Modularity, Supersymmetric Black Holes and The Ramanujan Tau Function

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Abstract

In this report, we consider modular forms, Jacobi forms and Siegel modular forms, and the application of these in computing black hole degeneracies for extremal BPS black holes. We give the definition and some properties of Eisenstein series, the η function, the modular discriminant and the Jacobi Theta functions and discuss their applications to the thermodynamics of string theoretic black holes. We compute the Fourier coefficients of the Igusa cusp form of weight 10 which correspond to the degeneracies of these extremal BPS black holes. We also apply numerical methods to consider an extension of Lehmer's conjecture on the Ramanujan's τ function and look at the surprising behaviour of integers $k : \tau(k) > 2k^{11/2}$.

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Chapter 1

Introduction

In the study of the entropy of supersymmetric black holes, one encounters complex-valued functions called modular forms. The study of modular forms began with their connection to elliptic functions and has since seen surprising applications in various fields such as number theory (modular forms were an integral part of Andrew Wiles' proof of Fermat's Last Theorem [1]), group theory, and the interest of this report, string theoretic black holes.

Modular forms appear in the study of integer partitions as generating functions. More precisely if $p(n)$ denotes the number of ways a positive integer n can be written as a sum of positive integers (where we do not take into account the ordering of the summands), then $p(n)$ is given implicitly by [2]

$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{j=1}^{\infty} \frac{1}{1-q^j}.$$

The product $\prod_{j=1}^{\infty} \frac{1}{1-q^j}$ is called the generating function of $p(n)$ and while it is not explicitly a modular form it is related to the eta function, $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n)$ where $q = e^{2\pi i \tau}$ which is intimately related to the study of modular forms.

The theory of integer partitions plays an important role in string theory. For example, when considering a single closed bosonic string compactified on a torus, integer partitions are used to determine the number of possible microstates of the string at a particular energy level [3]. Furthermore, the Elliptic Genus of K3 derived in [3] is in terms of Jacobi Forms (a concept related to modular forms) called the Jacobi Theta Functions. Modular forms appear in the study of the thermodynamics of string-theoretic black holes, as the Fourier series of the inverse of a Siegel Modular Form determine the degeneracies of extremal black holes in the theory [4].

At the end of this report, we perform some numerical analysis of the Ramanujan τ function, defines implicitly as the Fourier coefficients of η^{24}

$$\eta^{24}(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n)\, q^n.$$

The properties of this τ function have greatly interested mathematicians over the last 100 years since its definition by Ramanujan in 1916 [5]. Ramanujan made several conjectures regarding this τ function, which have all been subsequently proven. Another conjecture about the τ function, which is as of now unproven, was given in 1947 by Lehmer [6]. He conjectured that $\tau(n)$ was non-vanishing for all n. Through computational methods, we consider some variations of this lemma and also consider other properties of the τ function inspired by the conjectures of Ramanujan.

In chapter 2 we give some first examples and results of modular forms. We discuss modular transformations, Eisenstein series, and the Dedekind η function. We define Jacobi forms and consider the Jacboi-Theta functions in chapter 3. The connection between Siegel modular forms, a generalisation of modular forms, and black holes in string theory is discussed in chapter 4. Numerical analysis of the Ramanujan Tau function is given in chapter 5. Finally, in chapter 6 we conclude our results.

Chapter 2

Modular Forms

2.1 The Modular Group and Modular Forms

Definition 1. *The full modular group Γ is the group of transformations on the upper half plane, $\mathbb{H} := \{\tau \in \mathbb{C} : \operatorname{Im}(\tau) > 0\}$, of the form*

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}$$

where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$, with the binary operation defined by composition.

A transformation in this group

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}$$

can naturally be associated with a matrix in $\mathrm{SL}_2(\mathbb{Z})$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

As such we write $A\tau = \frac{a\tau+b}{c\tau+d}$. We also note that for $A \in \mathrm{SL}_2(\mathbb{Z})$ we have $A\tau = -A\tau$. Therefore, we instead see transformations as elements of the group $\mathrm{PSL}_2(\mathbb{Z}) := \mathrm{SL}_2(\mathbb{Z}) / \{\pm I\}$.

We can also consider congruence subgroups of Γ.

Definition 2. *For $N \in \mathbb{N}$ we define the congruence subgroup of level N as*

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \mod N \right\}$$

Note that we have $\Gamma = \Gamma_0(1)$.

