-compact four-dimensional Minkowski spacetime, and X_6 is a compact Calabi–Yau manifold of three complex dimensions [4]. In Chapter 6, we will consider black holes on the spacetime with the internal manifold $X_6 = \text{K3} \times T^2$.

The compactification of certain spacetime dimensions can be achieved using *identifications*, where points in a space are identified with each other according to certain equivalence relations. If M is a manifold and G is a set of equivalence relations, then the quotient space M/G consists of the set of equivalence classes of M. More explicitly, if G is a discrete isometry group, then a point in M/G corresponds to a point in M, plus all of its images under the action of the isometries [1]. The *fundamental domain* of an identification is a subset of the entire space such that

1. No two distinct points in the fundamental domain are identified, and
2. Any point in the entire space is identified with some point in the fundamental domain.

To provide a simple example, consider the xy-plane subject to the identifications

$$(x, y) \sim (x + 2\pi a, y), \qquad (x, y) \sim (x, y + 2\pi b). \tag{1.3}$$

In this case, the fundamental domain is the rectangle $[0, 2\pi a] \times [0, 2\pi b]$. The horizontal edges of this square are identified, as are the vertical edges. The resulting compactified space is the two-torus T^2 [10]. This procedure is illustrated in Figure 1.2.

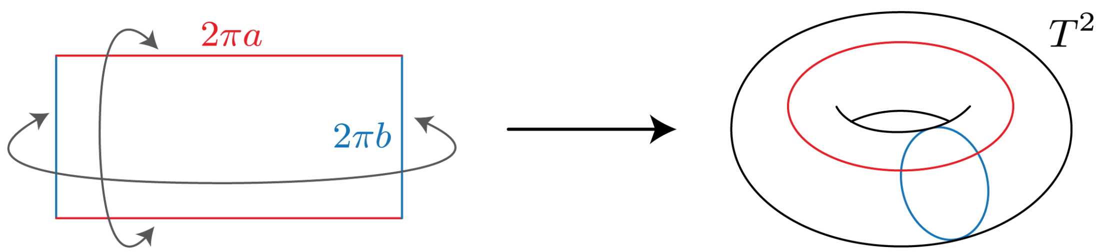

Figure 1.2: We can visualise the application of the given identification (1.3) to the fundamental domain as 'gluing' the horizontal (red) sides together to form a cylinder, and then bending this cylinder to glue the vertical (blue) sides together. The resulting compactified space is the two-torus T^2, where the large (red) circle has radius a and the smaller (blue) circle has radius b.

Some identifications have *fixed points*, which are points that are related to themselves by the identification [10]. This gives rise to the closely related concept of an *orbifold*. A quotient space M/G of a manifold M by a set of discrete symmetries G is an *orbifold* of M if any of the identifications in G have fixed points. In that case, the quotient space has singularities. Away from these singularities, the orbifold M/G is locally indistinguishable from the original manifold M [1].

In general, there are two types of physical states that occur in the spectrum of free strings on an orbifold M/G:

- **Untwisted states** are states that exist on M and are invariant under the group G.
- **Twisted states** are new closed string states that appear only after orbifolding. Consider the theory of closed strings on M. A string in such a theory must satisfy the boundary condition $X^\mu(\sigma) = X^\mu(\sigma+2\pi)$, meaning that it must start and end at the same point. A string that connects a point of M to one of its identifications under G is not an allowed configuration on M, since it is open. However, such a configuration is permissible on M/G, where it corresponds to a closed-string configuration. Mathematically, this corresponds to the condition
$$X^\mu(\sigma+2\pi) = gX^\mu(\sigma) \tag{1.4}$$
for some $g \in G$. There are generally several twisted sectors, which can be labelled by the corresponding group elements g. The untwisted states correspond to $g = 1$ [1].

Strings can propagate consistently on spaces with orbifold singularities provided that twisted sectors are taken into account. This allows a new string theory, defined on M/G, to be produced from a string theory defined on M. The string theory on the orbifold typically exhibits more breaking of supersymmetries, making it more desirable as a description of our physical reality.

In Chapter 5, we will see how the manifold K3 can be realised as an orbifold $T^4/\mathbb{Z}_2$, where the group action of $\mathbb{Z}_2 = \{1, -1\}$ corresponds to the identification of antipodal points. In general, the cyclic group $\mathbb{Z}_N$ is generated by $g_0 = e^{2\pi i/N}$, so $\mathbb{Z}_N$ has elements $g_0^k = e^{2\pi ik/N}$ for $0 \leq k \leq N-1$ [1]. In Chapter 6, we will be concerned with $\mathbb{Z}_N$ *Chaudhuri–Hockney–Lykken (CHL) models*, first introduced in [11], which are obtained by taking the $\mathbb{Z}_N$ quotient of heterotic string theory compactified on T^6.

The concept of identifications is also related to the idea of *moduli*, which are the parameters labelling the geometry of a manifold. Notable examples are the parameters for the geometry of compactification. The set of values that the parameters can take is known as the *moduli space* [2, 10]. For example, the only modulus of the two-torus T^2 defined by (1.3) is the ratio $\tau = a/b$, and tori with ratio τ and $1/\tau$ are conformally equivalent (see section 1.1.4 below). Hence the tori with $0 < \tau \leq 1$ are conformally equivalent to tori with $1 \leq \tau < \infty$. It can be shown that there is no further conformal map that produces a further identification, so that the moduli space of two-tori can be chosen to be either the interval $0 < \tau \leq 1$ or the interval $1 \leq \tau < \infty$, of which latter is the canonical choice [10].

1.1.4 Conformal Field Theory

A *conformal transformation* of spacetime is a change of coordinates $x^\mu \to \tilde{x}^\mu$ such that the metric transforms according to
$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \omega^2(x) g_{\mu\nu}, \tag{1.5}$$
where $\omega(x)$ is some non-vanishing function and x refers to the set of spacetime coordinates x^μ. This transformation is essentially a local rescaling. A key property of conformal transformations is that they leave null curves invariant. To show this, we recall that a curve $x^\mu(\lambda)$ is null with respect to $g_{\mu\nu}$ if and only if its tangent vector is null with respect to $g^{\mu\nu}$:
$$g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0. \tag{1.6}$$
This then implies that
$$\tilde{g}_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = \omega^2(x) g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0, \tag{1.7}$$
so any curve which is null with respect to $g_{\mu\nu}$ will also be null with respect to the transformed metric $\tilde{g}_{\mu\nu}$. This means that light cones are invariant under conformal transformations [12].

A *conformal field theory* (CFT) is a field theory which is invariant under conformal transformations. This means that the physics described by the theory looks the same at all length scales [3]. Such theories are important in string theory because the reparameterisation invariance of the (τ, σ) coordinate system on string worldsheets is equivalent to conformal invariance. [13]. The quantum field theory on the string worldsheet is thus a two-dimensional CFT [14].

A *superconformal field theory* (SCFT) is a conformal field theory which also incorporates supersymmetry. Such theories are naturally incorporated into superstring theory.

1.2 Introduction to Black Holes

1.2.1 Classical Black Holes

In general relativity, a *black hole* is an asymptotically flat spacetime that contains a region which is not in the backwards light cone of future timelike infinity. The boundary of such a region is called the *event horizon* of the black hole [4]. The term 'black hole' is derived from the fact that it is impossible to see inside such a region, since nothing can escape from behind the event horizon.

1.2.1.1 Schwarzschild Black Holes

In classical general relativity, the unique spherically symmetric vacuum solution of the Einstein equation $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = 0$ is the Schwarzschild metric, given in spherical coordinates (t, r, θ, ϕ) as

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2\, d\Omega^2, \tag{1.8}$$

where M is identified as the mass of the gravitating object and $d\Omega^2$ is the metric on S^2,

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2\,. \tag{1.9}$$

This metric has a true singularity at $r = 0$. It also seems to be singular at $r = 2GM$, where $g_{00} \to 0$ and $g_{rr} \to \infty$, but this is in fact merely a coordinate singularity and can be removed by a coordinate change. The surface defined by $r = 2GM$ is the event horizon of the Schwarzschild black hole [4, 12].

Aside from the radius or area of the event horizon, another important black hole parameter is the surface gravity κ. In a static, asymptotically flat spacetime, the surface gravity is the acceleration experienced by a static observer near the horizon, as measured by a static observer at infinity [12]. For the Schwarzschild black hole, $\kappa = 1/4GM$ [4].

1.2.1.2 Black Hole Thermodynamics

In classical general relativity, black holes obey three main mechanical laws. In brief, the zeroth law of black hole mechanics states that the surface gravity κ of a stationary black hole is constant on the event horizon; the first law is a statement of the conservation of energy or, equivalently, conservation of mass; and the second law states that the area of a black hole is nondecreasing [4, 12]. These three laws bear a close resemblance to the first three laws of thermodynamics, as is summarised in Table 1.1.

Law	**Thermodynamics**	**Black Hole Mechanics**
Zeroth	The temperature T of a body in thermal equilibrium is constant throughout the body.	The surface gravity κ of a stationary black hole is constant on the event horizon.
First	The energy of a closed system is conserved. Mathematically, $$dE = T dS + \mu\, dQ + \Omega\, dJ, \tag{1.10}$$ where S is the entropy of the system, Q is the charge, μ is the chemical potential, J is the angular momentum, and Ω is the angular velocity.	The energy of a black hole is conserved. Mathematically, $$dM = \frac{\kappa}{8\pi G} dA + \mu\, dQ + \Omega\, dJ, \tag{1.11}$$ where M is the mass of the black hole, and A is the area, μ is the chemical potential, J is the spin, and Ω is the angular velocity.
Second	In any physical process, the total entropy S of the universe never decreases: $$\Delta S \geq 0.$$	In any physical process, the total area A of a black hole never decreases: $$\Delta A \geq 0.$$

Table 1.1: The first three laws of thermodynamics, alongside the corresponding laws of black hole mechanics. The information presented here is taken from [4], where a similar table also appears.

As we will see in the next section, the correspondence between black hole mechanics and thermodynamics goes deeper than a mere superficial similarity. In fact, a semiclassical treatment shows that the area of a black hole can be considered a measure of its entropy [4, 12].

1.2.2 Semiclassical Black Holes

If an object possessing entropy falls into a classical black hole, then the net entropy of the outside world decreases. Classical black holes therefore violate the second law of thermodynamics. This violation can be resolved by making the assumption that the black hole itself possesses entropy. It then follows from thermodynamic principles that if a black hole has energy E and entropy S, then it must also have a temperature T derived from

$$\frac{1}{T} = \frac{\partial S}{\partial E}. \tag{1.12}$$

But if a black hole has a temperature, then it must radiate like any other thermal body. For a classical black hole, this is impossible since nothing can escape the event horizon. This can be resolved through a semiclassical description of black holes.

In the semiclassical treatment, the classical description of spacetime in general relativity is retained, while fields such as the electromagnetic field are treated quantum mechanically. When quantum effects are included, black holes can indeed radiate, as was shown by Hawking [15]. Quantum theory predicts the constant production of particle-antiparticle pairs, even in vacuum. Near the event horizon, an antiparticle can occasionally fall into the black hole while its partner particle escapes to infinity. This provides a heuristic explanation of Hawking radiation. A detailed calculation shows that the spectrum emitted by the black hole is thermal, with a temperature known as the *Hawking temperature*:

$$T = \frac{\kappa}{2\pi}, \tag{1.13}$$

where κ is the surface gravity of the black hole. For a Schwarzschild black hole of mass M and area A, for which $\mu = \Omega = 0$, the first law of thermodynamics, given by (1.11), is

$$dM = TdS = \frac{\kappa}{8\pi G} dA. \tag{1.14}$$

We can use this to determine a formula for the entropy of a black hole in terms of its area:

$$S = \frac{A}{4G}. \tag{1.15}$$

This is known as the *Bekenstein–Hawking entropy* [4, 12]. It is notable that the entropy is proportional to the area of the black hole rather than its volume, as would typically be expected in thermodynamics.

1.2.2.1 Reissner–Nordström Black Holes

The most general static, spherically symmetric, *charged* solution of the Einstein–Maxwell field equations is the Reissner–Nordström metric, given by

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2+P^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2+P^2}{r^2}\right)^{-1} dr^2 + r^2\, d\Omega^2, \tag{1.16}$$

where we have chosen units in which $G = 1$. In this solution, Q is the total electric charge of the black hole, and P is the total magnetic charge. String theory allows for the existence of *dyons*, which are hypothetical particles which carry both electric and magnetic charge [16], so it is pertinent to include a magnetic charge here, even though magnetic monopoles have not been observed experimentally [12]. The event horizon is located at r such that $g^{rr} = 0$, or

$$1 - \frac{2M}{r} + \frac{Q^2+P^2}{r^2} = 0 \iff \Delta(r) \coloneqq r^2 - 2Mr + Q^2 + P^2 = 0. \tag{1.17}$$

The function $\Delta(r)$ is plotted in Figure 1.3. The quadratic equation $\Delta(r) = 0$ has two solutions:

$$r_\pm = M \pm \sqrt{M^2 - (Q^2+P^2)}. \tag{1.18}$$

Hence the Reissner–Nordström black hole has an outer event horizon at radius r_+ and an inner event horizon at r_-, and the area of the black hole is $A = \pi r_+^2$.

The temperature and entropy of such a black hole are then [4]

$$T = \frac{\sqrt{M^2 - (Q^2+P^2)}}{2\pi\left(2M\left(M + \sqrt{M^2-(Q^2+P^2)}\right) - (Q^2+P^2)\right)} \tag{1.19a}$$

$$S = \pi\left(M + \sqrt{M^2 - (Q^2+P^2)}\right). \tag{1.19b}$$

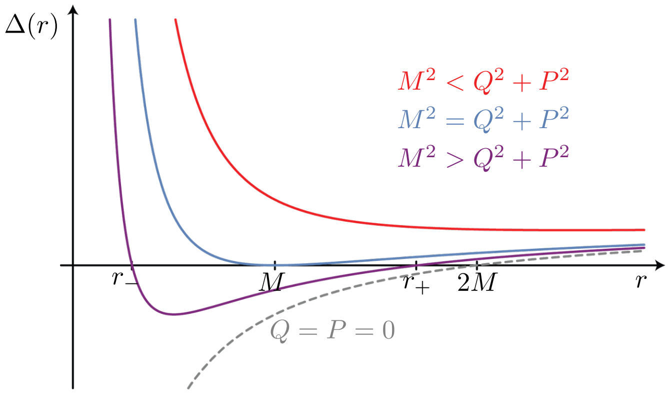

Figure 1.3: The left-hand side $\Delta(r)$ of the Reissner–Nordström quadratic equation (1.17) for different ratios of M^2 to $Q^2 + P^2$. The zeroes of this function correspond to event horizons of the black hole. The case $M^2 < Q^2 + P^2$ includes a naked singularity and is generally considered to be unphysical [12], while the case $M^2 = Q^2 + P^2$ corresponds to an extremal black hole, which is of interest in this report. The corresponding function for the Schwarzschild solution, in which the black hole is uncharged, is shown as a dashed line. This figure is based on a graph from [12].

1.2.2.2 Extremal Black Holes

For the definitions of temperature and entropy of a Reissner–Nordström black hole to be physically sensible, the mass M of the black hole must satisfy the constraint $M^2 \geq Q^2 + P^2$. Black holes for which this bound is saturated, with $M = \sqrt{Q^2 + P^2}$, are called *extremal black holes*. In this case, $r_+ = r_- = \sqrt{Q^2 + P^2}$, so that the inner and outer event horizons coincide and the metric (1.16) assumes the form

$$ds^2 = -\left(1 - \frac{\sqrt{Q^2 + P^2}}{r}\right) dt^2 + \left(1 - \frac{\sqrt{Q^2 + P^2}}{r}\right)^{-1} dr^2 + r^2 \, d\Omega^2 \,. \tag{1.20}$$

In this extremal limit, we note that the temperature of the black hole approaches zero, whereas the entropy has a finite limit of $\pi\sqrt{Q^2 + P^2}$. Thus extremal black holes have a large entropy even though they are stable against Hawking radiation [4].

Extremal black holes are of relevance in this work because they can preserve certain symmetries in supersymmetric theories [12]. In Chapter 6, we will compute the microstate degeneracy for a class of extremal supersymmetric black holes in string theory.

1.2.3 String Theoretic Black Holes

The entropy of black holes provides a way to test the validity of theories of quantum gravity, such as string theory. For such a theory to be consistent, it must predict that the thermodynamic entropy of a black hole is equal to its statistical entropy. One of the successes of string theory is that it describes the thermodynamic entropy of certain supersymmetric black holes as a logarithm of the microscopic degeneracy of states d, as required by the Boltzmann relation of statistical mechanics, $S = \ln d$. The constraints imposed on the entropy of black holes provide a very useful guide in quantum gravity research, where direct experimental guidance is extremely limited [4].

1.3 Outline

The primary aim of this project is to compute the number of microstates of a class of supersymmetric black holes in string theory. In the pursuit of this goal, we find it necessary to introduce several concepts and definitions from quantum field theory, string theory and number theory.

In Chapter 2, we provide a brief introduction to quantum field theory. This includes a discussion of the canonical quantisation of the Klein-Gordon and Dirac fields, as well as a short explanation of renormalisation.

Chapter 3 contains the computation of the partition functions for free closed bosonic and fermionic strings. The results obtained here are of relevance to the computation of the elliptic genus of K3 in Chapter 5.

Chapter 4 contains the formal definitions of the mathematical objects used in this work, including modular forms, Siegel modular forms, Eisenstein series, and Jacobi forms. We also define the Dedekind eta function and the Jacobi theta functions.

In Chapter 5, we compute the elliptic genus of K3 by realising K3 as a SCFT in the orbifold limit $T^4/\mathbb{Z}_2$. This computation uses some of the results obtained for the string partition functions in Chapter 3. We also briefly discuss the generalisation to the twisted elliptic genus of K3.

Chapter 6 constitutes the main part of the project, in which we compute the number of microstates of quarter-BPS extremal black holes in $\mathcal{N} = 4$ supersymmetric type II string theory compactified on $\mathrm{K3} \times T^2$. We also extend this computation to the $2A$ orbifold of $\mathrm{K3} \times T^2$. In both cases our results are found to agree with the literature [17].

In Chapter 7, we introduce mock modular forms and discuss the results from [18] which have a physical interpretation in terms of the region of the moduli space in which single-centred black hole configurations may be found.

Finally, in Chapter 8, we present our conclusions and discuss possible future topics of interest in this research area.

Chapter 2

Introduction to Quantum Field Theory

Quantum field theory arises out of the combination of three major areas in physics: quantum mechanics, field theory, and special relativity. It underlies modern elementary particle physics, as described within the framework of the Standard Model, as well as providing insight into a wide range of other areas of physics, including nuclear physics, atomic physics, condensed matter physics, and astrophysics. By most standards, the theory of quantum electrodynamics (QED) is the most successful of all physical theories, with its predictions agreeing with experimental results to astonishing levels of accuracy. To provide just one example, the QED prediction for the Landé g-factor of the electron agrees with the experimentally-measured value to eight significant figures [5].

However, there remain a number of outstanding problems in quantum field theory. The most glaring of these is the absence of a well-defined quantum theory of gravity. Quantum theories that incorporate gravity as a weak-coupling field theory are found to be nonrenormalisable, as discussed below, and it seems probable that quantum field theory will break down at the Planck scale of very small spacetime distances. One of the most ambitious goals of modern physics is finding a *theory of everything* that successfully reconciles quantum field theory and the theory of gravity. String theory has emerged as a strong possible candidate for this theory [5].

In this chapter, we discuss some of the basic concepts of quantum field theory. We begin by discussing the canonical quantisation of the Klein-Gordon and Dirac fields, and follow this with a brief review of renormalisation. The concepts introduced here will be of use in Chapter 3, when we come to examine the quantisation of bosonic and fermionic strings.

2.1 Canonical Quantisation

Quantisation is the process whereby a classical theory is converted to a quantum theory. In preparation for our discussion of the quantisation of strings in Chapter 3, we first recall the steps involved in quantising the two simplest relativistic fields: the Klein–Gordon field and the Dirac field. We will see that Klein–Gordon particles are bosons, whereas Dirac particles are fermions. The material in this section is based on [5].

2.1.1 The Classical Klein–Gordon Field

The classical Klein–Gordon field ϕ in four-dimensional spacetime is described by the Lagrangian density

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 \\ &= \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2. \end{aligned} \tag{2.1}$$

The equations of motion can be obtained from the Lagrangian density using the Euler-Lagrange equations:

$$0 = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = \left(\partial^\mu\partial_\mu + m^2\right)\phi. \tag{2.2}$$

This is the *Klein–Gordon equation.* The momentum density conjugate to ϕ is

$$\pi(\mathbf{x}) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}(\mathbf{x})} = \dot{\phi}(\mathbf{x}), \tag{2.3}$$

which allows us to construct the Hamiltonian density:

$$\mathcal{H} = \pi(\mathbf{x})\dot{\phi}(\mathbf{x}) - \mathcal{L} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2. \tag{2.4}$$

Hence the Hamiltonian is

$$H = \int \mathcal{H}\, d^3x = \int \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right] d^3x\,. \tag{2.5}$$

It is also useful to recall that the energy-momentum tensor of the field ϕ is defined by

$$T^{\mu}{}_{\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \mathcal{L}\delta^{\mu}{}_{\nu}. \tag{2.6}$$

For the Klein-Gordon field, we have

$$T^{0}{}_{0} = \frac{\partial\mathcal{L}}{\partial\dot{\phi}}\dot{\phi} - \mathcal{L} = \mathcal{H}, \tag{2.7a}$$

$$T^{0}{}_{i} = \frac{\partial\mathcal{L}}{\partial\dot{\phi}}\partial_i\phi = \pi\partial_i\phi. \tag{2.7b}$$

Thus the conserved charge associated with time translations is the spatial integral over T^{00}, which is simply the Hamiltonian H, while the conserved charges associated with spatial translations are

$$P^i = \int T^{0i}\, d^3x = -\int \pi\partial_i\phi\, d^3x, \tag{2.8}$$

which are interpreted as the components of the physical momentum carried by the field.

Finally, note that if we expand ϕ in Fourier space as

$$\phi(\mathbf{x},t) = \int \frac{d^3p}{(2\pi)^3}\, e^{i\mathbf{p}\cdot\mathbf{x}}\, \phi(\mathbf{p},t), \tag{2.9}$$

then the Klein–Gordon equation becomes

$$\left[\frac{\partial^2}{\partial t^2} + \left(|\mathbf{p}|^2 + m^2\right)\right]\phi(\mathbf{p},t) = 0. \tag{2.10}$$

This is the equation of motion for a simple harmonic oscillator with frequency

$$\omega_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}. \tag{2.11}$$

2.1.2 Quantisation of the Klein–Gordon Field

We will quantise the Klein–Gordon field in the Schrödinger picture, where ϕ and π do not depend on time. To do this, we promote ϕ and π to operators and impose the following commutation relations:

$$\begin{aligned}[\phi(\mathbf{x}),\pi(\mathbf{y})] &= i\delta^{(3)}(\mathbf{x}-\mathbf{y})\\ [\phi(\mathbf{x}),\phi(\mathbf{y})] &= [\phi(\mathbf{x}),\phi(\mathbf{y})] = 0.\end{aligned} \tag{2.12}$$

Next, we aim to find the spectrum of the Hamiltonian. With this goal in mind, we expand the field and its conjugate momentum density in Fourier space as

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \tag{2.13a}$$

$$\pi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\left(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} - a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right). \tag{2.13b}$$

For convenience, we can rearrange these to isolate the exponential factor:

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger\right)e^{i\mathbf{p}\cdot\mathbf{x}} \tag{2.14a}$$

$$\pi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\left(a_{\mathbf{p}} - a_{-\mathbf{p}}^\dagger\right)e^{i\mathbf{p}\cdot\mathbf{x}}. \tag{2.14b}$$

Taking the inverse Fourier transform, we have

$$\begin{aligned}
\int \phi(\mathbf{x}) e^{i\mathbf{p}'\cdot\mathbf{x}} d^3x &= \int\int \frac{d^3p\, d^3x}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a_{\mathbf{p}} + a^\dagger_{-\mathbf{p}}\right) e^{i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}} \\
&= \int\int \frac{d^3p\, d^3x}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a_{\mathbf{p}} + a^\dagger_{-\mathbf{p}}\right) e^{i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}} \\
&= \int \frac{d^3p}{\sqrt{2\omega_{\mathbf{p}}}} \left(a_{\mathbf{p}} + a^\dagger_{-\mathbf{p}}\right) \delta^{(3)}(\mathbf{p}+\mathbf{p}') \\
&= \frac{1}{\sqrt{2\omega_{\mathbf{p}'}}} \left(a_{-\mathbf{p}'} + a^\dagger_{\mathbf{p}'}\right) \qquad (2.15)
\end{aligned}$$

and

$$\begin{aligned}
\int \pi(\mathbf{x}) e^{i\mathbf{p}'\cdot\mathbf{x}} d^3x &= \int\int \frac{d^3p\, d^3x}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\mathbf{p}}}{2}} \left(a_{\mathbf{p}} - a^\dagger_{-\mathbf{p}}\right) e^{i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}} \\
&= -i \int\int \frac{d^3p\, d^3x}{(2\pi)^3} \sqrt{\frac{\omega_{\mathbf{p}}}{2}} \left(a_{\mathbf{p}} - a^\dagger_{-\mathbf{p}}\right) e^{i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}} \\
&= -i \int d^3p \sqrt{\frac{\omega_{\mathbf{p}}}{2}} \left(a_{\mathbf{p}} - a^\dagger_{-\mathbf{p}}\right) \delta^{(3)}(\mathbf{p}+\mathbf{p}') \\
&= -i \sqrt{\frac{\omega_{\mathbf{p}'}}{2}} \left(a_{-\mathbf{p}'} - a^\dagger_{\mathbf{p}'}\right). \qquad (2.16)
\end{aligned}$$

Hence we obtain the following linear system (where we have relabelled $\mathbf{p}'$ as $\mathbf{p}$):

$$\begin{aligned}
a_{-\mathbf{p}} + a^\dagger_{\mathbf{p}} &= \sqrt{2\omega_{\mathbf{p}}} \int \phi(\mathbf{x}) e^{i\mathbf{p}\cdot\mathbf{x}}\, d^3x \\
a_{-\mathbf{p}} - a^\dagger_{\mathbf{p}} &= i \sqrt{\frac{2}{\omega_{\mathbf{p}}}} \int \pi(\mathbf{x}) e^{i\mathbf{p}\cdot\mathbf{x}}\, d^3x\,. \qquad (2.17)
\end{aligned}$$

This can easily be solved to find that

$$a_{\mathbf{p}} = \sqrt{\frac{1}{2\omega_{\mathbf{p}}}} \int (\omega_{\mathbf{p}}\phi(\mathbf{x}) + i\pi(\mathbf{x})) e^{-i\mathbf{p}\cdot\mathbf{x}}\, d^3x \qquad (2.18a)$$

$$a^\dagger_{\mathbf{p}} = \sqrt{\frac{1}{2\omega_{\mathbf{p}}}} \int (\omega_{\mathbf{p}}\phi(\mathbf{x}) - i\pi(\mathbf{x})) e^{i\mathbf{p}\cdot\mathbf{x}}\, d^3x\,. \qquad (2.18b)$$

Hence imposing the canonical commutation relations, we find that

$$\begin{aligned}
\left[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}\right] &= \frac{1}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}} \int\int [\omega_{\mathbf{p}}\phi(\mathbf{x}) + i\pi(\mathbf{x}), \omega_{\mathbf{p}'}\phi(\mathbf{x}') - i\pi(\mathbf{x}')] e^{-i\mathbf{p}\cdot\mathbf{x} + i\mathbf{p}'\cdot\mathbf{x}'}\, d^3x\, d^3x' \\
&= \frac{i}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}} \int\int (-\omega_{\mathbf{p}}[\phi(\mathbf{x}), \pi(\mathbf{x}')] + \omega_{\mathbf{p}'}[\pi(\mathbf{x}), \phi(\mathbf{x}')]) e^{-i\mathbf{p}\cdot\mathbf{x} + i\mathbf{p}'\cdot\mathbf{x}'}\, d^3x\, d^3x' \\
&= \frac{i}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}} \int\int \left(-i\omega_{\mathbf{p}}\delta^{(3)}(\mathbf{x}-\mathbf{x}') - i\omega_{\mathbf{p}'}\delta^{(3)}(\mathbf{x}-\mathbf{x}')\right) e^{-i\mathbf{p}\cdot\mathbf{x} + i\mathbf{p}'\cdot\mathbf{x}'}\, d^3x\, d^3x' \\
&= \frac{\omega_{\mathbf{p}} + \omega_{\mathbf{p}'}}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}} \int e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\, d^3x \\
&= \frac{\omega_{\mathbf{p}} + \omega_{\mathbf{p}'}}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}} (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{p}') \\
&= (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{p}'). \qquad (2.19)
\end{aligned}$$

To find the quantised Hamiltonian, we begin by computing the gradient of ϕ:

$$\nabla\phi = \int \frac{d^3p}{(2\pi)^3} \frac{-i\mathbf{p}}{\sqrt{2\omega_{\mathbf{p}}}} \left(a_{\mathbf{p}} + a^\dagger_{-\mathbf{p}}\right) e^{i\mathbf{p}\cdot\mathbf{x}}, \qquad (2.20)$$

which means that

$$(\nabla\phi)^2 = \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{-\mathbf{p}\cdot\mathbf{p'}}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p'}}}} \left(a_{\mathbf{p}} + a^\dagger_{-\mathbf{p}}\right)\left(a_{\mathbf{p'}} + a^\dagger_{-\mathbf{p'}}\right) e^{i\left(\mathbf{p}+\mathbf{p'}\right)\cdot\mathbf{x}}. \tag{2.21}$$

The Hamiltonian can now be written as

$$\begin{aligned}
H &= \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right] \\
&= \int d^3x \int \frac{d^3p\, d^3p'}{(2\pi)^6} e^{i(\mathbf{p}+\mathbf{p'})\cdot\mathbf{x}} \left[-\frac{\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p'}}}}{4}\left(a_{\mathbf{p}} - a^\dagger_{-\mathbf{p}}\right)\left(a_{\mathbf{p'}} - a^\dagger_{-\mathbf{p'}}\right) + \frac{-\mathbf{p}\cdot\mathbf{p'}+m^2}{4\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p'}}}}\left(a_{\mathbf{p}} + a^\dagger_{-\mathbf{p}}\right)\left(a_{\mathbf{p'}} + a^\dagger_{-\mathbf{p'}}\right)\right] \\
&= \int \frac{d^3p\, d^3p'}{(2\pi)^3} \frac{\delta(\mathbf{p}+\mathbf{p'})}{4\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p'}}}} \left[-\omega_{\mathbf{p}}\omega_{\mathbf{p'}}\left(a_{\mathbf{p}} - a^\dagger_{-\mathbf{p}}\right)\left(a_{\mathbf{p'}} - a^\dagger_{-\mathbf{p'}}\right) + (-\mathbf{p}\cdot\mathbf{p'}+m^2)\left(a_{\mathbf{p}} + a^\dagger_{-\mathbf{p}}\right)\left(a_{\mathbf{p'}} + a^\dagger_{-\mathbf{p'}}\right)\right] \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{\omega_{\mathbf{p}}}{4}\left[-\left(a_{\mathbf{p}} - a^\dagger_{-\mathbf{p}}\right)\left(a_{-\mathbf{p}} - a^\dagger_{\mathbf{p}}\right) + \left(a_{\mathbf{p}} + a^\dagger_{-\mathbf{p}}\right)\left(a_{-\mathbf{p}} + a^\dagger_{\mathbf{p}}\right)\right] \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{\omega_{\mathbf{p}}}{2}\left(a_{\mathbf{p}}a^\dagger_{\mathbf{p}} + a^\dagger_{\mathbf{p}}a_{\mathbf{p}}\right) \\
&= \int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}}\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} + \frac{1}{2}\left[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}}\right]\right),
\end{aligned} \tag{2.22}$$

where we have used the fact that $\omega_{\mathbf{p}} = |\mathbf{p}|^2 + m^2$. The second term here is proportional to $\delta(0)$, which is infinite and comes from summing over all modes of the zero-point energies $\omega_{\mathbf{p}}/2$. However, since only energy differences from the ground state are physically meaningful, this infinite term can simply be dropped from all calculations. This can be seen as a form of renormalisation, which will be discussed further at the end of this chapter.

We can now evaluate the commutator

$$\begin{aligned}
\left[H, a^\dagger_{\mathbf{p}}\right] &= \int \frac{d^3p'}{(2\pi)^3}\omega_{\mathbf{p'}}\left[a^\dagger_{\mathbf{p'}}a_{\mathbf{p'}}, a^\dagger_{\mathbf{p}}\right] \\
&= \int \frac{d^3p'}{(2\pi)^3}\omega_{\mathbf{p'}}a^\dagger_{\mathbf{p'}}\left[a_{\mathbf{p'}}, a^\dagger_{\mathbf{p}}\right] \\
&= \int \frac{d^3p'}{(2\pi)^3}\omega_{\mathbf{p'}}a^\dagger_{\mathbf{p'}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p'}) \\
&= \omega_{\mathbf{p}}a^\dagger_{\mathbf{p}},
\end{aligned} \tag{2.23}$$

and similarly

$$[H, a_{\mathbf{p}}] = -\omega_{\mathbf{p}}a_{\mathbf{p}}. \tag{2.24}$$

We define the ground state or *vacuum* to be the state $|0\rangle$ such that $a_{\mathbf{p}}|0\rangle = 0$ for all $\mathbf{p}$, which has energy $E = 0$. All other energy eigenstates can be formed by acting on $|0\rangle$ with creation operators $a^\dagger_{\mathbf{p}}$, and the state $a^\dagger_{\mathbf{p}}a^\dagger_{\mathbf{q}}\cdots|0\rangle$ is an energy eigenstate with eigenvalue $\omega_{\mathbf{p}}+\omega_{\mathbf{q}}+\ldots$ We have thus succeeded in finding the spectrum of the Klein–Gordon Hamiltonian. We note that the creation operator $a^\dagger_{\mathbf{p}}$ can be applied to a state to create arbitrarily many particles in a single mode $\mathbf{p}$. This means that the occupation number of each state in the Klein–Gordon theory is unbounded, so that Klein–Gordon particles obey *Bose–Einstein statistics*.

2.1.3 The Classical Dirac Field

To discuss the Dirac field, it is convenient to introduce the set of four four-dimensional *Dirac matrices*, which can be written in 2×2 block form as

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{2.25}$$

where σ^i are the Pauli sigma matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.26}$$

These matrices satisfy the Dirac algebra,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \tag{2.27}$$

and provide a representation of the Lorentz algebra with generators

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]. \tag{2.28}$$

Explicitly, the generators of Lorentz boosts are

$$S^{0i} = \frac{i}{4}\left[\gamma^0, \gamma^i\right] = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}, \tag{2.29}$$

while the generators of rotations are

$$S^{ij} = \frac{i}{4}\left[\gamma^i, \gamma^j\right] = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}. \tag{2.30}$$

A four-component field that transforms under boosts and rotations according to these rules is called a *Dirac spinor*. Since S^{0i} is not Hermitian, $\psi^\dagger\psi$ is not a Lorentz invariant. It is therefore useful to define $\bar{\psi} \equiv \psi^\dagger\gamma^0$, so that $\bar{\psi}\psi$ is Lorentz invariant. The Lagrangian density for the Dirac field is then

$$\begin{aligned} \mathcal{L} &= \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \\ &= i\bar{\psi}\gamma^0\dot{\psi} + i\bar{\psi}\boldsymbol{\gamma}\cdot\nabla\psi - m\bar{\psi}\psi. \end{aligned} \tag{2.31}$$

The equation of motion for $\bar{\psi}$ derived from this is

$$0 = \frac{\partial\mathcal{L}}{\partial\bar{\psi}} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\right) = (i\gamma^\mu\partial_\mu - m)\psi, \tag{2.32}$$

which is the *Dirac equation*. The canonical momentum density conjugate to ψ is

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\psi^\dagger\gamma^0\gamma^0 = i\psi^\dagger, \tag{2.33}$$

so that the Dirac Hamiltonian density is easily found to be

$$\begin{aligned} \mathcal{H} &= \pi(x)\dot{\psi}(x) - \mathcal{L} \\ &= i\psi^\dagger\dot{\psi} - i\bar{\psi}\gamma^0\dot{\psi} - i\bar{\psi}\boldsymbol{\gamma}\cdot\nabla\psi + m\bar{\psi}\psi \\ &= \bar{\psi}(-i\boldsymbol{\gamma}\cdot\nabla + m)\psi. \end{aligned} \tag{2.34}$$

The Dirac Hamiltonian is thus

$$H = \int d^3x\,\bar{\psi}(-i\boldsymbol{\gamma}\cdot\nabla + m)\psi. \tag{2.35}$$

We note that the Dirac equation implies the Klein–Gordon equation, since acting on the left by $(-i\gamma^\mu\partial_\mu - m)$ gives

$$\begin{aligned} 0 &= (-i\gamma^\mu\partial_\mu - m)(i\gamma^\nu\partial_\nu - m)\psi \\ &= \left(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2\right)\psi \\ &= \left(\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu\partial_\nu + m^2\right)\psi \\ &= \left(\partial^2 + m^2\right)\psi. \end{aligned} \tag{2.36}$$

We can therefore look for plane wave solutions of the Dirac equation. For positive frequencies, these have the form

$$\psi(x) = u(p)e^{-ip\cdot x}, \tag{2.37}$$

where $p^2 = m^2$ and $p^0 > 0$. Substituting this ansatz into the Dirac equation, we find that

$$(\gamma^\mu p_\mu - m)u(p) = 0.$$

The solution of this equation is

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\bar{\sigma}}\xi \end{pmatrix},$$

where $\sigma \equiv (1, \boldsymbol{\sigma}), \bar{\sigma} \equiv (1, -\boldsymbol{\sigma})$ and ξ is any two-component spinor, conventionally normalised such that $\xi^\dagger \xi = 1$. There are thus two linearly independent positive-frequency solutions:

$$u^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\xi^s \\ \sqrt{p \cdot \bar{\sigma}}\xi^s \end{pmatrix}, \quad s = 1, 2, \tag{2.38}$$

where $\xi^1$and ξ^2 are orthonormal two-component basis spinors, so that the solutions are normalised according to

$$\bar{u}^r(p)u^s(p) = 2m\delta^{rs} \quad \text{or} \quad u^{r\dagger}(p)u^s(p) = 2E_{\mathbf{p}}\delta^{rs}. \tag{2.39}$$

Similarly, negative-frequency solutions have the form

$$\psi(x) = v(p)e^{ip\cdot x}, \tag{2.40}$$

with two linearly independent solutions,

$$v^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\eta^s \\ -\sqrt{p \cdot \bar{\sigma}}\eta^s \end{pmatrix}, \quad s = 1, 2, \tag{2.41}$$

where η is another basis of two component spinors and $\bar{v}^r(p)v^s(p) = -2m\delta^{rs}$.

$$\bar{v}^r(p)v^s(p) = -2m\delta^{rs} \quad \text{or} \quad v^{r\dagger}(p)v^s(p) = 2E_{\mathbf{p}}\delta^{rs}. \tag{2.42}$$

We note that while $u^{r\dagger}(p)v^s(p) \neq 0$ and $v^{r\dagger}(p)u^s(p) \neq 0$ in general, we do have the relation

$$u^{r\dagger}(\mathbf{p})v^s(-\mathbf{p}) = u^{r\dagger}(\mathbf{p})v^s(-\mathbf{p}) = 0. \tag{2.43}$$

2.1.4 Quantisation of the Dirac Field

Expanding ψ and $\psi^\dagger$ in the basis obtained above, we have

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \big(a^s_{\mathbf{p}} u^s(p) e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}} v^s(p) e^{ip\cdot x}\big), \tag{2.44a}$$

$$\psi^\dagger(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \big(b^s_{\mathbf{p}} v^{s\dagger}(p) e^{-ip\cdot x} + a^{s\dagger}_{\mathbf{p}} u^{s\dagger}(p) e^{ip\cdot x}\big). \tag{2.44b}$$

Taking the inverse Fourier transform at $t = 0$, we have

$$\begin{aligned}
\int \psi(\mathbf{x})e^{i\mathbf{q}\cdot\mathbf{x}}\, d^3x &= \int\int \frac{d^3p\, d^3x}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \big(a^s_{\mathbf{p}} u^s(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}} + b^{s\dagger}_{\mathbf{p}} v^s(\mathbf{p}) e^{-i\mathbf{p}\cdot\mathbf{x}}\big) e^{i\mathbf{q}\cdot\mathbf{x}} \\
&= \int \frac{d^3p}{\sqrt{2E_{\mathbf{p}}}} \sum_s \Big(a^s_{\mathbf{p}} u^s(\mathbf{p}) \delta^{(3)}(\mathbf{p}+\mathbf{q}) + b^{s\dagger}_{\mathbf{p}} v^s(\mathbf{p}) \delta^{(3)}(\mathbf{p}-\mathbf{q})\Big) \\
&= \frac{1}{\sqrt{2E_{\mathbf{q}}}} \sum_s \big(a^s_{-\mathbf{q}} u^s(-\mathbf{q}) + b^{s\dagger}_{\mathbf{q}} v^s(\mathbf{q})\big),
\end{aligned} \tag{2.45}$$

and

$$\begin{aligned}
\int \psi^\dagger(\mathbf{x})e^{i\mathbf{q}\cdot\mathbf{x}}\, d^3x &= \int\int \frac{d^3p\, d^3x}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \big(b^s_{\mathbf{p}} v^{s\dagger}(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}} + a^{s\dagger}_{\mathbf{p}} u^{s\dagger}(\mathbf{p}) e^{-i\mathbf{p}\cdot\mathbf{x}}\big) e^{i\mathbf{q}\cdot\mathbf{x}} \\
&= \int \frac{d^3p}{\sqrt{2E_{\mathbf{p}}}} \sum_s \Big(b^s_{\mathbf{p}} v^{s\dagger}(\mathbf{p}) \delta^{(3)}(\mathbf{p}+\mathbf{q}) + a^{s\dagger}_{\mathbf{p}} u^{s\dagger}(\mathbf{p}) \delta^{(3)}(\mathbf{p}-\mathbf{q})\Big) \\
&= \frac{1}{\sqrt{2E_{\mathbf{q}}}} \sum_s \big(b^s_{-\mathbf{q}} v^{s\dagger}(-\mathbf{q}) + a^{s\dagger}_{\mathbf{q}} u^{s\dagger}(\mathbf{q})\big).
\end{aligned} \tag{2.46}$$