2.1.1 Modular Forms

Definition 3. *A modular form $f : \mathbb{H} \to \mathbb{C}$ of weight k and level N is a holomorphic function on $\mathbb{H}$ and at $i\infty$ such that*

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and f is bounded as $\tau \to i\infty$.

A cusp form is a modular form such that $\lim_{\tau \to i\infty} f(\tau) = 0$.

In general, when we say a modular form, we refer to a modular form of level 1 unless stated otherwise.

Theorem 2.1.1. *Let f be a modular form of weight k and $\tau \in \mathbb{H}$. Then $f(\tau + 1) = f(\tau)$ and $f\left(-\frac{1}{\tau}\right) = \tau^k f(\tau)$*

Proof. This follows clearly by considering the transformation rule for modular forms and the matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

□

Theorem 2.1.2. *A modular form f has a Fourier series of the form $\sum_{n=0}^{\infty} a_n q^n$ where $q = e^{2\pi i \tau}$.*

Proof. Consider the map $Q : \mathbb{H} \to A, Q(\tau) = e^{2\pi i \tau}$ where $A = \{z \in \mathbb{C} : 0 < |z| < 1\}$ is the punctured open unit disk. From the modular form f define $g : A \to \mathbb{C}$ by $g(q) = f(\tau)$. Note that such a g is well defined as if $q = q'$ where $q = e^{2\pi i \tau}$ and $q' = e^{2\pi i \tau'}$ we have τ and τ' differ by an integer. So by periodicity of f, $f(\tau) = f(\tau')$.

The function g is analytic on all of A and as such has Laurent expansion of the form $\sum_{n=-\infty}^{\infty} a_n q^n$ for $|q| < 1$. So we have $f(\tau) = \sum_{n=-\infty}^{\infty} a_n q^n$. As f is bounded as $\tau \to i\infty$ or equivalently $q \to 0$ we see $a_n = 0, \forall n < 0$. □

Definition 4. *We denote by $M_k(\Gamma_0(N))$ the space of all modular forms of weight k and level N. The ring of modular forms of level N, $M(\Gamma_0(N))$ is defined as $M(\Gamma_0(N)) = \bigoplus_k M_k(\Gamma_0(N))$.*

In [7] Apostol shows that for each k the space $M_k(\Gamma)$ is finite-dimensional. He also shows the following:

Theorem 2.1.3. *For k odd, $k < 0$, or $k = 2$ we have that $M_k(\Gamma) = \{0\}$.*

It is important to note that this result generally does not hold for modular forms of level $N > 1$.

We now state an important result from Sturum [8, 9] that states that modular forms are uniquely determined by their "first few" Fourier coefficients.

Theorem 2.1.4. *Let $f, g \in M_k(\Gamma_0(N))$ with Fourier expansions $f(\tau) = \sum_{n=0}^{\infty} a_n q^n$ and $g(\tau) = \sum_{n=0}^{\infty} b_n q^n$ such that $a_n = b_n$ $\forall n \leq \frac{km}{12}$, where $m = [\Gamma : \Gamma_0(N)]$, is the index of $\Gamma_0(N)$ in Γ. Then $f = g$.*

Note that for prime N we have the index $m = [\Gamma : \Gamma_0(N)] = N$.

This result is very useful as it shows that at a particlar weight and level "few" modaular forms exist.

2.2 Eisenstein Series

Definition 5. *For even $r > 2$ and $\tau \in \mathbb{H}$ we define the Eisenstein Series of order r as*

$$G_r(\tau) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m+n\tau)^r}.$$

One can also similarly define the Eisenstein series G_2, but this is not a modular form. However, it is an example of a mock modular form (as the function $E_2(\tau) - \frac{3}{2\,\mathrm{Im}(\tau)}$ transforms as a modular form of weight 2).

As an Eisenstein series is a modular form, it has a Fourier series which we now consider. First, we state the Lipschitz summation formula from [10] that we will need.

Lemma 2.2.1. *For $k > 2$ and $\tau \in \mathbb{H}$ we have*

$$\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{r=1}^{\infty} r^{k-1} e^{2\pi i z r}.$$

Theorem 2.2.2. *The Eisenstein Series G_4 and G_6 have the Fourier series*

$$G_4(\tau) = 2\zeta(4)\left(1 + 240\sum_{n=1}^{\infty}\sigma_3(n)\,q^n\right)$$
$$G_6(\tau) = 2\zeta(6)\left(1 - 504\sum_{n=1}^{\infty}\sigma_5(n)\,q^n\right)$$

where $q = e^{2\pi i\tau}$ and σ_α is the divisor function defined as $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$.