We now isolate the operators using relations (2.39), (2.42) and (2.43) as follows:

$$\begin{aligned}
\int u^{r\dagger}(-\mathbf{q})\psi(\mathbf{x})e^{i\mathbf{q}\cdot\mathbf{x}}\, d^3x &= \frac{1}{\sqrt{2E_{\mathbf{q}}}} \sum_s u^{r\dagger}(-\mathbf{q})\big(a^s_{-\mathbf{q}} u^s(-\mathbf{q}) + b^{s\dagger}_{\mathbf{q}} v^s(\mathbf{q})\big) \\
&= \frac{1}{\sqrt{2E_{\mathbf{q}}}} \sum_s a^s_{-\mathbf{q}} (2E_{-\mathbf{q}}\delta^{rs}) \\
&= \sqrt{2E_{-\mathbf{q}}}\, a^r_{-\mathbf{q}},
\end{aligned}$$

and similarly

$$\int v^{r\dagger}(\mathbf{q})\psi(\mathbf{x})e^{i\mathbf{q}\cdot\mathbf{x}}\,d^3x = \sqrt{2E_{\mathbf{q}}}b_{\mathbf{q}}^{r\dagger}.$$

For $\psi^\dagger$, we also have

$$\int \psi^\dagger(\mathbf{x})v^r(-\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{x}}\,d^3x = \sqrt{2E_{-\mathbf{q}}}b_{-\mathbf{q}}^{r}$$
$$\int \psi^\dagger(\mathbf{x})u^r(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{x}}\,d^3x = \sqrt{2E_{\mathbf{q}}}a_{\mathbf{q}}^{r\dagger}.$$

Hence we obtain the expressions

$$a_{\mathbf{q}}^r = \frac{1}{\sqrt{2E_{\mathbf{q}}}}\int u^{r\dagger}(\mathbf{q})\psi(\mathbf{x})e^{-i\mathbf{q}\cdot\mathbf{x}}\,d^3x, \qquad b_{\mathbf{q}}^r = \frac{1}{\sqrt{2E_{\mathbf{q}}}}\int \psi^\dagger(\mathbf{x})v^r(\mathbf{q})e^{-i\mathbf{q}\cdot\mathbf{x}}\,d^3x,$$
$$a_{\mathbf{q}}^{r\dagger} = \frac{1}{\sqrt{2E_{\mathbf{q}}}}\int \psi^\dagger(\mathbf{x})u^r(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{x}}\,d^3x, \qquad b_{\mathbf{q}}^{r\dagger} = \frac{1}{\sqrt{2E_{\mathbf{q}}}}\int v^{r\dagger}(\mathbf{q})\psi(\mathbf{x})e^{i\mathbf{q}\cdot\mathbf{x}}\,d^3x\,. \tag{2.47}$$

We now aim to quantise the Dirac field. Unlike in the Klein–Gordon case, we cannot impose commutation relations on the Dirac field, as this would lead to states of negative energy. We instead impose the *anticommutation* relations

$$\begin{aligned}\left\{\psi_a(\mathbf{x}),\psi_b^\dagger(\mathbf{y})\right\} &= \delta^{(3)}(\mathbf{x}-\mathbf{y})\delta_{ab}\\ \{\psi_a(\mathbf{x}),\psi_b(\mathbf{y})\} &= \left\{\psi_a^\dagger(\mathbf{x}),\psi_b^\dagger(\mathbf{y})\right\} = 0.\end{aligned} \tag{2.48}$$

We then find that

$$\begin{aligned}\left\{a_{\mathbf{p}}^r, a_{\mathbf{q}}^{s\dagger}\right\} &= \frac{1}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}}\int\int d^3x\,d^3x'\,u^{r\dagger}(\mathbf{p})\left\{\psi(\mathbf{x}),\psi^\dagger(\mathbf{x}')\right\}u^s(\mathbf{q})e^{-i\mathbf{p}\cdot\mathbf{x}+i\mathbf{q}\cdot\mathbf{x}'}\\
&= \frac{1}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}}\int\int d^3x\,d^3x'\,u^{r\dagger}(\mathbf{p})\delta^{(3)}(\mathbf{x}-\mathbf{x}')u^s(\mathbf{q})e^{-i\mathbf{p}\cdot\mathbf{x}+i\mathbf{q}\cdot\mathbf{x}'}\\
&= \frac{1}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}}u^{r\dagger}(\mathbf{p})u^s(\mathbf{q})\int d^3x\,e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\\
&= \frac{1}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}}u^{r\dagger}(\mathbf{p})u^s(\mathbf{q})(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\\
&= \frac{(2\pi)^3}{2E_{\mathbf{p}}}u^{r\dagger}(\mathbf{p})u^s(\mathbf{p})\delta^{(3)}(\mathbf{p}-\mathbf{q})\\
&= (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\delta^{rs},\end{aligned} \tag{2.49}$$

and similarly

$$\left\{b_{\mathbf{p}}^r, b_{\mathbf{q}}^{s\dagger}\right\} = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\delta^{rs}, \tag{2.50}$$

while all other anticommutators are zero; for example

$$\begin{aligned}\left\{a_{\mathbf{p}}^r, b_{\mathbf{q}}^{s}\right\} &= \frac{1}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}}\int\int d^3x\,d^3x'\,u^{r\dagger}(\mathbf{p})\left\{\psi(\mathbf{x}),\psi^\dagger(\mathbf{x}')\right\}v^s(\mathbf{q})e^{-i\mathbf{p}\cdot\mathbf{x}-i\mathbf{q}\cdot\mathbf{x}'}\\
&= \frac{1}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}}\int\int d^3x\,d^3x'\,u^{r\dagger}(\mathbf{p})\delta^{(3)}(\mathbf{x}-\mathbf{x}')v^s(\mathbf{q})e^{-i\mathbf{p}\cdot\mathbf{x}-i\mathbf{q}\cdot\mathbf{x}'}\\
&= \frac{1}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}}u^{r\dagger}(\mathbf{p})v^s(\mathbf{q})\int d^3x\,e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}\\
&= \frac{1}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}}u^{r\dagger}(\mathbf{p})v^s(\mathbf{q})(2\pi)^3\delta^{(3)}(\mathbf{p}+\mathbf{q})\\
&= \frac{(2\pi)^3}{2E_{\mathbf{p}}}u^{r\dagger}(\mathbf{p})v^s(-\mathbf{p})\delta^{(3)}(\mathbf{p}+\mathbf{q})\\
&= 0.\end{aligned} \tag{2.51}$$

The relations $\{a^{r\dagger}_{\mathbf{p}}, a^{s\dagger}_{\mathbf{q}}\} = \{b^{r\dagger}_{\mathbf{p}}, b^{s\dagger}_{\mathbf{q}}\} = 0$ for the creation operators imply that $\left(a^{s\dagger}_{\mathbf{p}}\right)^2 = \left(b^{s\dagger}_{\mathbf{p}}\right)^2 = 0$, so that each mode $\mathbf{p}$ can only be occupied by a maximum of one particle. We hence conclude that Dirac particles obey *Fermi–Dirac statistics*.

We note that the Dirac Hamiltonian can be written as

$$H = \int d^3x\, \bar{\psi}(-i\boldsymbol{\gamma}\cdot\nabla + m)\psi = \int d^3x\, \psi^\dagger h_D \psi, \tag{2.52}$$

where we define

$$h_D = -i\gamma^0\boldsymbol{\gamma}\cdot\nabla + m\gamma^0. \tag{2.53}$$

We know that $u(p)e^{-ip\cdot x}$ is a solution of the free-field Dirac equation:

$$0 = (i\gamma^\mu\partial_\mu - m)u(p)e^{-ip\cdot x} = \big(i\gamma^0\partial_0 + i\boldsymbol{\gamma}\cdot\nabla - m\big)u(p)e^{-ip\cdot x},$$

so that

$$(-i\boldsymbol{\gamma}\cdot\nabla + m)u(p)e^{-ip\cdot x} = i\gamma^0\partial_0 u(p)e^{-ip\cdot x} = p^0\gamma^0 u(p)e^{-ip\cdot x} = E_{\mathbf{p}}\gamma^0 u(p)e^{-ip\cdot x}.$$

Multiplying both sides of this equation on the left by γ^0, we find that

$$h_D u(p)e^{-ip\cdot x} = E_{\mathbf{p}}\gamma^0 u(p)e^{-ip\cdot x},$$

so $u(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}$ is an eigenfunction of h_D with eigenvalue $E_{\mathbf{p}}$. Similarly, since $v(p)e^{ip\cdot x}$ is also a solution of the Dirac equation, we find that $v(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}$ is an eigenfunction of h_D with eigenvalue $-E_{\mathbf{p}}$. We hence have

$$h_D\psi = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_{\mathbf{p}}}{2}}\sum_s \big(a^s_{\mathbf{p}}u^s(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}} - b^{s\dagger}_{\mathbf{p}}v^s(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}\big). \tag{2.54}$$

The quantised Dirac Hamiltonian is thus

$$\begin{aligned}
H &= \int d^3x\, \psi^\dagger h_D \psi \\
&= \int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6}\frac{1}{2}\sqrt{\frac{E_{\mathbf{p}}}{E_{\mathbf{q}}}}\sum_{r,s}\big(b^r_{\mathbf{q}}v^{r\dagger}(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{x}} + a^{r\dagger}_{\mathbf{q}}u^{r\dagger}(\mathbf{q})e^{-i\mathbf{q}\cdot\mathbf{x}}\big)\big(a^s_{\mathbf{p}}u^s(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}} - b^{s\dagger}_{\mathbf{p}}v^s(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}\big) \\
&= \int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6}\frac{1}{2}\sqrt{\frac{E_{\mathbf{p}}}{E_{\mathbf{q}}}}\sum_{r,s}\Big(b^r_{\mathbf{q}}v^{r\dagger}(\mathbf{q}) + a^{r\dagger}_{-\mathbf{q}}u^{r\dagger}(-\mathbf{q})\Big)\Big(a^s_{\mathbf{p}}u^s(\mathbf{p}) - b^{s\dagger}_{-\mathbf{p}}v^s(-\mathbf{p})\Big)e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} \\
&= \int \frac{d^3p\, d^3q}{(2\pi)^3}\frac{1}{2}\sqrt{\frac{E_{\mathbf{p}}}{E_{\mathbf{q}}}}\sum_{r,s}\Big(b^r_{\mathbf{q}}v^{r\dagger}(\mathbf{q}) + a^{r\dagger}_{-\mathbf{q}}u^{r\dagger}(-\mathbf{q})\Big)\Big(a^s_{\mathbf{p}}u^s(\mathbf{p}) + b^{s\dagger}_{-\mathbf{p}}v^s(-\mathbf{p})\Big)\delta^{(3)}(\mathbf{p}+\mathbf{q}) \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2}\sum_{r,s}\big(b^r_{-\mathbf{p}}v^{r\dagger}(-\mathbf{p}) + a^{r\dagger}_{\mathbf{p}}u^{r\dagger}(\mathbf{p})\big)\Big(a^s_{\mathbf{p}}u^s(\mathbf{p}) + b^{s\dagger}_{-\mathbf{p}}v^s(-\mathbf{p})\Big) \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2}\sum_{r,s}2E_{\mathbf{p}}\delta^{rs}\Big(a^{r\dagger}_{\mathbf{p}}a^s_{\mathbf{p}} - b^r_{-\mathbf{p}}b^{s\dagger}_{-\mathbf{p}}\Big) \\
&= \int \frac{d^3p}{(2\pi)^3}\sum_s E_{\mathbf{p}}\big(a^{s\dagger}_{\mathbf{p}}a^s_{\mathbf{p}} + b^{s\dagger}_{\mathbf{p}}b^s_{\mathbf{p}} - \{b^s_{\mathbf{p}}, b^{s\dagger}_{\mathbf{p}}\}\big) \\
&= \int \frac{d^3p}{(2\pi)^3}\sum_s E_{\mathbf{p}}\big(a^{s\dagger}_{\mathbf{p}}a^s_{\mathbf{p}} + b^{s\dagger}_{\mathbf{p}}b^s_{\mathbf{p}}\big),
\end{aligned} \tag{2.55}$$

where we have neglected the infinite constant arising from $\{b^s_{\mathbf{p}}, b^{s\dagger}_{\mathbf{p}}\} = (2\pi)^3\delta^{(3)}(0)$. The vacuum $|0\rangle$ is defined to be the state such that

$$a^s_{\mathbf{p}}|0\rangle = b^s_{\mathbf{p}}|0\rangle = 0,$$

which corresponds to the lowest energy state with $E = 0$. Thus both $a^\dagger_{\mathbf{p}}$ and $b^{s\dagger}_{\mathbf{p}}$ create particles with energy $E_{\mathbf{p}}$ and momentum $\mathbf{p}$. Particles created by $a^\dagger_{\mathbf{p}}$ are called *fermions*, while particles created by $b^{s\dagger}_{\mathbf{p}}$ are called *antifermions*.

2.2 Renormalisation

In quantum field theory, no exactly solvable interacting field theories are known in more than two dimensions. In order to study interacting fields, we instead treat the interaction term H_{int} in the Hamiltonian as a perturbation and compute physical quantities using perturbation theory. However, higher terms in perturbation theory involve integrals over the four-momenta of virtual particles, which are often formally divergent. The problematic divergences are *ultraviolet divergences*, where integrals are divergent in the region of large momentum. This signals that quantities calculated in a quantum field theory depend on a very large momentum scale called the ultraviolet cutoff [5].

In order for a quantum field theory to be sensible, all physical quantities must assume finite values, and divergences are considered acceptable only if they do not appear in physical predictions. A method of resolving the divergences that occur in physically measurable quantities is therefore necessary. This method involves renormalising the physical quantities such as mass and charge to finite values, and is known as *renormalisation*. The simplest example involves simply cutting off the divergent integral at some large but finite momentum Λ. At the end of the calculation, the limit $\Lambda \to \infty$ is taken. If the physical quantities in the theory are found to be independent of Λ, then the theory is said to be *renormalisable* [5, 19].

The nonrenormalisability of gravity is a notorious problem that arises in the quest to unify general relativity and quantum field theory. When gravity is incorporated as a weak-coupling field theory and treated in perturbation theory using Feynman diagrams, the resulting divergences are found to be nonrenormalisable [5, 20]. At present, the only known way to cut off these divergences without compromising the consistency of the theory is string theory [2], to which we turn in the following chapter.

Chapter 3

String Partition Functions

An important quantity in statistical physics is the *partition function* Z, which relates the microscopic configurations of a system to its macroscopic thermodynamic variables, such as energy and entropy. In quantum field theory, the partition function Z for a field theory governed by a Hamiltonian H is defined as

$$Z = \mathrm{tr}\big(e^{-\beta H}\big), \tag{3.1}$$

where the trace is taken over all states in the theory. In this chapter, we compute the partition functions for closed bosonic and fermionic strings, omitting the zero modes for simplicity. These results will be used in Chapter 5 to compute the elliptic genus of K3, which is needed to find the entropy of black holes in certain string theories.

3.1 Partition Function for the Bosonic String

Following [21], the Lagrangian for a closed bosonic string is

$$L = \frac{1}{4\pi\alpha'}\int_0^{2\pi} \partial_\alpha X \partial^\alpha X \, d\sigma, \tag{3.2}$$

where $X(\tau, \sigma)$ is the bosonic field of the two-dimensional worldsheet [1], and α' is the Regge slope, a parameter related to the string tension T by $T = 1/2\pi\alpha'$. In two-dimensional Minkowski space with $(+, -)$ metric, the Lagrangian can be written as

$$L = \frac{1}{4\pi\alpha'}\int_0^{2\pi} (\dot{X}^2 - X'^2) \, d\sigma, \tag{3.3}$$

where the dot denotes a derivative with respect to , and the prime denotes a derivative with respect to σ. The Lagrangian can also be expressed as

$$L = \int_0^{2\pi} \mathcal{L} \, d\sigma, \tag{3.4}$$

where $\mathcal{L}$ is the Lagrangian density,

$$\mathcal{L} = \frac{1}{4\pi\alpha'}\partial_\alpha X \partial^\alpha X = \frac{1}{4\pi\alpha'}(\dot{X}^2 - X'^2). \tag{3.5}$$

We note the similarity to the Lagrangian density of the Klein-Gordon field, (2.1), as would be expected for a string theory that gives rise to bosonic modes. The corresponding classical Hamiltonian is

$$H = \frac{\partial L}{\partial \dot{X}}\dot{X} - L = \frac{1}{4\pi\alpha'}\int_0^{2\pi} (\dot{X}^2 + X'^2) \, d\sigma, \tag{3.6}$$

and the canonical momentum density is

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{X}} = \frac{1}{2\pi\alpha'}\dot{X}. \tag{3.7}$$

From the Lagrangian density, we can evaluate the Euler-Lagrange equations,

$$0 = \partial_\tau\left(\frac{\partial \mathcal{L}}{\partial \dot{X}}\right) - \frac{\partial \mathcal{L}}{\partial X} = \frac{1}{4\pi\alpha'}\partial_\alpha \partial^\alpha X, \tag{3.8}$$

so that the equations of motion are

$$\partial_\alpha \partial^\alpha X = 0, \tag{3.9}$$

Using the method of separation of variables, we assume that X can be expressed in the form $X(\tau,\sigma) = a(\tau)b(\sigma)$. Upon substituting this into the EOM, we obtain

$$\frac{\ddot{a}}{a} = \frac{b''}{b}. \tag{3.10}$$

Since this must be true for all τ, σ, these expressions must be equal to some constant, say k^2. The solutions for nonzero k (up to an overall constant factor) are then

$$\begin{aligned} a &= e^{-ik\tau} \\ b &= e^{\pm ik\sigma}, \end{aligned} \tag{3.11}$$

so that the solution for a particular nonzero value of k is

$$X_k(\tau,\sigma) = \alpha_k e^{-ik\tau+ik\sigma} + \tilde{\alpha}_k e^{-ik\tau-ik\sigma}, \tag{3.12}$$

where α_k and $\tilde{\alpha}_k$ are constants. Turning to the special case $k = 0$, we see that we instead get linear solutions

$$\begin{aligned} a &= \beta\sigma + \gamma \\ b &= p\tau + x, \end{aligned} \tag{3.13}$$

where β, γ, p, x are constants. Note that x and p can be interpreted as the centre of mass position and momentum of the string.

Imposing the periodic boundary condition for a closed string, $X_k(\sigma) = X_k(\sigma + 2\pi)$, we see that k in (3.12) must be an integer, while in (3.13) we must have $\beta = 0, \gamma = 1$. The general solution of the equations of motion can therefore be expanded in Fourier modes as

$$X = i\sqrt{\frac{\alpha'}{2}}\left(\sum_{k\neq 0} \frac{\alpha_k}{k} e^{-ik\tau+ik\sigma} + \sum_{k\neq 0} \frac{\tilde{\alpha}_k}{k} e^{-ik\tau-ik\sigma}\right) + \alpha' p\tau + x, \tag{3.14}$$

where k ranges over all nonzero integers, and we have chosen the normalisation for later convenience. Note that the modes α_k are left-moving, while the modes $\tilde{\alpha}_k$ are right-moving. The modes α_k and $\tilde{\alpha}_k$ act as creation operators for $k < 0$ and annihilation operators for $k > 0$ [13].

We now carry out the quantisation of the bosonic string. For simplicity, in the subsequent discussion we consider equal times and set $\tau = 0$. Imposing the equal-time canonical commutation relation $[X(\sigma), \Pi(\sigma')] = i\delta(\sigma - \sigma')$, we get

$$\left[X(\sigma), \dot{X}(\sigma')\right] = 2\pi i\alpha' \delta(\sigma - \sigma'). \tag{3.15}$$

The derivative of X with respect to τ is

$$\dot{X} = \sqrt{\frac{\alpha'}{2}} \sum_{k\neq 0} e^{-ik\tau}\left(\alpha_k e^{ik\sigma} + \tilde{\alpha}_k e^{-ik\sigma}\right) + \alpha' p, \tag{3.16}$$

so at $\tau = 0$,

$$X = i\sqrt{\frac{\alpha'}{2}} \sum_{k\neq 0} \frac{1}{k}\left(\alpha_k e^{ik\sigma} + \tilde{\alpha}_k e^{-ik\sigma}\right) + x \tag{3.17a}$$

$$\dot{X} = \sqrt{\frac{\alpha'}{2}} \sum_{k\neq 0}\left(\alpha_k e^{ik\sigma} + \tilde{\alpha}_k e^{-ik\sigma}\right) + \alpha' p. \tag{3.17b}$$

Since the zero modes commute with the rest of the modes, their contribution may be neglected throughout the computation and simply added at the end, so we will now drop x and $\alpha' p$ from our calculation.

We now wish to isolate the modes α_k and $\tilde{\alpha}_k$. To this end, we multiply both sides of these equations by $e^{ik'\sigma}$ and integrate over σ. For the first equation, this yields

$$\begin{aligned}
\int_0^{2\pi} X e^{-ik'\sigma}\, d\sigma &= i\sqrt{\frac{\alpha'}{2}} \sum_{k\neq 0} \frac{1}{k} \int_0^{2\pi} \left(\alpha_k e^{ik\sigma} + \tilde{\alpha}_k e^{-ik\sigma}\right) e^{-ik'\sigma}\, d\sigma \\
&= i\sqrt{\frac{\alpha'}{2}} \sum_{k\neq 0} \frac{1}{k} \int_0^{2\pi} \left(\alpha_k e^{i\left(k-k'\right)\sigma} + \tilde{\alpha}_k e^{-i\left(k+k'\right)\sigma}\right) d\sigma \\
&= i\sqrt{\frac{\alpha'}{2}} \sum_{k\neq 0} \frac{2\pi}{k} (\alpha_k \delta_{k,k'} + \tilde{\alpha}_k \delta_{k,-k'}) \\
&= \frac{2\pi i}{k'} \sqrt{\frac{\alpha'}{2}} (\alpha_{k'} - \tilde{\alpha}_{-k'}),
\end{aligned} \tag{3.18}$$

and similarly for the second, we obtain

$$\begin{aligned}
\int_0^{2\pi} \dot{X} e^{-ik'\sigma}\, d\sigma &= \sqrt{\frac{\alpha'}{2}} \sum_{k\neq 0} \int_0^{2\pi} \left(\alpha_k e^{ik\sigma} + \tilde{\alpha}_k e^{-ik\sigma}\right) e^{-ik'\sigma}\, d\sigma \\
&= \sqrt{\frac{\alpha'}{2}} \sum_{k\neq 0} \int_0^{2\pi} \left(\alpha_k e^{i\left(k-k'\right)\sigma} + \tilde{\alpha}_k e^{-i\left(k+k'\right)\sigma}\right) d\sigma \\
&= \sqrt{\frac{\alpha'}{2}} \sum_{k\neq 0} 2\pi (\alpha_k \delta_{k,k'} + \tilde{\alpha}_k \delta_{k,-k'}) \\
&= 2\pi \sqrt{\frac{\alpha'}{2}} (\alpha_{k'} + \tilde{\alpha}_{-k'}).
\end{aligned} \tag{3.19}$$

This gives us the following system of equations, where for simplicity we replace k' with k:

$$\alpha_k - \tilde{\alpha}_{-k} = \frac{1}{2\pi i \sqrt{\frac{\alpha'}{2}}} k \int_0^{2\pi} X e^{-ik\sigma}\, d\sigma \tag{3.20a}$$

$$\alpha_k + \tilde{\alpha}_{-k} = \frac{1}{2\pi \sqrt{\frac{\alpha'}{2}}} \int_0^{2\pi} \dot{X} e^{-ik\sigma}\, d\sigma\,. \tag{3.20b}$$

This system can be easily solved, and we find that the modes can be expressed as

$$\alpha_k = \frac{1}{4\pi i \sqrt{\frac{\alpha'}{2}}} \left(k \int_0^{2\pi} X e^{-ik\sigma}\, d\sigma + i \int_0^{2\pi} \dot{X} e^{-ik\sigma}\, d\sigma \right) \tag{3.21a}$$

$$\tilde{\alpha}_k = \frac{1}{4\pi i \sqrt{\frac{\alpha'}{2}}} \left(-k \int_0^{2\pi} X e^{-ik\sigma}\, d\sigma + i \int_0^{2\pi} \dot{X} e^{-ik\sigma}\, d\sigma \right). \tag{3.21b}$$

We can now use these expressions to evaluate the commutators of the modes. Firstly, for the left-moving modes

α_k, we obtain

$$\begin{aligned}
[\alpha_k, \alpha_{k'}] &= \frac{1}{(4\pi i)^2 \frac{\alpha'}{2}} \left[\int_0^{2\pi} \Big(kX(\sigma) + i\dot{X}(\sigma)\Big) e^{-ik\sigma}\, d\sigma, \int_0^{2\pi} \Big(k'X(\sigma') + i\dot{X}(\sigma')\Big) e^{-ik'\sigma'}\, d\sigma' \right] \\
&= -\frac{1}{8\pi^2\alpha'} \int_0^{2\pi}\int_0^{2\pi} \Big[kX(\sigma) + i\dot{X}(\sigma), k'X(\sigma') + i\dot{X}(\sigma')\Big] e^{-ik\sigma - ik'\sigma'} d\sigma d\sigma' \\
&= -\frac{i}{8\pi^2\alpha'} \int_0^{2\pi}\int_0^{2\pi} \Big(k\Big[X(\sigma), \dot{X}(\sigma')\Big] + k'\Big[\dot{X}(\sigma), X(\sigma')\Big]\Big) e^{-ik\sigma - ik'\sigma'} d\sigma d\sigma' \\
&= -i\frac{k-k'}{8\pi^2\alpha'} \int_0^{2\pi}\int_0^{2\pi} 2\pi i\alpha'\delta(\sigma - \sigma') e^{-ik\sigma - ik'\sigma'} d\sigma d\sigma' \\
&= \frac{k-k'}{4\pi} \int_0^{2\pi} e^{-i(k+k')\sigma} d\sigma \\
&= \frac{k-k'}{4\pi} 2\pi \delta_{k,-k'} \\
&= k\delta_{k,-k'},
\end{aligned} \tag{3.22}$$

where we used the fact that $[X(\sigma), X(\sigma')] = \Big[\dot{X}(\sigma), \dot{X}(\sigma')\Big] = 0$. Similarly for the right-moving modes $\tilde{\alpha}_k$, we find that

$$[\tilde{\alpha}_k, \tilde{\alpha}_{k'}] = k\delta_{k,-k'}. \tag{3.23}$$

Finally, the commutator between left-moving and right-moving modes is

$$\begin{aligned}
[\alpha_k, \tilde{\alpha}_{k'}] &= \frac{1}{(4\pi i)^2 \frac{\alpha'}{2}} \left[\int_0^{2\pi} \Big(kX(\sigma) + i\dot{X}(\sigma)\Big) e^{-ik\sigma}\, d\sigma, \int_0^{2\pi} \Big(-k'X(\sigma') + i\dot{X}(\sigma')\Big) e^{-ik'\sigma'}\, d\sigma' \right] \\
&= -\frac{1}{8\pi^2\alpha'} \int_0^{2\pi}\int_0^{2\pi} \Big[kX(\sigma) + i\dot{X}(\sigma), -k'X(\sigma') + i\dot{X}(\sigma')\Big] e^{-ik\sigma - ik'\sigma'} d\sigma d\sigma' \\
&= -\frac{i}{8\pi^2\alpha'} \int_0^{2\pi}\int_0^{2\pi} \Big(k\Big[X(\sigma), \dot{X}(\sigma')\Big] - k'\Big[\dot{X}(\sigma), X(\sigma')\Big]\Big) e^{-ik\sigma - ik'\sigma'} d\sigma d\sigma' \\
&= -i\frac{k+k'}{8\pi^2\alpha'} \int_0^{2\pi}\int_0^{2\pi} 2\pi i\alpha'\delta(\sigma - \sigma') e^{-ik\sigma - ik'\sigma'} d\sigma d\sigma' \\
&= \frac{k+k'}{4\pi} \int_0^{2\pi} e^{-i(k+k')\sigma} d\sigma \\
&= \frac{k+k'}{4\pi} 2\pi \delta_{k,-k'} \\
&= 0.
\end{aligned} \tag{3.24}$$

To find the partition function, we return to our classical Hamiltonian (3.6). To evaluate this, we need the spatial derivative of X, which is given by

$$X' = \sqrt{\frac{\alpha'}{2}} \sum_{k\neq 0} \big(-\alpha_k e^{ik\sigma} + \tilde{\alpha}_k e^{-ik\sigma}\big). \tag{3.25}$$

Thus the quantised Hamiltonian is

$$\begin{aligned}
H &= \frac{1}{4\pi\alpha'}\frac{\alpha'}{2} \int_0^{2\pi} \sum_{k,k'\neq 0} \Big[\big(\alpha_k e^{ik\sigma} + \tilde{\alpha}_k e^{-ik\sigma}\big)\Big(\alpha_{k'} e^{ik'\sigma} + \tilde{\alpha}_{k'} e^{-ik'\sigma}\Big) \\
&\qquad + \big(-\alpha_k e^{ik\sigma} + \tilde{\alpha}_k e^{-ik\sigma}\big)\Big(-\alpha_{k'} e^{ik'\sigma} + \tilde{\alpha}_{k'} e^{-ik'\sigma}\Big)\Big] d\sigma \\
&= \frac{1}{8\pi} 2\pi \sum_{k,k'} 2[\alpha_k\alpha_{k'} + \tilde{\alpha}_k\tilde{\alpha}_{k'}]\delta_{k,-k'} \\
&= \frac{1}{2}\sum_k (\alpha_k\alpha_{-k} + \tilde{\alpha}_k\tilde{\alpha}_{-k}).
\end{aligned} \tag{3.26}$$

Restricting k to positive integers only, we have

$$\begin{aligned} H &= \frac{1}{2}\sum_{k=1}^{\infty}(\alpha_k\alpha_{-k} + \tilde{\alpha}_k\tilde{\alpha}_{-k} + \alpha_{-k}\alpha_k + \tilde{\alpha}_{-k}\tilde{\alpha}_k) \\ &= \sum_{k=1}^{\infty}\left(\frac{1}{2}([\alpha_k,\alpha_{-k}] + [\tilde{\alpha}_k,\tilde{\alpha}_{-k}]) + \alpha_{-k}\alpha_k + \tilde{\alpha}_{-k}\tilde{\alpha}_k\right). \end{aligned} \tag{3.27}$$

We now split this into two parts, one associated with left-moving modes α_k, and another with right-moving modes $\tilde{\alpha}_k$:

$$H_L = \sum_k\left(\frac{1}{2}[\alpha_k,\alpha_{-k}] + \alpha_{-k}\alpha_k\right) = \frac{1}{2}\sum_{k=1}^{\infty}k + \sum_{k=1}^{\infty}\alpha_{-k}\alpha_k \tag{3.28a}$$

$$H_R = \sum_k\left(\frac{1}{2}[\tilde{\alpha}_k,\tilde{\alpha}_{-k}] + \tilde{\alpha}_{-k}\tilde{\alpha}_k\right) = \frac{1}{2}\sum_{k=1}^{\infty}k + \sum_{k=1}^{\infty}\tilde{\alpha}_{-k}\tilde{\alpha}_k. \tag{3.28b}$$

We now need to evaluate the divergent sum $\sum_{k=1}^{\infty} k$, which we can do using zeta function regularisation. The Riemann zeta function is defined as

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z} \tag{3.29}$$

for $z \in \mathbb{C}$ such that $\mathrm{Re}(z) > 1$. The zeta function has a unique analytic continuation to $z = -1$, where it takes the value $-1/12$ [1, 3]. We hence have the result

$$\sum_{k=1}^{\infty} k = -\frac{1}{12}, \tag{3.30}$$

which leads to

$$H_L = -\frac{1}{24} + \sum_{k=1}^{\infty}\alpha_{-k}\alpha_k, \qquad H_R = -\frac{1}{24} + \sum_{k=1}^{\infty}\tilde{\alpha}_{-k}\tilde{\alpha}_k. \tag{3.31}$$

We note that $\alpha_{-k}\alpha_k$ and $\tilde{\alpha}_{-k}\tilde{\alpha}_k$ act as number operators in the left- and right-moving sectors, respectively. Hence for left-moving modes, the contribution to the partition function is

$$\begin{aligned} \mathrm{tr}\left(e^{-\beta_L H_L}\right) &= \mathrm{tr}\left(\exp\left(-\beta_L\left(-\frac{1}{24} + \sum_{k=1}^{\infty}\alpha_{-k}\alpha_k\right)\right)\right) \\ &= e^{\beta_L/24}\prod_{k=1}^{\infty}\mathrm{tr}\left(e^{-\beta_L\alpha_{-k}\alpha_k}\right) \\ &= e^{\beta_L/24}\prod_{k=1}^{\infty}\sum_{n=0}^{\infty}e^{-n\beta_L k} \\ &= e^{\beta_L/24}\prod_{k=1}^{\infty}\frac{1}{1-e^{-\beta_L k}}, \end{aligned} \tag{3.32}$$

and similarly for right-moving modes,

$$\mathrm{tr}\left(e^{-\beta_R H_R}\right) = e^{\beta_R/24}\prod_{k=1}^{\infty}\frac{1}{1-e^{-\beta_R k}}, \tag{3.33}$$

In the context of conformal field theories, β_R is the complex conjugate of β_L. We now identify a complex parameter τ such that $\beta_L = -2\pi i\tau$, and hence $\beta_R = 2\pi i\bar{\tau}$, where $\bar{\tau}$ is the complex conjugate of τ. (Note that this τ is a new parameter, and is not the timelike coordinate used earlier in the Lagrangian.) This allows us to write the partition function as

$$\begin{aligned}
Z(\tau) &= \mathrm{tr}\big(e^{-\beta_L H_L - \beta_R H_R}\big) \\
&= e^{-2\pi i\tau/24} \prod_{k=1}^{\infty} \frac{1}{1-e^{2\pi i\tau k}} e^{2\pi i\bar{\tau}/24} \prod_{k'=1}^{\infty} \frac{1}{1-e^{-2\pi i\bar{\tau}k'}} \\
&= q^{-1/24}\bar{q}^{-1/24} \prod_{k=1}^{\infty} \frac{1}{(1-q^k)(1-\bar{q}^k)} \\
&= \frac{1}{|\eta(\tau)|^2},
\end{aligned} \tag{3.34}$$

where we have defined $q = e^{2\pi i\tau}$ (and its complex conjugate $\bar{q} = e^{-2\pi i\bar{\tau}}$), and η is the Dedekind eta function,

$$\eta(\tau) = q^{1/24} \prod_{k=1}^{\infty} \big(1-q^k\big) \tag{3.35}$$

which is discussed further in Chapter 4.

3.1.1 Anti-Periodic Boundary Conditions

In Chapter 5, we will consider string theory on a $\mathbb{Z}_2$ orbifold. In this case, in addition to the periodic boundary conditions considered above, we can also have anti-periodic boundary conditions, with $X(\sigma) = -X(\sigma+2\pi)$. The main difference in this case is that k in (3.12) will take half-odd-integer values. For simplicity, we replace k with $k-1/2$, where k is again an integer. This gives

$$\big[\alpha_{k+1/2}, \alpha_{-k-1/2}\big] = k + \frac{1}{2}, \tag{3.36}$$

where k runs over all integers including zero, and

$$\big[\alpha_{k-1/2}, \alpha_{-k+1/2}\big] = k - \frac{1}{2}, \tag{3.37}$$

where k runs over positive or negative integers only. We see that the zero modes do not contribute in this case.

For the partition function, we get a similar result to the periodic case, except that the zero point energy is different:

$$\mathrm{tr}\big(e^{-\beta_L H_L}\big) = e^{-\beta_L/48} \prod_{k=1}^{\infty} \frac{1}{1-e^{-\beta_L(k-1/2)}} \tag{3.38a}$$

$$\mathrm{tr}\big(e^{-\beta_R H_R}\big) = e^{-\beta_R/48} \prod_{k=1}^{\infty} \frac{1}{1-e^{-\beta_R(k-1/2)}}, \tag{3.38b}$$

where we used the following definition from [2]:

$$\sum_{k=1}^{\infty}(k-\theta) = \frac{1}{24} - \frac{1}{8}(2\theta-1)^2. \tag{3.39}$$

Thus the partition function in this case is

$$\begin{aligned}
Z(\tau) &= e^{-\beta_L/48} e^{-\beta_R/48} \prod_{k=1}^{\infty} \frac{1}{1-e^{-\beta_L(k-1/2)}} \frac{1}{1-e^{-\beta_R(k-1/2)}} \\
&= q^{1/48}\bar{q}^{1/48} \prod_{k=1}^{\infty} \frac{1}{\big(1-q^{k-1/2}\big)\big(1-\bar{q}^{k-1/2}\big)}.
\end{aligned} \tag{3.40}$$

3.2 Partition Function for the Fermionic String

Our discussion of the fermionic string will be based on [1], and will follow the same lines as our discussion of the Dirac field in Chapter 2. We begin by introducing the two-dimensional Dirac matrices ρ^0, ρ^1, which satisfy the Dirac algebra

$$\big\{\rho^\alpha, \rho^\beta\big\} = 2\eta^{\alpha\beta}, \tag{3.41}$$

and can be represented by

$$\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.42}$$

To move from bosonic string theory to superstring theory, the bosonic field $X(\tau, \sigma)$ of the previous section must be paired with a fermionic field, which we denote by $\Psi(\tau, \sigma)$. This is a two-component spinor, which we write as

$$\Psi = \begin{pmatrix} \psi \\ \tilde{\psi} \end{pmatrix}, \tag{3.43}$$

while its Dirac conjugate $\bar{\Psi}$ is given by

$$\bar{\Psi} = i\psi^\dagger \rho^0 = i\begin{pmatrix} \tilde{\psi}^* & -\psi^* \end{pmatrix}. \tag{3.44}$$

We will consider the case of Majorana fermions, which are fermions that are their own antiparticles. In the representation of the Dirac algebra given above, a Majorana spinor is equivalent to a real spinor, so that $\psi^* = \psi$ and $\tilde{\psi}^* = \tilde{\psi}$, although we still consider these to be independent variables when deriving the equations of motion. The action for a closed fermionic string is then

$$S = \frac{i}{4\pi} \int\int \bar{\Psi}\rho^\alpha \partial_\alpha \Psi \, d\sigma \, d\tau, \tag{3.45}$$

where we have now made the conventional choice $\alpha' = 1/2$. In superstring theory, this action would be added to the bosonic string action to obtain the superstring action. The integrand can be written out explicitly as

$$\begin{aligned} \bar{\Psi}\rho_\alpha \partial^\alpha \Psi &= i\begin{pmatrix} \tilde{\psi}^* & -\psi^* \end{pmatrix} \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \partial_\tau + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_\sigma \right] \begin{pmatrix} \psi \\ \tilde{\psi} \end{pmatrix} \\ &= i\begin{pmatrix} \tilde{\psi}^* & -\psi^* \end{pmatrix} \begin{pmatrix} 0 & -\partial_\tau + \partial_\sigma \\ \partial_\tau + \partial_\sigma & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \tilde{\psi} \end{pmatrix} \\ &= i\Big(-\psi^*(\partial_\tau + \partial_\sigma)\psi - \tilde{\psi}^*(\partial_\tau - \partial_\sigma)\tilde{\psi} \Big). \end{aligned}$$

This allows the action to be written as

$$S = \frac{1}{4\pi} \int\int \Big(\psi^*(\partial_\tau + \partial_\sigma)\psi + \tilde{\psi}^*(\partial_\tau - \partial_\sigma)\tilde{\psi} \Big) \, d\sigma \, d\tau, \tag{3.46}$$

so that the Lagrangian density is

$$\mathcal{L} = \frac{1}{4\pi} \Big(\psi^*(\partial_\tau + \partial_\sigma)\psi + \tilde{\psi}^*(\partial_\tau - \partial_\sigma)\tilde{\psi} \Big). \tag{3.47}$$

Recalling the reality condition for Majorana spinors, the canonical momenta are

$$\Pi(\psi) = \frac{\partial \mathcal{L}}{\partial(\partial_\tau \psi)} = \frac{1}{4\pi}\psi^* = \frac{1}{4\pi}\psi, \tag{3.48a}$$

$$\tilde{\Pi}(\tilde{\psi}) = \frac{\partial \mathcal{L}}{\partial\Big(\partial_\tau \tilde{\psi}\Big)} = \frac{1}{4\pi}\tilde{\psi}^* = \frac{1}{4\pi}\tilde{\psi}. \tag{3.48b}$$

Thus the classical Hamiltonian for the closed free fermionic string is

$$\begin{aligned} H &= \frac{\partial L}{\partial(\partial_\tau \psi)} \partial_\tau \psi + \frac{\partial L}{\partial(\partial_\tau \tilde{\psi})} \partial_\tau \tilde{\psi} - L \\ &= \frac{1}{4\pi} \int_0^{2\pi} \psi \partial_\tau \psi \, d\sigma + \frac{1}{4\pi i} \int_0^{2\pi} \tilde{\psi} \partial_\tau \tilde{\psi} \, d\sigma - L \\ &= \frac{1}{4\pi} \int_0^{2\pi} (-\psi \partial_\sigma \psi + \tilde{\psi} \partial_\sigma \tilde{\psi}) \, d\sigma \, . \end{aligned} \tag{3.49}$$

The equations of motion for ψ and $\tilde{\psi}$ are obtained from the Euler-Lagrange equations for ψ^* and $\tilde{\psi}^*$,

$$\begin{aligned} 0 &= \partial_\tau \left(\frac{\partial \mathcal{L}}{\partial(\partial_\tau \psi^*)} \right) + \partial_\sigma \left(\frac{\partial \mathcal{L}}{\partial(\partial_\sigma \psi^*)} \right) - \frac{\partial \mathcal{L}}{\partial \psi^*} = -\frac{1}{4\pi}(\partial_\tau + \partial_\sigma)\psi, \\ 0 &= \partial_\tau \left(\frac{\partial \mathcal{L}}{\partial\Big(\partial_\tau \tilde{\psi}^*\Big)} \right) + \partial_\sigma \left(\frac{\partial \mathcal{L}}{\partial\Big(\partial_\sigma \tilde{\psi}^*\Big)} \right) - \frac{\partial \mathcal{L}}{\partial \tilde{\psi}^*} = -\frac{1}{4\pi}(\partial_\tau - \partial_\sigma)\tilde{\psi}. \end{aligned}$$

Thus the equations of motion are

$$(\partial_\tau + \partial_\sigma)\psi = 0 \tag{3.50a}$$

$$(\partial_\tau - \partial_\sigma)\tilde{\psi} = 0. \tag{3.50b}$$

3.2.1 Periodic Boundary Conditions

The remainder of this section closely follows [21]. We first solve the equations of motion for the case of periodic boundary conditions, $\psi(\sigma) = \psi(\sigma + 2\pi)$ and $\tilde{\psi}(\sigma) = \tilde{\psi}(\sigma + 2\pi)$; the case of anti-periodic boundary conditions is very similar and will be discussed briefly at the end of this section. Using the method of separation of variables, we assume that ψ can be expressed in the form $\psi(\tau, \sigma) = a(\tau)b(\sigma)$. Upon substituting this into the equations of motion, we obtain

$$\dot{a}b + b'a = 0 \Longrightarrow \frac{\dot{a}}{a} = -\frac{b'}{b}, \tag{3.51}$$

and we choose this to be equal to $-ik$ for some constant k. Up to an overall constant, this yields the solutions

$$\begin{aligned} a &= e^{-ik\tau} \\ b &= e^{ik\sigma}, \end{aligned} \tag{3.52}$$

where for periodic boundary conditions, we require $k \in \mathbb{Z}$. Thus choosing a convenient normalisation, the mode expansion of ψ is

$$\psi = \sqrt{-i} \sum_{k \in \mathbb{Z}} \psi_k e^{-ik\tau + ik\sigma}. \tag{3.53}$$

Similarly for $\tilde{\psi}$, we find that

$$\tilde{\psi} = \sqrt{-i} \sum_{k \in \mathbb{Z}} \tilde{\psi}_k e^{-ik\tau - ik\sigma}. \tag{3.54}$$

To find the commutator between the modes, we repeat the same procedure as for the bosonic string, multiplying both sides by $e^{-ik'\sigma}$ and integrating over σ for $\tau = 0$. From the equation for ψ, we obtain

$$\begin{aligned} \int_0^{2\pi} \psi e^{-ik'\sigma} d\sigma &= \int_0^{2\pi} \sqrt{-i} \sum_{k \in \mathbb{Z}} \psi_k e^{ik\sigma} e^{-ik'\sigma}\, d\sigma \\ &= \sqrt{-i} \sum_{k \in \mathbb{Z}} \psi_k \int_0^{2\pi} e^{i(k-k')\sigma}\, d\sigma \\ &= \sqrt{-i} \sum_{k \in \mathbb{Z}} \psi_k 2\pi \delta_{k,k'} \\ &= 2\pi \sqrt{-i} \psi_{k'}, \end{aligned} \tag{3.55}$$

and similarly the equation for $\tilde{\psi}$ yields

$$\begin{aligned} \int_0^{2\pi} \tilde{\psi} e^{-ik'\sigma} d\sigma &= \int_0^{2\pi} \sqrt{-i} \sum_{k \in \mathbb{Z}} \tilde{\psi}_k e^{-ik\sigma} e^{-ik'\sigma}\, d\sigma \\ &= \sqrt{-i} \sum_{k \in \mathbb{Z}} \tilde{\psi}_k \int_0^{2\pi} e^{-i(k+k')\sigma}\, d\sigma \\ &= \sqrt{-i} \sum_{k \in \mathbb{Z}} \tilde{\psi}_k 2\pi \delta_{k,-k'} \\ &= 2\pi \sqrt{-i} \tilde{\psi}_{-k'}. \end{aligned} \tag{3.56}$$

Hence the modes can be expressed as

$$\psi_k = \frac{1}{2\pi\sqrt{-i}} \int_0^{2\pi} \psi e^{-ik\sigma}\, d\sigma \tag{3.57a}$$

$$\tilde{\psi}_k = \frac{1}{2\pi\sqrt{-i}} \int_0^{2\pi} \tilde{\psi} e^{ik\sigma}\, d\sigma\,. \tag{3.57b}$$

We now quantise the fermionic string by imposing the anticommutation relation $\{\psi(\sigma), \psi(\sigma')\} = -2\pi i\delta(\sigma - \sigma')$, where the factor of -2π has been chosen for later convenience. This implies that the anticommutator between the left-moving modes is

$$\begin{aligned}
\{\psi_k, \psi_{k'}\} &= \frac{i}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\{\psi(\sigma), \psi(\sigma')\}e^{-ik\sigma - ik'\sigma'}\,d\sigma\,d\sigma' \\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}\delta(\sigma - \sigma')e^{-ik\sigma - ik'\sigma'}\,d\sigma\,d\sigma' \\
&= \frac{1}{2\pi}\int_0^{2\pi}e^{-i\left(k+k'\right)\sigma}\,d\sigma \\
&= \delta_{k,-k'},
\end{aligned} \tag{3.58}$$

and similarly the anticommutator between the right-moving modes is

$$\begin{aligned}
\left\{\tilde{\psi}_k, \tilde{\psi}_{k'}\right\} &= \frac{i}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\left\{\tilde{\psi}(\sigma), \tilde{\psi}(\sigma')\right\}e^{ik\sigma + ik'\sigma'}\,d\sigma\,d\sigma' \\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}\delta(\sigma - \sigma')e^{ik\sigma + ik'\sigma'}\,d\sigma\,d\sigma' \\
&= \frac{1}{2\pi}\int_0^{2\pi}e^{-i\left(k+k'\right)\sigma}\,d\sigma \\
&= \delta_{k,-k'}.
\end{aligned} \tag{3.59}$$

This gives the result

$$\{\psi_k, \psi_{-k}\} = \left\{\tilde{\psi}_k, \tilde{\psi}_{-k}\right\} = 1, \tag{3.60}$$

while the rest of the anticommutators are zero; for example,

$$\left\{\tilde{\psi}_k, \psi_{k'}\right\} = \frac{i}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\left\{\tilde{\psi}(\sigma), \psi(\sigma')\right\}e^{-ik\left(\sigma - \sigma'\right)}\,d\sigma\,d\sigma' = 0. \tag{3.61}$$

To quantise our Hamiltonian (3.49), we need to compute the spatial derivatives

$$\partial_\sigma\psi = i\sqrt{-i}\sum_{k\in\mathbb{Z}}k\psi_k e^{-ik\tau + ik\sigma}, \tag{3.62a}$$

$$\partial_\sigma\tilde{\psi} = -i\sqrt{-i}\sum_{k\in\mathbb{Z}}k\tilde{\psi}_k e^{-ik\tau - ik\sigma}. \tag{3.62b}$$

Thus for $\tau = 0$, the quantised Hamiltonian is

$$\begin{aligned}
H &= \frac{1}{4\pi}\int_0^{2\pi}(-\psi\partial_\sigma\psi + \tilde{\psi}\partial_\sigma\tilde{\psi})\,d\sigma \\
&= \frac{1}{4\pi}\int_0^{2\pi}(-(\sqrt{-i}\sum_k\psi_k e^{-ik\sigma})(i\sqrt{-i}\sum_{k'}k'\psi_{k'}e^{-ik'\sigma}) + (\sqrt{-i}\sum_k\tilde{\psi}_k e^{ik\sigma})(-i\sqrt{-i}\sum_{k'}k'\tilde{\psi}_{k'}e^{ik'\sigma}))\,d\sigma \\
&= -\frac{1}{4\pi}\sum_{k,k'}k'\int_0^{2\pi}(\psi_k\psi_{k'}e^{-i\left(k+k'\right)\sigma} + \tilde{\psi}_k\tilde{\psi}_{k'}e^{i\left(k+k'\right)\sigma})\,d\sigma \\
&= -\frac{1}{2}\sum_{k,k'}k'(\psi_k\psi_{k'} + \tilde{\psi}_k\tilde{\psi}_{k'})\delta_{k,-k'} \\
&= \frac{1}{2}\sum_k k(\psi_k\psi_{-k} + \tilde{\psi}_k\tilde{\psi}_{-k}).
\end{aligned} \tag{3.63}$$

Restricting k to positive integers only, we can rewrite this as

$$\begin{aligned}
H &= \frac{1}{2}\sum_{k=1}^{\infty}k(\psi_k\psi_{-k} + \tilde{\psi}_k\tilde{\psi}_{-k} - \psi_{-k}\psi_k - \tilde{\psi}_{-k}\tilde{\psi}_k) \\
&= \sum_{k=1}^{\infty}k\left(-\frac{1}{2}\left(\{\psi_k, \psi_{-k}\} + \left\{\tilde{\psi}_k, \tilde{\psi}_{-k}\right\}\right) + \psi_{-k}\psi_k + \tilde{\psi}_{-k}\tilde{\psi}_k\right).
\end{aligned} \tag{3.64}$$

Splitting this into left and right parts, we have

$$\begin{aligned} H_L &= \sum_{k=1}^{\infty} k\left(-\frac{1}{2}\{\psi_k, \psi_{-k}\} + \psi_{-k}\psi_k\right) \\ &= \sum_{k=1}^{\infty} k\left(-\frac{1}{2} + \psi_{-k}\psi_k\right) \\ &= \frac{1}{24} + \sum_{k=1}^{\infty} k\psi_{-k}\psi_k, \end{aligned} \tag{3.65}$$

and similarly

$$H_R = \sum_{k=1}^{\infty} k\left(-\frac{1}{2}\left\{\tilde{\psi}_k, \tilde{\psi}_{-k}\right\} + \tilde{\psi}_{-k}\tilde{\psi}_k\right) = \frac{1}{24} + \sum_{k=1}^{\infty} k\tilde{\psi}_{-k}\tilde{\psi}_k. \tag{3.66}$$

We can now compute the partition function for the fermionic string. The contribution from the left-moving modes is

$$\begin{aligned} \mathrm{tr}\big[e^{-\beta_L H_L}\big] &= \mathrm{tr}\left(\exp\left(-\beta_L\left(\frac{1}{24} + \sum_{k=1}^{\infty} k\psi_{-k}\psi_k\right)\right)\right) \\ &= e^{-\beta_L/24}\mathrm{tr}\left(\exp\left(-\beta_L\sum_{k=1}^{\infty} k\psi_{-k}\psi_k\right)\right) \\ &= e^{-\beta_L/24}\prod_{k=1}^{\infty}\mathrm{tr}\big(e^{-\beta_L k\psi_{-k}\psi_k}\big) \\ &= e^{-\beta_L/24}\prod_{k=1}^{\infty}\big(1 + e^{-\beta_L k}\big), \end{aligned} \tag{3.67}$$

since fermionic states can only have occupation numbers of 0 or 1. We get a similar result for the right-moving modes, so that the partition function is

$$\begin{aligned} Z(\tau) &= e^{-\beta_L/24}\prod_{k=1}^{\infty}\big(1 + e^{-\beta_L k}\big)e^{-\beta_R/24}\prod_{k'=1}^{\infty}\left(1 + e^{-\beta_R k'}\right) \\ &= q^{1/24}\bar{q}^{1/24}\prod_{k=1}^{\infty}\big(1 + q^k\big)\big(1 + \bar{q}^k\big) \end{aligned} \tag{3.68}$$

where, as in the bosonic case, we have set $q = e^{2\pi i\tau} = e^{-\beta_L/24}$, $\bar{q} = e^{-2\pi i\bar{\tau}} = e^{-\beta_R/24}$.

3.2.2 Anti-Periodic Boundary Conditions

For anti-periodic boundary conditions, as in the bosonic case, the only major difference is that k will take on half-odd-integer values. This means that only the zero point energy must be modified, so that the partition function is

$$\begin{aligned} Z(\tau) &= e^{\beta_L/48}\prod_{k=1}^{\infty}\left(1 + e^{-\beta_L(k-1/2)}\right)e^{\beta_R/48}\prod_{k'=1}^{\infty}\left(1 + e^{-\beta_R\left(k'-1/2\right)}\right) \\ &= e^{-2\pi i\tau/48}\prod_{k=1}^{\infty}\left(1 + e^{2\pi i\tau(k-1/2)}\right)e^{2\pi i\bar{\tau}/48}\prod_{k'=1}^{\infty}\left(1 + e^{-2\pi i\bar{\tau}\left(k'-1/2\right)}\right) \\ &= q^{-1/48}\bar{q}^{-1/48}\prod_{k=1}^{\infty}\left(1 + q^{k-1/2}\right)\left(1 + \bar{q}^{k-1/2}\right), \end{aligned} \tag{3.69}$$

where we have again replaced k by $k - 1/2$, so that k remains an integer.

Chapter 4

Modular Forms and Jacobi Forms

In this chapter, we give a brief introduction to some of the mathematical objects which will be used in the computation of black hole entropy in later chapters, including modular forms, Eisenstein series and Jacobi forms.

4.1 Modular Forms

The material in this section is primarily based on [21, 22].

Definition 4.1. The two-dimensional *special linear group* $SL(2,\mathbb{Z})$ is defined to be the group of 2×2 matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{4.1}$$

where a, b, c, d are integers, with $\det A = ad - bc = 1$.

Definition 4.2. The *modular group* Γ is the set of all Möbius transformations of the form

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \tag{4.2}$$

where a, b, c, d are integers with $ad - bc = 1$. The modular group is isomorphic to the two-dimensional projective special linear group $PSL(2,\mathbb{Z})$, which is the quotient of $SL(2,\mathbb{Z})$ by its centre $\{I, -I\}$, i.e. the group of of 2×2 matrices $A \in SL(2,\mathbb{Z})$ where A is identified with $-A$. The modular group Γ is generated by the two matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{4.3}$$

whose actions on $\mathbb{H}$ are given by

$$T : \tau \mapsto \tau + 1, \qquad S : \tau \mapsto -\frac{1}{\tau}. \tag{4.4}$$

An important class of subgroups of the full modular group consists of the congruence subgroups. The most relevant of these in this work is

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c = 0 \ (\text{mod}\, N) \right\}. \tag{4.5}$$

Definition 4.3. In this work, following [21], a function f will be termed a *modular function* if it satisfies the following conditions:

1. f is meromorphic in the upper half-plane $\mathbb{H} = \{\tau \in \mathbb{C} \mid \text{Im}(\tau) > 0\}$.
2. f transforms under $SL(2,\mathbb{Z})$ or its subgroups according to

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau), \tag{4.6}$$

where k is an integer or half-integer called the *weight* of f. In some cases, we will also allow the inclusion of a phase factor $\zeta(a, b, c, d)$ which depends on the transformation matrix.

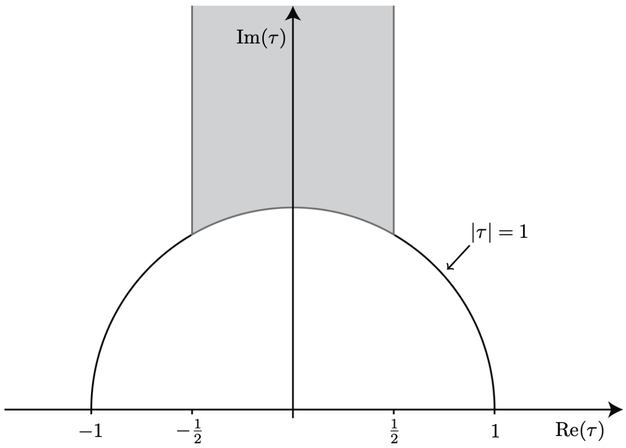

Figure 4.1: The fundamental domain R_Γ of the modular group Γ, shown as the shaded region. It is defined by the requirements $|\tau| > 1$ and $|\mathrm{Re}(\tau)| < 1/2$. It can easily be checked that this region satisfies the definition of a fundamental domain given in the Introduction [22]; in particular, it satisfies the requirements that no two points in R_Γ are equivalent under Γ, and any point in $\mathbb{H}$ is equivalent to some point in R_Γ under Γ. This plot is based on an image from [22].

Definition 4.4. A *modular form* satisfies the same conditions as a modular function, in addition to the stronger requirement that it be holomorphic on $\mathbb{H}$ rather than just meromorphic.

A modular form f satisfies the condition $f(\tau) = f(\tau+1)$, and hence can be written as a Fourier series

$$f(\tau) = \sum_{n\in\mathbb{Z}} a(n)q^n, \qquad q = e^{2\pi i\tau}, \tag{4.7}$$

and is bounded as $\mathrm{Im}(\tau) \to \infty$. Additionally, following [18],

- If $a(0) = 0$, then f vanishes at infinity and is called a *cusp form;*
- If $a(n) = 0$ for $n < -N$ for some $N \geq 0$, then f is called a *weakly holomorphic modular form.*

We also define a function $f : \mathbb{H} \to \mathbb{C}$ to be an *almost holomorphic modular form* (or a nearly holomorphic modular form) if it satisfies the modular transformation property (4.6) and is a polynomial in $1/\mathrm{Im}(\tau)$ with coefficients that are holomorphic functions of q.

Finally, a function is called a *quasi-modular form* if it is the holomorphic part of an almost holomorphic modular form [23].

4.2 Siegel Modular Forms

The material in this section is based on [4, 23]. We only give the definition of Siegel modular forms of degree two, since the more general degree-g definition will not be relevant to the physical applications in this project.

Definition 4.5. Let $Sp(2,\mathbb{Z})$ be the group of 4×4 symplectic matrices. This consists of matrices

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{4.8}$$

where A, B, C, D are 2×2 matrices with integer entries, and

$$AB^T = BA^T, \qquad CD^T = DC^T, \qquad AD^T - BC^T = \mathbf{1}. \tag{4.9}$$

Define the *Siegel upper half-plane of degree two*, denoted by $\mathbb{H}_2$, to be the set of 2×2 symmetric matrices over $\mathbb{C}$ whose imaginary part is positive-definite. This consists of matrices

$$\Omega = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix}, \tag{4.10}$$

with $\tau, \sigma \in \mathbb{H}$ and

$$\det(\mathrm{Im}(\Omega)) > 0. \tag{4.11}$$

A holomorphic function $\Phi : \mathbb{H}_2 \to \mathbb{C}$ is said to be a *Siegel modular form* of weight k (and degree two) if

$$\Phi\Big((A\Omega+B)(C\Omega+D)^{-1}\Big) = (\det(C\Omega+D))^k\Phi(\Omega) \tag{4.12}$$

for all $\Omega \in Sp(2,\mathbb{Z})$.

A Siegel modular form Φ can be expanded in a Fourier series as

$$\Phi(\Omega) = \sum_{m,n,p\in\mathbb{Z}} g(m,n,r) q^m y^n p^r, \qquad q = e^{2\pi i\tau},\ y = e^{2\pi i z},\ p = e^{2\pi i\sigma}. \tag{4.13}$$

Siegel modular forms will be used in the computation of the degeneracy of black hole microstates in Chapter 6, where this degeneracy is obtained from the Fourier coefficients of the Igusa cusp form of weight ten.

4.3 Eisenstein Series

Definition 4.6. The *Eisenstein series* of weight $2k$ is defined for $k \geq 2$, $\tau \in \mathbb{H}$ as

$$G_{2k}(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m+n\tau)^{2k}}. \tag{4.14}$$

G_{2k} is holomorphic on $\mathbb{H}$ and satisfies (4.6), so it is a modular form [22]. The Eisenstein series $G_{2k}(\tau)$ for all $k \geq 2$ has the Fourier series expansion

$$G_{2k}(\tau) = 2\zeta(2k)\left(1 + c_{2k}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n\right), \tag{4.15}$$

where $q = e^{2\pi i\tau}$ is the nome, $\sigma_k(n) = \sum_{d|n} d^k$ is the divisor sum function, and the coefficients are

$$c_{2k} = \frac{(2\pi i)^{2k}}{(2k-1)!\zeta(2k)}. \tag{4.16}$$

For simplicity, we will use the notation

$$E_{2k}(\tau) = \frac{G_{2k}(\tau)}{2\zeta(2k)}, \tag{4.17}$$

and will subsequently refer to this as simply the Eisenstein series of weight $2k$. The main Eisenstein series used in this work are

$$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n, \tag{4.18a}$$

$$E_4(\tau) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n, \tag{4.18b}$$

$$E_6(\tau) = 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n. \tag{4.18c}$$

We note that E_4 and E_6 are modular forms of weights four and six respectively, while $E_2(\tau)$ is a quasi-modular form, since the non-holomorphic function $E_2(\tau) - 3/\pi\mathrm{Im}(\tau)$ transforms as a modular form of weight two. Any modular form can be written in terms of E_4 and E_6 [21].

We can use the definitions above to define the function

$$\mathcal{E}_N(\tau) = \frac{1}{N-1}(NE_2(N\tau) - E_2(\tau)), \tag{4.19}$$

which transforms as a modular form of weight two under the congruence subgroup $\Gamma_0(N)$ and satisfies

$$\mathcal{E}_N(\tau+1) = \mathcal{E}_N(\tau), \qquad \mathcal{E}_N\left(-\frac{1}{\tau}\right) = -\frac{\tau^2}{N}\mathcal{E}_N\left(\frac{\tau}{N}\right). \tag{4.20}$$

One can show [21] that this is equivalent to

$$\mathcal{E}_N(\tau) = \frac{12i}{\pi(N-1)}\partial_\tau[\ln\eta(\tau) - \ln\eta(N\tau)]. \tag{4.21}$$

4.4 Jacobi Forms

Definition 4.7. Consider a holomorphic function $\varphi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ such that

1. φ transforms under $SL(2,\mathbb{Z})$ as
$$\varphi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e^{\frac{2\pi icz}{c\tau+d}} \varphi(\tau,z), \tag{4.22}$$
where k is an integer; i.e. φ is modular in its first argument.

2. For all $\lambda, \mu \in \mathbb{Z}$, φ satisfies
$$\varphi(\tau, z+\lambda\tau+\mu) = e^{-2\pi im\left(\lambda^2\tau+2\lambda z\right)}\varphi(\tau,z), \tag{4.23}$$
where m is an integer; i.e. φ is elliptic in its second argument.

These properties imply that φ has a Fourier expansion of the form
$$\varphi(\tau,z) = \sum_{n=0}^{\infty} \sum_{\substack{r\in\mathbb{Z} \\ r^2\leq 4nm}} c(n,r)e^{2\pi i(n\tau+rz)}. \tag{4.24}$$

The second condition (4.23) implies that the Fourier coefficients satisfy the periodicity property
$$c(n,r) = C(\Delta,r), \qquad \Delta = 4nm - r^2, \tag{4.25}$$
where Δ is known as the *discriminant* of φ, and $C(\Delta,r)$ depends only on $r (\operatorname{mod} 2m)$ [18, 24]. Following [18], the function $\varphi(\tau,z)$ is called:

- A *holomorphic Jacobi form*, or simply a Jacobi form, of weight k and index m if
$$c(n,r) = 0 \text{ unless } \Delta \geq 0;$$

- A *Jacobi cusp form* of weight k and index m if
$$c(n,r) = 0 \text{ unless } \Delta > 0;$$

- A *weak Jacobi form* of weight k and index m if
$$c(n,r) = 0 \text{ unless } n \geq 0;$$

- A *weakly holomorphic Jacobi form* of weight k and index m if
$$c(n,r) = 0 \text{ unless } n \geq n_0 \text{ for some } n \in \mathbb{Z}.$$

We also define a *meromorphic Jacobi form* to be a function $\varphi(\tau,z)$ satisfying properties (4.22) and (4.23), but where φ need only be meromorphic in z and weakly holomorphic in τ [25].

4.4.1 Theta Expansion of Jacobi Forms

If $\varphi(\tau,z)$ is a Jacobi form, then the property (4.23) implies that its Fourier expansion can be written in the form
$$\varphi(\tau,z) = \sum_{\ell\in\mathbb{Z}} q^{\ell^2/4m} h_\ell(\tau) e^{2\pi i\ell z}, \tag{4.26}$$
where $h_\ell(\tau)$ is defined by
$$h_\ell(\tau) = \sum_{\Delta} C(\Delta,\ell) q^{\Delta/4m}, \qquad \ell \in \mathbb{Z}/2m\mathbb{Z}. \tag{4.27}$$

The periodicity property also allows (4.26) to be rewritten as the *theta expansion*
$$\varphi(\tau,z) = \sum_{\ell\in\mathbb{Z}/2m\mathbb{Z}} h_\ell(\tau)\theta_{m,\ell}(\tau,z), \tag{4.28}$$

where $\theta_{m,\ell}(\tau,z)$ is the index-m *generalised theta function*:

$$\theta_{m,\ell}(\tau,z) := \sum_{\substack{r\in\mathbb{Z} \\ r\equiv\ell\,(\mathrm{mod}2m)}} q^{r^2/4m}y^r = \sum_{n\in\mathbb{Z}} q^{(\ell+2mn)^2/4m}y^{\ell+2mn}. \tag{4.29}$$

The coefficients $h_\ell(\tau)$ are modular forms of weight $k=1/2$ [18].

In string theory, Jacobi forms usually arise as elliptic genera of two-dimensional SFCTs with a worldsheet supersymmetry of $(2,2)$ or greater. In Chapter 5, we will show that the elliptic genus of K3 is a Jacobi form of index one and weight zero, so it will be useful to briefly take a closer look at Jacobi forms of index one. If $m=1$, then the Fourier coefficients can simply be written as a function of a single argument, $c(n,r)=C(4n-r^2)$, since $4n-r^2$ fully determines the value of $r(\mathrm{mod}\,2)$ [18].

4.4.2 Multiplicative Lift of a Jacobi Form

Following [26], let $\varphi(\tau,z)$ be a Jacobi form of weight zero and index t, with a Fourier expansion given by (4.24). Assuming that the coefficients $c(n,r)$ are integers, we define the *multiplicative lift*, or Borcherds lift, of φ to be

$$\Phi(\rho,\sigma,v) = e^{2\pi i(\alpha\rho+\beta\sigma+\gamma v)} \prod_{\substack{k,l,j\in\mathbb{Z},k,l\geq 0 \\ j<0 \text{ for } k=l=0}} (1-\exp(2\pi i(k\sigma+l\rho+jv)))^{c(kl,j)}, \tag{4.30}$$

where

$$\alpha = \frac{1}{24}\sum_r c(0,r), \qquad \beta = \frac{1}{2}\sum_{r>0} rc(0,r), \qquad \gamma = \frac{1}{4}\sum_r r^2 c(0,r). \tag{4.31}$$

This is termed a 'lift' because it uses the Fourier coefficients of a function of two variables $\varphi(\tau,z)$ to define a function of three variables $\Phi(\rho,\sigma,v)$. The descriptor 'multiplicative' is used because the construction provides a product representation of $\Phi(\rho,\sigma,v)$; an *additive lift* can be used to find a sum representation, but this will not be of relevance in this work.

We will use this definition in Chapter 6, where we will see that the microstate degeneracy of certain black holes can be computed from the Fourier coefficients of the reciprocal of a multiplicative lift.

4.5 Special Functions

Below we give an introduction to some of the special functions used in this work.

4.5.1 The Dedekind Eta Function

Definition 4.8. Following [22], the *Dedekind eta function* is defined as

$$\eta(\tau) = q^{1/24}\prod_{k=1}^{\infty}(1-q^k), \tag{4.32}$$

where $q=e^{2\pi i\tau}$, and the domain of definition is the upper half-plane $\mathbb{H}=\{\tau \mid \mathrm{Im}(\tau)>0\}$. The eta function satisfies the following transformation properties:

$$\eta(\tau+1) = e^{\pi i/12}\eta(\tau), \qquad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau). \tag{4.33}$$

It can also be expanded in an infinite series as

$$\eta(\tau) = q^{1/24}\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{3n^2-n}{2}}. \tag{4.34}$$

We have already encountered this function in the computation of the partition function of the bosonic string in Chapter 3.

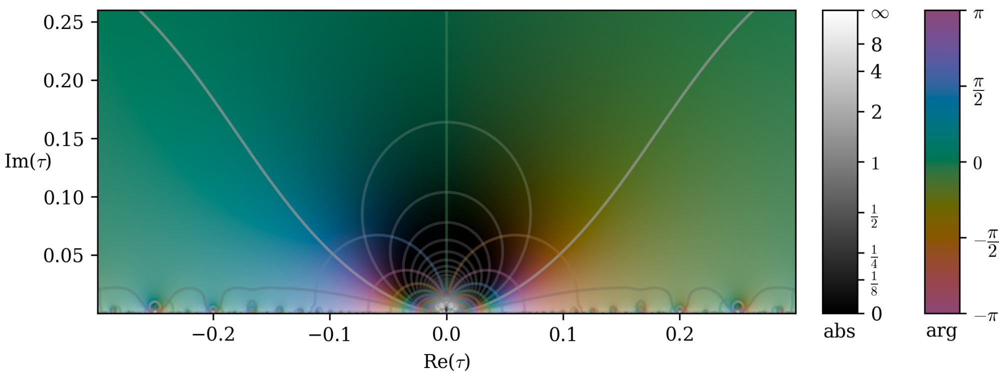

Figure 4.2: A plot of the Dedekind eta function $\eta(\tau)$ in the upper half of the complex plane. The plot uses domain colouring, where the complex phase is represented by the colour and the magnitude is represented by the brightness. It was generated in Python using the `mpmath` [47] and `cplot` [48] libraries.

4.5.2 The Jacobi Theta Functions

Definition 4.9. The *Jacobi theta functions* θ_i are a set of four complex functions, which can be represented by infinite products as follows:

$$\theta_1(\tau, z) = 2q^{1/8}\sin(\pi z)\prod_{m=1}^{\infty}(1-q^m)(1-q^m y)\big(1-q^m y^{-1}\big), \tag{4.35a}$$

$$\theta_2(\tau, z) = 2q^{1/8}\cos(\pi z)\prod_{m=1}^{\infty}(1-q^m)(1+q^m y)\big(1+q^m y^{-1}\big), \tag{4.35b}$$

$$\theta_3(\tau, z) = \prod_{m=1}^{\infty}(1-q^m)\Big(1+q^{m-1/2}y\Big)\Big(1+q^{m-1/2}y^{-1}\Big), \tag{4.35c}$$

$$\theta_4(\tau, z) = \prod_{m=1}^{\infty}(1-q^m)\Big(1-q^{m-1/2}y\Big)\Big(1-q^{m-1/2}y^{-1}\Big), \tag{4.35d}$$

where $q = e^{2\pi i\tau}$ and $y = e^{2\pi i z}$ [27].

The theta functions have the series representations

$$\theta_1(\tau, z) = -\sum_{k\in\mathbb{Z}}(-1)^{k-1/2}q^{(k+1/2)^2/2}y^{k+1/2}, \tag{4.36a}$$

$$\theta_2(\tau, z) = \sum_{k\in\mathbb{Z}}q^{(k+1/2)^2/2}y^{k+1/2}, \tag{4.36b}$$

$$\theta_3(\tau, z) = \sum_{k\in\mathbb{Z}}q^{k^2/2}y^{k}, \tag{4.36c}$$

$$\theta_4(\tau, z) = \sum_{k\in\mathbb{Z}}(-1)^k q^{k^2/2}y^{k}, \tag{4.36d}$$

and satisfy the following transformation rules:

$$\theta_1(\tau+1, z) = e^{\pi i/4}\theta_1(\tau, z), \qquad \theta_1\left(-\frac{1}{\tau}, -\frac{z}{\tau}\right) = -i(-i\tau)^{1/2}e^{\pi i z^2/\tau}\theta_1(\tau, z), \tag{4.37a}$$

$$\theta_2(\tau+1, z) = e^{\pi i/4}\theta_2(\tau, z), \qquad \theta_2\left(-\frac{1}{\tau}, -\frac{z}{\tau}\right) = (-i\tau)^{1/2}e^{\pi i z^2/\tau}\theta_4(\tau, z), \tag{4.37b}$$

$$\theta_3(\tau+1, z) = \theta_4(\tau, z), \qquad \theta_3\left(-\frac{1}{\tau}, -\frac{z}{\tau}\right) = (-i\tau)^{1/2}e^{\pi i z^2/\tau}\theta_3(\tau, z), \tag{4.37c}$$

$$\theta_4(\tau+1, z) = \theta_3(\tau, z), \qquad \theta_4\left(-\frac{1}{\tau}, -\frac{z}{\tau}\right) = (-i\tau)^{1/2}e^{\pi i z^2/\tau}\theta_2(\tau, z). \tag{4.37d}$$

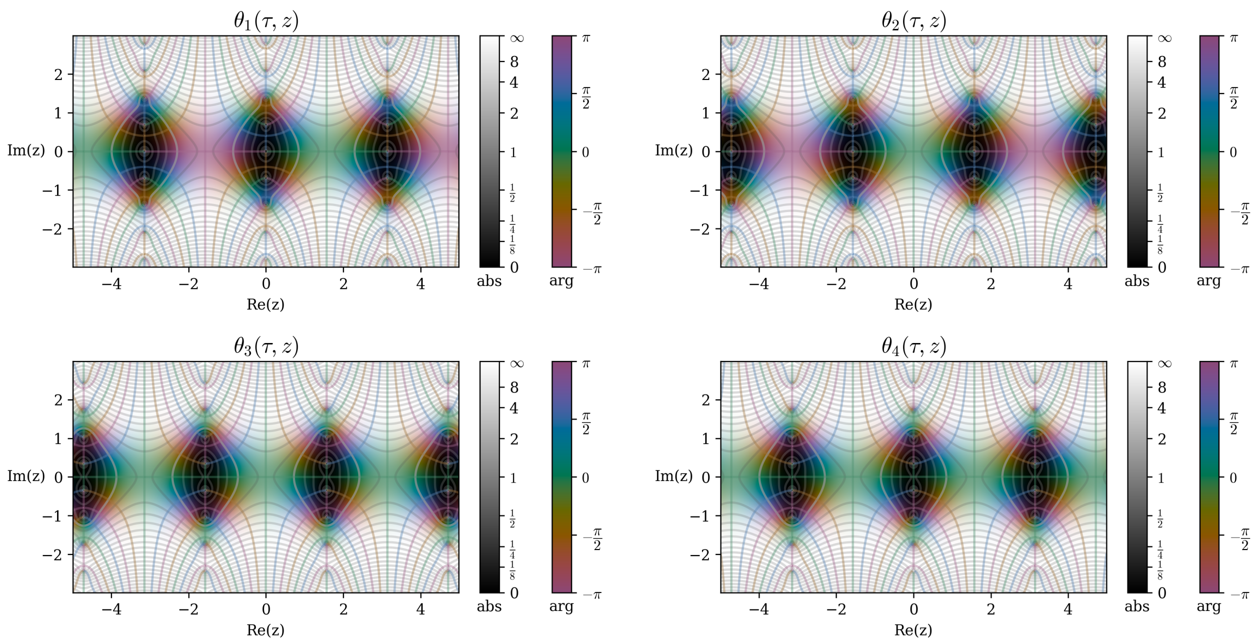

Figure 4.3: Plots of the Jacobi theta functions $\theta_i(\tau, z)$ as functions of z for $q = e^{2\pi i\tau} = 1/4$. The plots were generated in Python using the `mpmath` [47] and `cplot` [48] libraries.

For later convenience, we define

$$A(\tau, z) = \frac{\theta_2(\tau, z)^2}{\theta_2(\tau, 0)^2} + \frac{\theta_3(\tau, z)^2}{\theta_3(\tau, 0)^2} + \frac{\theta_4(\tau, z)^2}{\theta_4(\tau, 0)^2}, \tag{4.38}$$

which is a Jacobi form of index one and weight zero, and

$$B(\tau, z) = \frac{\theta_1(\tau, z)^2}{\eta(\tau)^6}, \tag{4.39}$$

which is a weak Jacobi form of weight -2 and index one [21]. These functions will be used in the computation of the elliptic genus and twisted elliptic genus of K3 in the following chapter.

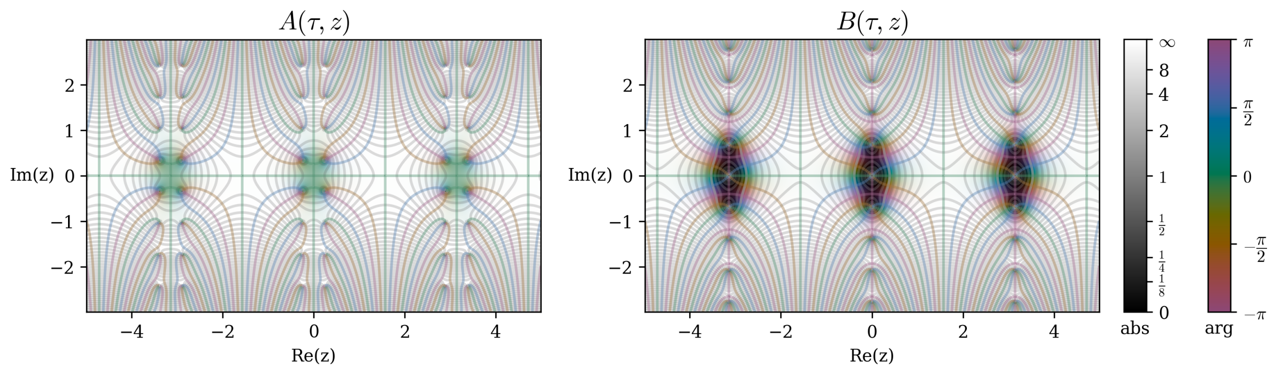

Figure 4.4: Plots of the Jacobi forms $A(\tau, z)$ and $B(\tau, z)$ as functions of z for $q = e^{2\pi i\tau} = 1/4$. The plots were generated in Python using the `mpmath` [47] and `cplot` [48] libraries.

Chapter 5

The Elliptic Genus of K3

The *elliptic genus* of a string theory defined on a given target space is a genus one path integral with certain twisted boundary conditions, which can be expressed as a trace over states. The elliptic genus is a topological invariant of the target space, and the elliptic genus of a supersymmetric model is conformally invariant even if the underlying model is not [28, 29]. In this chapter, we begin by discussing a simpler quantity known as the Witten index, following which we compute the elliptic genus of the manifold K3 by realising it as a SCFT in the orbifold limit $T^4/\mathbb{Z}_2$. We conclude the chapter by generalising the elliptic genus to obtain the twisted elliptic genus of K3.

5.1 The Witten Index

In a supersymmetric theory, the bosonic and fermionic ground states need not be paired, but each *excited* energy eigenstate must have paired bosonic and fermionic states. If the spectrum of the Hamiltonian is deformed continuously while preserving supersymmetry, then the excited states must move in bosonic-fermionic pairs. This means that, even if the spectrum of the Hamiltonian changes, the number of bosonic ground states minus the number of fermionic states is invariant. This invariant is called the *Witten index*, and can be represented as

$$\mathrm{tr}\Big[(-1)^F e^{-\beta H}\Big], \tag{5.1}$$

This is the desired invariant because the pairs of excited states cancel out when weighted with $(-1)^F$, leaving only the ground states; it is hence independent of β. We also remark that for the supersymmetric sigma model, the Witten index is equal to the Euler characteristic $\chi(M)$ of the underlying manifold M, which is a topological invariant [30, 31]. The elliptic genus discussed Section 5.2 is a generalisation of this invariant.

5.1.1 Witten Index of the Closed Fermionic String

As a simple example, following [21] we can easily compute the Witten index of the closed fermionic string:

$$\mathrm{tr}\Big[(-1)^{F_L+F_R} e^{-\beta_L H_L - \beta_R H_R}\Big]. \tag{5.2}$$

For periodic boundary conditions, we have

$$\begin{aligned}
\mathrm{tr}\Big[(-1)^{F_L} e^{-\beta_L H_L}\Big] &= \mathrm{tr}\left((-1)^{F_L} \exp\left(-\beta_L\left(\frac{1}{24}+\sum_{k=1}^{\infty} k\psi_{-k}\psi_k\right)\right)\right) \\
&= e^{-\beta_L/24}\mathrm{tr}\left((-1)^{F_L} \exp\left(-\beta_L \sum_{k=1}^{\infty} k\psi_{-k}\psi_k\right)\right) \\
&= e^{-\beta_L/24}\prod_{k=1}^{\infty}\left(1-e^{-\beta_L k}\right),
\end{aligned}$$

since the occupation number for fermionic states can only be 0 or 1. We notice that this is the same as the partition function from Chapter 3, except that the $(-1)^{F_L}$ operator has changed the sign of the occupied states at each level [32]. We get a similar result for the right-moving modes, so that the Witten index is

$$
\begin{aligned}
\operatorname{tr}\Big[(-1)^{F_L+F_R}e^{-\beta_L H_L-\beta_R H_R}\Big] &= e^{-\beta_L/24}\prod_{k=1}^{\infty}\big(1-e^{-\beta_L k}\big)e^{-\beta_R/24}\prod_{k'=1}^{\infty}\Big(1-e^{-\beta_R k'}\Big)\\
&= e^{2\pi i\tau/24}\prod_{k=1}^{\infty}\big(1-e^{2\pi i\tau k}\big)e^{-2\pi i\bar\tau/24}\prod_{k'=1}^{\infty}\Big(1-e^{-2\pi i\bar\tau k'}\Big)\\
&= |\eta(\tau)|^2.
\end{aligned}
$$

For anti-periodic boundary conditions, as usual only the zero point energy is different, so the Witten index is

$$
\begin{aligned}
\operatorname{tr}\Big[(-1)^{F_L+F_R}e^{-\beta_L H_L-\beta_R H_R}\Big] &= e^{-\beta_L/48}\prod_{k=1}^{\infty}\Big(1-e^{-\beta_L(k-1/2)}\Big)e^{-\beta_R/48}\prod_{k'=1}^{\infty}\Big(1-e^{-\beta_R\left(k'-1/2\right)}\Big)\\
&= e^{2\pi i\tau/48}\prod_{k=1}^{\infty}\Big(1-e^{2\pi i\tau(k-1/2)}\Big)e^{-2\pi i\bar\tau/48}\prod_{k'=1}^{\infty}\Big(1-e^{-2\pi i\bar\tau\left(k'-1/2\right)}\Big).
\end{aligned}
\tag{5.3}
$$

The results obtained here will be useful in the following section when including the fermionic contribution to the elliptic genus of K3.

5.2 Elliptic Genus of K3

Definition 5.1. A K3 surface is a simply-connected compact manifold of two complex dimensions with a trivial canonical bundle, i.e. with a nowhere-vanishing holomorphic two-form. Viewed as real four-manifolds, any two K3 surfaces are diffeomorphic to one another. Along with the four-tori T^4, K3 surfaces comprise all of the compact Calabi–Yau manifolds of two complex dimensions [33].

In the orbifold limit, a K3 surface can be described by $T^4/\mathbb{Z}_N$ for $N = 2, 3, 4, 6$. Here, we explicitly compute the elliptic genus of K3 by realising K3 as a SCFT in the orbifold limit $T^4/\mathbb{Z}_2$, based on [21, 28, 34]. We consider four bosons and four fermions compactified on a torus under the identification of antipodal points. This corresponds to $\mathcal{N} = 4$ supersymmetric string theory, where the physical coordinates consist of four variables of the X^μ type, which are $SU(2)$ singlets, and four Majorana spinors ψ^μ, which are $SU(2)$ doublets [32]. The generator g of the group action of $\mathbb{Z}_2$ can be given as

$$
g : (X^\mu, \psi^\mu) \mapsto (-X^\mu, -\psi^\mu), \tag{5.4}
$$

where $\mu = 1, 2, 3, 4$. The elliptic genus of K3 is then defined [29] to be the following trace, taken over the Ramond-Ramond sector:

$$
Z_{\mathrm{K3}}(\tau, z) = \frac{1}{2}\sum_{\substack{a,b=0\\(a,b)\neq(0,0)}}^{1}\operatorname{tr}_{\mathrm{RR}g^a}\Big[(-1)^{F_{\mathrm{K3}}+\bar F_{\mathrm{K3}}}g^b y^{J_{\mathrm{K3}}}q^{L_0-c/24}\bar q^{\bar L_0-\bar c/24}\Big]. \tag{5.5}
$$

Here, c and $\bar c$ are the central charges of the system, which are both equal to $3d$, where d is the complex dimension of the manifold. For K3, which is of complex dimension two, we have $c = \bar c = 6$ [3, 35]. The operators L_0 and $\bar L_0$ are, respectively, the left and right Hamiltonians with the zero point energy subtracted out, so that $H_L = L_0 - c/24$ and $H_R = \bar L_0 - \bar c/24$. Finally, $J_{\mathrm{K3}} = \pm 1$ for the left-moving fermions, which form a doublet in the $SU(2)$ representation. We have also defined $y = e^{2\pi i z}$.

We first consider the untwisted sector, where $a = 0$. For $(a, b) = (0, 1)$, the right-moving fermionic zero modes contribute a factor of 4, while the left-moving fermionic zero modes contribute a factor of

$$
\Big(y^{1/2} + y^{-1/2}\Big)^2 = 4\cos^2(\pi z).
$$

For the nonzero modes, the factors can be derived in a manner similar to that of the derivation of the partition functions performed in Chapter 3. For each real bosonic coordinate, we have $F_{\mathrm{K3}}, \bar F_{\mathrm{K3}}, J_{\mathrm{K3}} = 0$, and hence the contribution is

$$\begin{aligned}\operatorname{tr}_{\mathrm{RR}}\left[g q^{H_L}\bar{q}^{H_R}\right] &= (q\bar{q})^{-1/24}\operatorname{tr}\Big((-q)^{\sum_{k=1}^{\infty}\alpha_{-k}\alpha_k}\Big)\operatorname{tr}\Big((-\bar{q})^{\sum_{k=1}^{\infty}\tilde{\alpha}_{-k}\tilde{\alpha}_k}\Big)\\ &= |q|^{-1/12}\prod_{k=1}^{\infty}\sum_{n,m=0}^{\infty}\left(-q^k\right)^n\left(-\bar{q}^k\right)^m\\ &= |q|^{-1/12}\prod_{k=1}^{\infty}\frac{1}{1+q^k}\frac{1}{1+\bar{q}^k}.\end{aligned}$$

We thus see that the effect of the insertion of g has been to replace the minus signs in the final expression for the partition function by plus signs. This result implies that the four real bosonic coordinates contribute a factor

$$|q|^{-1/3}\prod_{k=1}^{\infty}\frac{1}{\left(1+q^k\right)^4}\frac{1}{\left(1+\bar{q}^k\right)^4}.$$

For each real fermionic coordinate with $J_{\mathrm{K3}}=+1$, we have

$$\begin{aligned}\operatorname{tr}_{\mathrm{RR}}\left[(-1)^{F_{\mathrm{K3}}+\bar{F}_{\mathrm{K3}}}g y^{J_{\mathrm{K3}}}q^{H_L}\bar{q}^{H_R}\right] &= |q|^{1/12}\operatorname{tr}\Big((-1)^{F_{\mathrm{K3}}}(-q)^{\sum_{k=0}^{\infty}k\psi_{-k}\psi_k}y^{J_{\mathrm{K3}}}\Big)\operatorname{tr}\Big((-1)^{\bar{F}_{\mathrm{K3}}}(-\bar{q})^{\sum_{k=0}^{\infty}k\tilde{\psi}_{-k}\tilde{\psi}_k}\Big)\\ &= |q|^{1/12}\prod_{k=1}^{\infty}\left(1+q^k y\right)\left(1+\bar{q}^k\right),\end{aligned}$$

while for each real fermionic coordinate with $J_{\mathrm{K3}}=-1$, we have

$$\operatorname{tr}_{\mathrm{RR}}\left[(-1)^{F_{\mathrm{K3}}+\bar{F}_{\mathrm{K3}}}g y^{J_{\mathrm{K3}}}q^{H_L}\bar{q}^{H_R}\right] = |q|^{1/12}\prod_{k=1}^{\infty}\left(1+q^k y^{-1}\right)\left(1+\bar{q}^k\right).$$

Since these occur in doublets, the total nonzero-mode fermionic contribution is

$$|q|^{1/3}\prod_{k=1}^{\infty}\left(1+q^k y\right)^2\left(1+q^k y^{-1}\right)^2\left(1+\bar{q}^k\right)^4.$$

Hence the total contribution to the elliptic genus of K3 for this case is

$$\frac{1}{2}\operatorname{tr}_{\mathrm{RR}g^0}\left[(-1)^{F_{\mathrm{K3}}+\bar{F}_{\mathrm{K3}}}g e^{2\pi i z J_{\mathrm{K3}}}q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}\right] = 8\cos^2(\pi z)\prod_{k=1}^{\infty}\frac{\left(1+q^k y\right)^2\left(1+q^k y^{-1}\right)^2}{\left(1+q^k\right)^4} = 8\frac{\theta_2(\tau,z)^2}{\theta_2(\tau,0)^2}, \quad (5.6)$$

where we notice that the terms involving $\bar{q}$, which come from the right-moving bosonic and fermionic contributions, have cancelled.

We now consider the twisted sector, where $a \neq 0$. In this case, $X(\sigma+2\pi)=-X(\sigma)$, i.e. we have the anti-periodic boundary conditions that we considered for the partition functions in Chapter 3, so we must make the modification $k \mapsto k-1/2$. We note that there are $2^4=16$ fixed points under the action of g, so there is an overall factor of 16. For $(a,b)=(1,0)$, i.e. with no insertion of g, we obtain

$$\frac{1}{2}\operatorname{tr}_{\mathrm{RR}g^1}\left[(-1)^{F_{\mathrm{K3}}+\bar{F}_{\mathrm{K3}}}e^{2\pi i z J_{\mathrm{K3}}}q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}\right] = 8\prod_{k=1}^{\infty}\frac{\left(1-q^{k-1/2}y\right)^2\left(1-q^{k-1/2}y^{-1}\right)^2}{\left(1-q^{k-1/2}\right)^4} = 8\frac{\theta_4(\tau,z)^2}{\theta_4(\tau,0)^2}. \quad (5.7)$$

For $(a,b)=(1,1)$, where there is an insertion of g, we have the same result, except that the minus signs are replaced by plus signs due to the action of g, just as we saw in the $(a,b)=(0,1)$ case:

$$\frac{1}{2}\operatorname{tr}_{\mathrm{RR}g^1}\left[(-1)^{F_{\mathrm{K3}}+\bar{F}_{\mathrm{K3}}}g e^{2\pi i z J_{\mathrm{K3}}}q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}\right] = 8\prod_{k=1}^{\infty}\frac{\left(1+q^{k-1/2}y\right)^2\left(1+q^{k-1/2}y^{-1}\right)^2}{\left(1+q^{k-1/2}\right)^4} = 8\frac{\theta_3(\tau,z)^2}{\theta_3(\tau,0)^2}. \quad (5.8)$$

Hence the elliptic genus of K3 is

$$Z_{\mathrm{K3}}(\tau,z) = 8\left(\frac{\theta_2(\tau,z)^2}{\theta_2(\tau,0)^2}+\frac{\theta_3(\tau,z)^2}{\theta_3(\tau,0)^2}+\frac{\theta_4(\tau,z)^2}{\theta_4(\tau,0)^2}\right) = 8A(\tau,z). \quad (5.9)$$

Thus the elliptic genus of K3 is a Jacobi form of index one and weight zero of the modular group. It satisfies the modular transformation property (4.22) as a consequence of the modular invariance of the path integral [18]. K3 can be realised in many different ways, including as a hypersurface in $\mathbb{C}P^4$, in the orbifold limits of $T^4/\mathbb{Z}_N$ for $N = 2, 3, 4, 6$, and numerous others. Since the elliptic genus is a topological invariant, it will remain the same for all of these constructions. We note that the supersymmetry of our theory means that $\bar{q}$ no longer features in the final expression, due to cancellation between the right-moving nonzero bosonic and fermionic modes. This means that $Z_{\mathrm{K3}}(\tau, z)$ is a holomorphic function of τ.

The elliptic genus can be used to find the Witten index of the theory, and hence the Euler characteristic of the underlying manifold K3, by setting $z = 0$ in (5.9). The Euler characteristic of K3 is thus seen to be

$$\chi(\mathrm{K3}) = 24. \tag{5.10}$$

The elliptic genus of K3 can be expanded in a series around $q = 0$ as

$$\begin{aligned} Z_{\mathrm{K3}}(\tau, z) &= 2y^{-1} + 20 + 2y \\ &\quad + \left(20y^{-2} - 128y^{-1} + 216 - 128y + 20y^2\right)q \\ &\quad + \left(2y^{-3} + 216y^{-2} - 1026y^{-1} + 1616 - 1026y + 216y^2 + 2y^3\right)q^2 + \mathcal{O}\left(q^3\right). \end{aligned} \tag{5.11}$$

The double expansion of Z_{K3} in q and y will be used in the computation of the degeneracy of black hole microstates in the following chapter.

5.3 Twisted Elliptic Genus of K3

The elliptic genus of K3 can be generalised to the orbifolds of K3, leading to the definition of the *twisted elliptic genus*. The twisted elliptic genus of K3 by an automorphism g' of order $\mathbb{Z}_N$ is defined as

$$F^{(r,s)}(\tau, z) = \frac{1}{N}\mathrm{tr}_{\mathrm{RR}g'^r}\left[(-1)^{F_{\mathrm{K3}}+\bar{F}_{\mathrm{K3}}} g'^s y^{J_{\mathrm{K3}}} q^{L_0 - c/24}\bar{q}^{\bar{L}_0 - \bar{c}/24}\right], \tag{5.12}$$

where the trace is taken over the Ramond-Ramond sector of the $\mathcal{N} = (4, 4)$ SCFT of K3 with central charge $(6, 6)$, and r, s are integers in the range from zero to $N - 1$. Under the modular group, the twisted elliptic genus transforms as

$$F^{(r,s)}\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = \exp\left(2\pi i \frac{cz^2}{c\tau + d}\right) F^{(cs+ar, ds+br)}(\tau, z). \tag{5.13}$$

For automorphisms g' belonging to the 26 conjugacy classes of the Mathieu group M_{24}, the value of $F^{(0,1)}$ was computed in [36, 37, 38]. In the case where N is prime, the transformation property (5.13) is sufficient for all of the sectors of the twisted elliptic genus to be found by relating them to $F^{(0,1)}$. However, for composite values of N, the various sectors break up into sub-orbits under the action of modular transformations and cannot be related to $F^{(0,1)}$. In these cases, the twisted elliptic genus can be determined using its correspondence with the cycle shape of M_{24}. These computations are performed in [21, 39].

For $N = 1$, the function $F^{(0,0)}(\tau, z)$ is simply the elliptic genus (5.9). In this work, our primary interest is in the cases where g' belongs to the conjugacy classes pA of M_{24}, with $p = 2, 3, 5, 7$, which correspond to the $\mathbb{Z}_p$ CHL models. For these cases with $N = 2$, the functions were computed explicitly in [34] as

$$F^{(0,0)}(\tau, z) = 4A(\tau, z) = 4\left(\frac{\theta_2(\tau, z)^2}{\theta_2(\tau, 0)^2} + \frac{\theta_3(\tau, z)^2}{\theta_3(\tau, 0)^2} + \frac{\theta_4(\tau, z)^2}{\theta_4(\tau, 0)^2}\right), \tag{5.14a}$$

$$F^{(0,1)}(\tau, z) = 4\frac{\theta_2(\tau, z)^2}{\theta_2(\tau, 0)^2}, \qquad F^{(1,0)}(\tau, z) = 4\frac{\theta_4(\tau, z)^2}{\theta_4(\tau, 0)^2}, \qquad F^{(1,1)}(\tau, z) = 4\frac{\theta_3(\tau, z)^2}{\theta_3(\tau, 0)^2}. \tag{5.14b}$$

This result will be used to compute the number of black hole microstates on the $2A$ orbifold of $\mathrm{K3} \times T^2$ in Chapter 6.

The same paper [34] found the following general forms of $F^{(r,s)}$ for the pA classes for $N = 2, 3, 5, 7$:

$$F^{(0,0)}(\tau, z) = \frac{8}{N}A(\tau, z), \tag{5.15a}$$

$$F^{(0,s)}(\tau, z) = \frac{8}{N(N+1)}A(\tau, z) - \frac{2}{N+1}B(\tau, z)\mathcal{E}_N(\tau) \text{ for } 1 \le s \le N - 1, \tag{5.15b}$$

$$F^{(r,rk)}(\tau, z) = \frac{8}{N(N+1)}A(\tau, z) - \frac{2}{N(N+1)}B(\tau, z)\mathcal{E}_N\left(\frac{\tau + k}{N}\right) \text{ for } 1 \le r, k \le N - 1, \tag{5.15c}$$

where $A(\tau, z), B(\tau, z)$ and $\mathcal{E}_N(\tau)$ are given by (4.38), (4.39) and (4.19), respectively.

Chapter 6

Black Hole Entropy

As discussed in the Introduction, the semiclassical theory of black holes predicts that a black hole is a radiating black body, with a finite temperature known as the Hawking temperature. This means that black holes are thermodynamic systems which possess an entropy given by the Bekenstein–Hawking formula. A major aim in quantum gravity research is to understand this entropy as a logarithm of the number of quantum microstates d associated with the black hole, so that the Bekenstein–Hawking entropy can be explained as a statistical entropy according to Boltzmann's formula $S_{\text{stat}} = \ln d$ [40].

This aim has already been achieved for the special class of extremal black holes in string theory. These black holes have zero temperature, and therefore emit no Hawking radiation. The analysis is simplest for a subset of extremal black holes known as *BPS black holes*, which are invariant under certain supersymmetry transformations. For a wide class of extremal BPS black holes in string theory, in the limit of large black hole size, the statistical entropy has been found to be equal to the Bekenstein–Hawking entropy, providing a microscopic explanation for the macroscopic thermodynamics [40, 41].

In this chapter, we compute the degeneracy of black hole microstates for quarter-BPS dyons in $\mathcal{N} = 4$ supersymmetric type II string theory compactified on $\mathrm{K3} \times T^2$, which is dual to heterotic string theory compactified on the six-torus T^6. We also count the degeneracies on the $2A$ orbifold of $\mathrm{K3} \times T^2$, which corresponds to the $\mathbb{Z}_2$ CHL model. Our results confirm those of [17].

6.1 Degeneracy of Black Hole Microstates

In classical general relativity, black holes are spherically symmetric and hence possess no angular momentum. However, in supersymmetric string theory, the black hole partially breaks the supersymmetry of the theory and thus supersymmetric excitations around the black hole include a set of fermionic zero modes. When these zero modes are quantised, angular momentum is imparted to the black hole. Since these fermionic zero modes lie outside the event horizon, supersymmetry ensures that the horizon remains spherically symmetric. It has been argued that a spherically symmetric horizon implies that the black hole represents a microcanonical ensemble of states, all of which carry zero angular momentum. In that case, only the fermion zero modes produced by the broken supersymmetry contribute to the angular momentum of the black hole. This leads to the conclusion that the helicity trace index of the black hole,

$$B_{2n} = \frac{1}{(2n)!} \mathrm{tr}\Big[(-1)^F \big(2J^3\big)^{2n}\Big], \tag{6.1}$$

is equal to $(-1)^n d$, where d is the degeneracy of black hole microstates. In the formula above, F is the fermion number operator, J^3 is the *helicity*, which is third component of the angular momentum of the black hole in its rest frame, $4n$ is the number of supersymmetries broken by the black hole, and the trace is taken over all states carrying a certain set of electric and magnetic charges [17].

Our focus will be on quarter-BPS black holes in $\mathcal{N} = 4$ supersymmetric string theories, which are states in which a quarter of the supersymmetry is preserved. This means that these black holes break 12 of the 16 total supersymmetries, so that the relevant helicity trace index is B_6. Since we consider a BPS state, the spectrum of the black hole is 'topological', meaning it does not change as the moduli are varied, except for jumps at certain walls in the moduli space [18], known as *walls of marginal stability*. We will carry out our computations for the *attractor chamber* of the moduli space bounded by the walls of marginal stability. In this chamber, only single-centred black holes contribute to the index B_6. We are considering BPS states, which in a given moduli space are fully determined by their charges [4]. We will represent a microstate of the black hole by $d(Q, P)$, where Q is the electric charge vector of the black hole, and P is the magnetic charge vector; both of these vectors are 28-dimensional. We therefore have

$$d(Q,P) = -B_6. \tag{6.2}$$

The helicity trace index B_6 has been computed for a wide range of $\mathcal{N}=4$ supersymmetric string theories. For theories represented as $\mathbb{Z}_N$ CHL models, obtained by taking the $\mathbb{Z}_N$ quotient of heterotic string theory compactified on T^6, one obtains

$$B_6 = \frac{1}{N}(-1)^{Q\cdot P}\int_{\mathcal{C}} e^{-\pi i\left(N\rho Q^2+\sigma P^2/N+2vQ\cdot P\right)}\frac{1}{\tilde{\Phi}(\rho,\sigma,v)}\,d\rho\,d\sigma\,dv, \tag{6.3}$$

where $\tilde{\Phi}(\rho,\sigma,v)$ is a known Siegel modular form, and $\mathcal{C}$ is a three-real-dimensional subspace of the three-complex-dimensional space labelled by $(\rho,\sigma,v)=(\rho_1+i\rho_2,\sigma_1+i\sigma_2,v_1+iv_2)$, with

$$\begin{aligned} &0\le\rho_1\le 1, \quad 0\le\sigma_1\le 1, \quad 0\le v_1\le 1,\\ &\rho_2 = M_1, \qquad \sigma_2 = M_2, \qquad v_2=-M_3, \end{aligned} \tag{6.4}$$

where M_1,M_2,M_3 are fixed large positive numbers, determined by the domain in which B_6 is to be evaluated. For a full derivation of this result, see [17].

6.2 Microstate Degeneracy for Type II String Theory on K3 $\times\, T^2$

We now compute the degeneracy of black hole microstates for heterotic string theory compactified on T^6, which corresponds to setting $N=1$ in formula (6.3). As discussed in the Introduction, string duality means that this compactification is equivalent to type II string theory compactified on K3 $\times\, T^2$ [8, 9]. This in turn means that the relevant Siegel modular form is the multiplicative lift of the elliptic genus of K3 [21]. This is given by the Igusa cusp form of weight ten, which has the product representation

$$\tilde{\Phi}_{10}(\rho,\sigma,v) = e^{2\pi i(\rho+\sigma+v)}\prod_{\substack{k,l,j\in\mathbb{Z},k,l\ge 0\\ j<0 \text{ for } k=l=0}}\left(1-e^{2\pi i(k\sigma+l\rho+jv)}\right)^{c\left(4kl-j^2\right)}, \tag{6.5}$$

where $c(4kl-j^2)$ are defined by the Fourier coefficients of the elliptic genus of K3. As we observed in Chapter 5, the elliptic genus of K3 can be expanded in a double expansion in q and y as

$$Z_{\mathrm{K3}}(\tau,z) = \sum_{n,j\in\mathbb{Z}} c(4n-j^2)q^n y^j, \tag{6.6}$$

so that the relevant expansion coefficients are $c(4n-j^2)$ with $n=kl$. The degeneracy of black hole microstates can then be found according to the formula

$$d(Q,P) = (-1)^{Q\cdot P+1}\int_{\mathcal{C}} e^{-\pi i\left(\rho Q^2+\sigma P^2+2vQ\cdot P\right)}\frac{1}{\tilde{\Phi}(\rho,\sigma,v)}\,d\rho\,d\sigma\,dv\,. \tag{6.7}$$

We will be considering the case

$$M_1,M_2\gg 0, \quad M_3\ll 0, \quad |M_3|\ll M_1,M_2. \tag{6.8}$$

This corresponds to the expansion of $1/\tilde{\Phi}_{10}$ first in powers of $e^{2\pi i\rho}, e^{2\pi i\sigma}$, and then in powers of $e^{-2\pi iv}$. Hence if we make the Fourier series expansion

$$\frac{1}{\tilde{\Phi}_{10}(\rho,\sigma,v)} = \sum_{m,n,r\in\mathbb{Z}} g(m,n,r)e^{2\pi i(m\rho+n\sigma+rv)}, \tag{6.9}$$

where $g(m,n,r)$ are the Fourier coefficients, we can find the degeneracy of black hole microstates as

$$\begin{aligned} d(Q,P) &= (-1)^{Q\cdot P+1}\sum_{m,n,r} g(m,n,r)\int_{\mathcal{C}} e^{2\pi i(m\rho+n\sigma+rv)}e^{-\pi i\left(\rho Q^2+\sigma P^2+2vQ\cdot P\right)}\,d\rho\,d\sigma\,dv\\ &= (-1)^{Q\cdot P+1}\sum_{m,n,r} g(m,n,r)\int_{\mathcal{C}} e^{2\pi i\left(\left(m-Q^2/2\right)\rho+\left(n-P^2/2\right)\sigma+(r-Q\cdot P)v\right)}\,d\rho\,d\sigma\,dv\\ &= (-1)^{Q\cdot P+1}g\left(\frac{Q^2}{2},\frac{P^2}{2},Q\cdot P\right). \end{aligned} \tag{6.10}$$

Thus the microstate degeneracy is given by the Fourier coefficients of $1/\tilde{\Phi}_{10}(\rho,\sigma,v)$ [17].

To compute the degeneracy $d(Q,P)$ for given values of $\left(Q^2,P^2,Q\cdot P\right)$, the product form of $1/\tilde{\Phi}_{10}$ was evaluated in Mathematica and its Fourier series coefficients were determined. The resulting values were multiplied by the corresponding sign $(-1)^{Q\cdot P+1}$ to find $d(Q,P)$. The results are shown in Table 6.1, and fully agree with those obtained in [17].

Table 6.1: The degeneracy $d(Q,P)$ of black hole microstates for type II string theory compactified on $\mathrm{K3}\times T^2$, corresponding to heterotic string theory compactified on T^6. The results are in agreement with those obtained in [17]. A few additional results not included in [17] are also shown.

$(Q^2,P^2)\backslash Q\cdot P$	-2	0	1	2	3	4	5
$(2,2)$	–209304	50064	25353	648	327	0	0
$(2,4)$	–2023536	1127472	561576	50064	8376	–648	0
$(4,4)$	–16620544	32861184	18458000	3859456	561576	12800	3272
$(2,6)$	–15493728	16491600	8533821	1127472	130329	–15600	972
$(4,6)$	–53249700	632078672	392427528	110910300	18458000	1127472	85176
$(6,6)$	2857656828	16193130552	11232685725	4173501828	920577636	110910300	8533821
$(6,8)$	91631080464	315614079072	233641003920	100673013264	26563753008	4173501828	392427528

6.3 Microstate Degeneracy for the $2A$ Orbifold of $\mathrm{K3}\times T^2$

Just as in the case of the elliptic genus, the twisted elliptic genus of K3 defined by (5.12) can be expanded in a double expansion in q and y as

$$F^{(r,s)}(\tau,z)=\sum_{j\in\mathbb{Z},n\in\mathbb{Z}/N} c^{(r,s)}(4n-j^2)q^n y^j, \tag{6.11}$$

where $c^{(r,s)}(4n-j^2)$ are the corresponding expansion coefficients. In analogy with (6.5), a Siegel modular form associated with this is

$$\tilde{\Phi}(\rho,\sigma,v)=e^{2\pi i\left(\tilde{\alpha}\rho+\tilde{\beta}\sigma+v\right)}\prod_{r=0}^{N-1}\prod_{\substack{k\in\mathbb{Z}+\frac{r}{N},l,j\in\mathbb{Z}\\ k,l\geq 0,j<0\text{ for }k=l=0}}\left(1-e^{2\pi i(k\sigma+l\rho+jv)}\right)^{\sum_{s=0}^{N-1}e^{2\pi isl/N}c^{(r,s)}\left(4kl-j^2\right)}. \tag{6.12}$$

Here, we will only be concerned with the CHL models corresponding to the pA conjugacy classes of K3 for $p=1,2,3,5,7$ (using the notation of [42]), for which

$$\tilde{\alpha}=1,\quad \tilde{\beta}=\frac{1}{N}. \tag{6.13}$$

Using formula (6.3), we see that the degeneracy of microstates is given by

$$d(Q,P)=(-1)^{Q\cdot P+1}g\left(\frac{N}{2}Q^2,\frac{1}{2N}P^2,Q\cdot P\right), \tag{6.14}$$

where $g(m,n,r)$ are the Fourier coefficients of $1/\tilde{\Phi}$. We will consider the simplest nontrivial case in this class, namely the $2A$ orbifold of $\mathrm{K3}\times T^2$, which corresponds to the $\mathbb{Z}_2$ CHL model. In this case, we have $N=2$, so that the relevant function is the weight-six Siegel modular form

$$\tilde{\Phi}_6(\rho,\sigma,v)=e^{2\pi i(\rho+\sigma/N+v)}\prod_{r=0}^{N-1}\prod_{\substack{k\in\mathbb{Z}+\frac{r}{N},l,j\in\mathbb{Z}\\ k,l\geq 0,j<0\text{ for }k=l=0}}\left(1-e^{2\pi i(k\sigma+l\rho+jv)}\right)^{\sum_{s=0}^{N-1}e^{2\pi isl/N}c^{(r,s)}\left(4kl-j^2\right)}, \tag{6.15}$$

and the functions $F^{(r,s)}$ are given by (5.14). Using the same procedure described in the previous section, the degeneracies $d(Q,P)$ were evaluated in Mathematica. The results are shown in Table 6.2, and again are in agreement with those reported in [17].

Table 6.2: The degeneracy $d(Q,P)$ of black hole microstates for the $\mathbb{Z}_2$ CHL model, corresponding to the $2A$ orbifold of K3 $\times$ T^2. Again, the results fully agree with those obtained in [17], and we have computed some additional values not given in that paper.

$(Q^2,P^2)\backslash Q\cdot P$	-2	0	1	2	3	4	5
$(1,2)$	–5410	2164	360	–2	0	0	0
$(1,4)$	–26464	18944	4352	160	0	0	0
$(2,4)$	–124160	198144	67008	6912	64	0	0
$(1,6)$	–114524	125860	36024	2164	52	0	0
$(2,6)$	–473088	1580672	671744	101376	4352	–16	0
$(2,8)$	–1235968	10586880	5189696	992256	67008	1152	0
$(3,6)$	–779104	15219528	7997655	1738664	149226	2164	3
$(3,8)$	9971200	122401792	72857944	19737088	2409072	101376	424

Chapter 7

Wall-Crossing and Mock Modular Forms

In Chapter 6, we computed the microstate degeneracy for single-centred black holes by working in the attractor chamber [17]. However, one may encounter cases where the macroscopic black hole configurations are no longer localised at a single point: as well as containing the single-centred black hole of interest, they may also include several multi-centred black holes. This gives rise to some computational difficulties, since upon crossing walls of marginal stability in the moduli space, the multi-centred configurations may split into their single-centred constituents, resulting in a jump in the index B_6. This phenomenon is known as *wall-crossing* [18].

The difficulties associated with counting the microstates, and hence computing the statistical entropy, of multi-centred black hole configurations means that it is useful to have a way of isolating the single-centred components at the microscopic level. In this chapter, we discuss results from [18] which show how the Fourier–Jacobi coefficients ψ_m of $1/\tilde{\Phi}_{10}$ can be decomposed into a finite part ψ_m^{F} and a polar part ψ_m^{P}, where the finite part corresponds to a region of the moduli space which contains the single-centred black hole configurations. Since we will find that *mock modular forms* and *mock Jacobi forms* play a significant role in this process, we begin the chapter by defining these functions and discussing some of their properties.

7.1 Mock Modular Forms and Mock Jacobi Forms

In this section we follow [18].

7.1.1 Mock Modular Forms

Definition 7.1. A *pure mock modular form* of weight $k \in \mathbb{Z}/2$ is defined to be the first member of a pair (h, g), where

1. h is a holomorphic function on $\mathbb{H}$ with at most exponential growth at all cusps;

2. The function $g(\tau)$, called the *shadow* of h, is a holomorphic modular form of weight $2-k$;

3. The sum $\hat{h} = h + g^*$, called the *completion* of h, transforms like a holomorphic modular form of weight k, i.e. $\hat{h}(\tau)/\theta(\tau)^{2k}$ is invariant under $\tau \to \gamma\tau$ for all $\tau \in \mathbb{H}$ and all γ in some congruence subgroup of $SL(2,\mathbb{Z})$.

Definition 7.2. A *mixed mock modular form* of weight $k|\ell \in \mathbb{Z}/2$ is defined to be a function $h(\tau)$, such that

1. h is a holomorphic function on $\mathbb{H}$ with at most polynomial growth at all cusps;

2. h has a *shadow* defined by $\sum_j f_j \bar{g}_j$, where $f_j(\tau)$ and $g_j(\tau)$ are holomorphic modular forms of weight ℓ and $2-k+\ell$ respectively;

3. h has a *completion* of the form $\hat{h} = h + \sum_j f_j \bar{g}_j$, which transforms like a holomorphic modular form of weight k.

7.1.2 Mock Jacobi Forms

Definition 7.3. A *pure mock Jacobi form* of weight k and index m is defined in [18] as a holomorphic function φ on $\mathbb{H} \times \mathbb{C}$ that satisfies the elliptic transformation property (4.23) and therefore has a Fourier expansion as in (4.24), and a theta expansion as in (4.28), but for which the coefficients $h_\ell(\tau)$ are now *mock* modular forms of weight $k - 1/2$, rather than modular forms, and the modular transformation property (4.22) is not satisfied by φ itself but instead by the *completed* function

$$\hat{\varphi}(\tau, z) = \sum_{\ell \in \mathbb{Z}/2m\mathbb{Z}} \hat{h}_\ell(\tau)\theta_{m.\ell}(\tau, z). \tag{7.1}$$

Definition 7.4. A *mixed mock Jacobi form* is defined in [18] to satisfy the same properties as a pure mock Jacobi form, except that the coefficients $h_\ell(\tau)$ of the theta expansions are now *mixed* mock modular forms.

7.1.3 Decomposition of Meromorphic Jacobi Forms

We now state a result from [18], which states that the pole-free part of a meromorphic Jacobi form can be separated from the remaining polar part.

Theorem 7.1. *If $\varphi(\tau, z)$ is a meromorphic Jacobi form, then it has the decomposition*

$$\varphi(\tau, z) = \varphi^{\mathrm{F}}(\tau, z) + \varphi^{\mathrm{P}}(\tau, z), \tag{7.2}$$

where φ^{F} is the finite part of φ, which has no poles, and φ^{P} is the polar part of φ, which has the same pole structure (in z) as φ.

7.2 Application to Black Holes

7.2.1 Fourier–Jacobi Coefficients on $\mathrm{K3} \times T^2$

As we saw in Chapter 6, the black hole microstate degeneracies for $\mathcal{N} = 4$ type II string theory compactified on $\mathrm{K3} \times T^2$ are given, up to a sign, by the Fourier coefficients of the reciprocal of the weight-ten Igusa cusp form, $\tilde{\Phi}_{10}$. The degeneracy for a given set of charges (in the attractor chamber) is therefore given by the contour integral (6.7). Although $1/\tilde{\Phi}_{10}$ has the same functional form throughout the moduli space, the microstate degeneracy is moduli-dependent because of the moduli-dependence of the contour of integration and the pole structure of $1/\tilde{\Phi}_{10}$.

Following [18], the reciprocal of $\tilde{\Phi}_{10}$ can be written as the expansion

$$\frac{1}{\tilde{\Phi}_{10}(q, p, y)} = \sum_{m=-1}^{\infty} \psi_m(\tau, z)p^m, \qquad q = e^{2\pi i\tau},\ p = e^{2\pi i\sigma},\ y = e^{2\pi i z}, \tag{7.3}$$

where $\eta(\tau)^{24}\psi_m(\tau, z)$ are meromorphic Jacobi forms of weight two with a double pole at $z = 0$ and no other poles, up to a translation by the period lattice. Note that we have made the notational changes $\sigma \to \tau, v \to z$ to conform with the standard presentation of Jacobi forms.

Using the Fourier expansion of $1/\tilde{\Phi}_{10}$ in Mathematica, we find that

$$\begin{aligned}\eta(\tau)^{24}\psi_{-1}(\tau, z) = \big(y^{-1} + 2y^{-2} + 3y^{-3} + \mathcal{O}(y^{-4})\big) + 2q \\ + q^2\big(3y + 3y^{-1} + \mathcal{O}(y^{-4})\big) + q^3\big(4y^2 + 4y^{-2} + \mathcal{O}(y^{-4})\big) + \mathcal{O}(q^4).\end{aligned} \tag{7.4}$$

By performing the expansion of $-1/B(\tau, z)$ as well, we confirm the result from [18] that

$$\eta(\tau)^{24}\psi_{-1}(\tau, z) = -\frac{1}{B(\tau, z)}. \tag{7.5a}$$

Similar expansions for subsequent values of m allow us to verify that

$$\eta(\tau)^{24}\psi_0(\tau, z) = -\frac{8A(\tau, z)}{B(\tau, z)}, \tag{7.5b}$$

$$\eta(\tau)^{24}\psi_1(\tau, z) = -\frac{36A(\tau, z)^2}{B(\tau, z)} - \frac{3}{4}E_4(q)B(\tau, z). \tag{7.5c}$$

Invoking Theorem 7.1, we note that we can decompose ψ_m into

$$\psi_m(\tau, z) = \psi_m^{\mathrm{F}}(\tau, z) + \psi_m^{\mathrm{P}}(\tau, z), \tag{7.6}$$

where $\psi^{\mathrm{F}}(\tau, z)$ has no poles, and $\psi^{\mathrm{P}}(\tau, z)$ has the same pole structure in z as $\psi_m(\tau, z)$. We note that $\psi_m(\tau, z)$ is the counting function of all asymptotic states from both single and multi-centred black hole configuration, and the contribution from $\psi^{\mathrm{F}}(\tau, z)$ comes from an area of the moduli space which encloses the region in which the positivity conjecture of [17] is satisfied. Thus the finite part ψ_m^{F} covers the region of the moduli space where the single-centred black holes are found.

It was shown in [18] that for $m > 0$, the decomposition of ψ_m for $1/\tilde{\Phi}_{10}$ has the finite part

$$\psi_m^{\mathrm{F}}(\tau, z) = \sum_{\ell \,(\mathrm{mod} 2m)} f_{m,\ell}^*(\tau)\theta_{m,\ell}(\tau, z), \tag{7.7}$$

where

$$f_{m,\ell}^*(\tau) = e^{-i\pi\ell^2\tau/2m} \int_{-\ell\tau/2m}^{-\ell\tau/2m+1} \psi_m(\tau, z)e^{-2\pi i\ell z} dz. \tag{7.8}$$

This finite part ψ_m^{F} is independent of the choice of contour in the moduli space, and is found to be a mock Jacobi form. Hence the microstate degeneracies of black holes corresponding to ψ_m^{F} with magnetic charge invariant m are Fourier coefficients of a mock Jacobi form of index m. It was also shown in the paper [18] that the polar part of ψ_m for $m > 0$ is

$$\psi_m^{\mathrm{P}}(\tau, z) = \frac{p_{24}(m+1)}{\eta(\tau)^{24}}\mathcal{A}_{2,m}(\tau, z), \tag{7.9}$$

where $p_{24}(m+1)$ is the coefficient of q^m in $1/\eta(\tau)^{24}$, and $\mathcal{A}_{2,m}$ is the Appell–Lerch sum

$$\mathcal{A}_{2,m}(\tau, z) = \sum_{s\in\mathbb{Z}} \frac{q^{ms^2+s}y^{2ms+1}}{(1-q^s y)^2}. \tag{7.10}$$

Note that this sum diverges for $m = -1$.

7.2.2 Fourier–Jacobi Coefficients on the $2A$ Orbifold of $\mathrm{K3} \times T^2$

As discussed in Chapter 6, the black hole microstate degeneracies on the $2A$ orbifold of $\mathrm{K3} \times T^2$ are given by the Fourier coefficients of $1/\tilde{\Phi}_6$. As in the previous section, we can write this as the following Fourier–Jacobi expansion:

$$\frac{1}{\tilde{\Phi}_6(q, p, y)} = \sum_{m=-1}^{\infty} \psi_m(\tau, z)p^{m/2}. \tag{7.11}$$

We note that $[\eta(\tau)\eta(2\tau)]^8\psi_m(\tau, z)$ are meromorphic Jacobi forms of weight two. Performing the Fourier expansion in Mathematica, we find that the Jacobi form corresponding to $m = -1$ can be expanded as

$$\begin{aligned}[\eta(\tau)\eta(2\tau)]^8\psi_{-1}(\tau, z) = &\left(y^{-1} + 2y^{-2} + 3y^{-3} + 4y^{-4} + 5y^{-5} + \mathcal{O}(y^{-6})\right)\\ &+ 2q^2 + 2q^4\left(3y + 3y^{-1} + \mathcal{O}(y^{-6})\right) + \mathcal{O}(q^6).\end{aligned} \tag{7.12}$$

By computing the expansion of $-1/B(2\tau, z)$ as well, we find that this result confirms the finding from [21, 43] that

$$[\eta(\tau)\eta(2\tau)]^8\psi_{-1}(\tau, z) = -\frac{1}{B(2\tau, z)}. \tag{7.13a}$$

We notice that this is identical to the $m = -1$ coefficient (7.5a) for the unorbifolded case, upon making the replacement $\tau \to 2\tau$. This means that the Appell-Lerch sum corresponds to (7.10) under the replacement $q \to q^2$, so the sum diverges in this case.

By similarly expanding the next two lowest-m coefficients, we also verify the following results from [43]:

$$[\eta(\tau)\eta(2\tau)]^8\psi_0(\tau, z) = -\frac{2F^{(0,1)}(2\tau, z)}{B(2\tau, z)}, \tag{7.13b}$$

$$[\eta(\tau)\eta(2\tau)]^8\psi_1(\tau, z) = -\frac{F^{(0,0)}(\tau, z) + F^{(0,1)}(\tau + 1/2, z) + 2\left[F^{(1,0)}(\tau, z)\right]^2 + F^{(1,0)}(4\tau, z^2)}{B(2\tau, z)}. \tag{7.13c}$$

For the case $m = 0$, we can use (5.15) to write out the twisted elliptic genus explicitly, giving

$$[\eta(\tau)\eta(2\tau)]^8 \psi_0(\tau, z) = -\frac{2F^{(0,1)}(2\tau, z)}{B(2\tau, z)} = \frac{8}{3}\frac{A(2\tau, z)}{B(2\tau, z)} + \frac{2}{3}\mathcal{E}_2(\tau). \tag{7.14}$$

It was shown in [43] that this meromorphic Jacobi form admits the decomposition

$$[\eta(\tau)\eta(2\tau)]^8 \psi_0^{\mathrm{F}}(\tau, z) = \frac{2}{3}(E_2(2\tau) - \mathcal{E}_2(\tau)), \tag{7.15a}$$

$$[\eta(\tau)\eta(2\tau)]^8 \psi_0^{\mathrm{P}}(\tau, z) = 8 \sum_{n\in\mathbb{Z}} \frac{q^{2n} y}{(1 - q^{2n} y)^2}. \tag{7.15b}$$

The decomposition for $m = 1$ was also successfully performed in [43]. These results indicate that the Fourier–Jacobi decomposition of meromorphic Jacobi forms introduced for K3 $\times\, T^2$ in [18] can be successfully extended to the $2A$ orbifold of K3 $\times\, T^2$, and in fact to higher orbifolds of K3 $\times\, T^2$ as well. This provides a useful method for separating the part of the Fourier–Jacobi coefficients that covers region of the moduli space where the single-centred black holes may be found.

Chapter 8

Conclusions

In this report, we have seen how string theory provides a well-defined method of computing the entropy of a certain class of extremal supersymmetric black holes as a logarithm of the number of microstates. The agreement of the entropy obtained via this method with the prediction of the Bekenstein–Hawking formula is a major success of string theory.

We began by providing a brief overview of string theory and classical general relativity in Chapter 1. Following this, in Chapter 2 we discussed some of the basic principles of quantum field theory, including canonical quantisation and renormalisation.

In Chapter 3, we computed the partition functions for the free closed bosonic and fermionic strings, and in the process encountered the Dedekind eta function. This was our first indication of the importance of modular forms in string theory. This led us to the formal definition of modular forms in Chapter 4, where we also defined other mathematical objects of importance in this area, including Siegel modular forms, Eisenstein series and Jacobi forms. In Chapter 5, we computed the elliptic genus of K3 by realising K3 as a SCFT in the orbifold limit $T^4/\mathbb{Z}_2$, and found that it took the form of a Jacobi form of index one and weight zero. We also briefly introduced the twisted elliptic genus of K3.

The main portion of the work was contained in Chapter 6, where we demonstrated the method of microstate-counting for the case of quarter-BPS extremal black holes in $\mathcal{N} = 4$ supersymmetric type II string theory compactified on $\mathrm{K3} \times T^2$. This involved using Mathematica to determine the Fourier coefficients of the reciprocal of the Igusa cusp form of weight ten, which corresponded (up to a sign) to the degeneracies of black holes of given electric and magnetic charge vectors. In addition, we performed the same analysis for the $2A$ orbifold of $\mathrm{K3} \times T^2$, where the relevant Siegel modular form was of weight six. The results confirmed those obtained previously by [17, 21].

Finally, in Chapter 7, we discussed how the Fourier–Jacobi coefficients of the reciprocal of the generating function are meromorphic Jacobi forms. We discussed results from [18] that allow such functions to be decomposed into a finite part and a polar part, where the finite part corresponds to a region of the moduli space that contains the single-centred black hole configurations. We concluded by giving a brief description of how these results were partially extended to the orbifolds of $\mathrm{K3} \times T^2$ in [43].

There are still many open questions in this field, and several possible future directions of research might be interesting to pursue. One area of future research would entail the extension of the Fourier–Jacobi decomposition of the meromorphic Jacobi forms ψ_m discussed in Chapter 7 to all positive values of m for the $2A$ orbifold of $\mathrm{K3} \times T^2$, as this has so far only been achieved in [43] for the cases $m = 0$ and $m = 1$.

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