Proof. We prove only the case of G_4 as the proof for G_6 follows similarly. Begin by separating the sum for G_4 into the cases of $m = 0$ and $m \neq 0$ so we have

$$\begin{aligned} G_4(\tau) &= \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{n^4} + \sum_{\substack{m,n \in \mathbb{Z} \\ m \neq 0}} \frac{1}{(m\tau + n)^4} \\ &= 2\zeta(4) + 2 \sum_{m=1}^{\infty} \sum_{n \in \mathbb{Z}} \frac{1}{(m\tau + n)^4} \end{aligned}$$

Using Lipschitz summation formula we get

$$\begin{aligned} G_4(\tau) &= 2\zeta(2) + 2 \sum_{m=1}^{\infty} \left(\frac{(-2\pi i)^4}{(4-1)!} \sum_{r=1}^{\infty} r^{k-1} q^{mr} \right) \\ &= 2\zeta(2) + 2\frac{16\pi^4}{3!} \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} r^{k-1} q^{mr} \\ &= 2\zeta(2) \left(1 + 240 \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} r^{k-1} q^{mr} \right) \end{aligned}$$

Finally noting that the double series $\sum_{m=1}^{\infty} \sum_{r=1}^{\infty} r^{k-1} q^{mr}$ is an example of a Lambert series and can be written as $\sum_{n=1}^{\infty} \sigma_3(n) q^n$ which was shown in [11] by Hardy completes the proof. $\square$

It is useful to further define $E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n$ and $E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n$ as the Fourier coefficients for these E_4, E_6 are all integers and the constant term is 1.

The importance of Eisenstein series is demonstrated in the following by Zaiger in [10].

Theorem 2.2.3. *A given modular form of weight k can be expressed as a $\mathbb{C}$-linear combination of terms of the form $E_4^{\alpha} E_6^{\beta}$ where $4\alpha + 6\beta = k$.*

This is extremely useful for computations of modular forms as the Fourier Series for Eisenstein Series can be easily implimented into code.

2.3 Dedekind η Function and the Discriminant

We now define the Dedekind η function and also the discriminant function. The η function plays an important role in bosonic string theory and the Fourier expansion of the discriminant function is of central importance in chapter 5.

Definition 6. *The Dedekind eta function is defined on $\mathbb{H}$ by*

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$$

where $q = e^{2\pi i \tau}$.

Lemma 2.3.1. *For $\tau \in \mathbb{H}$ we have*

$$\eta(\tau+1) = e^{(2\pi i)/24}\eta(\tau).$$

Proof.

$$\begin{aligned}\eta(\tau+1) &= q^{1/24}e^{(2\pi i)/24}\prod_{n=1}^{\infty}\left(1-q^n e^{2\pi i n}\right)\\ &= e^{(2\pi i)/24}\eta(\tau)\end{aligned}$$

□

From this result, it follows that η^{24} is periodic with period 1.

For the transformation of η under $\tau \to \frac{-1}{\tau}$ we have the following from Apostol [7]:

Theorem 2.3.2. *For $\tau \in \mathbb{H}$*

$$\eta\left(\frac{-1}{\tau}\right) = (-i\tau)^{1/2}\,\eta(\tau)$$

2.3.1 The Discriminant Function

Definition 7. *For $\tau \in \mathbb{H}$ we define the discriminant function, Δ, as*

$$\Delta(\tau) = \eta^{24}(\tau).$$

Theorem 2.3.3. *The Discriminant Function Δ can be expressed as*

$$\Delta(\tau) = \frac{1}{1728}\left(E_4^3(\tau) - E_6^2(\tau)\right).$$

Corollary 2.3.3.1. *Δ is a cusp form of weight 12.*

Chapter 3

Jacobi Forms

3.1 Jacobi Forms

Jacobi forms are functions of 2 complex variables that transform similarly to modular forms. Our main consideration is given to the Jacobi Theta forms which are examples of Jacobi forms.

Definition 8. *A Jacobi Form of weight k and index m is a holomorphic function $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ satisfying:*

1. $\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{\nu}{c\tau+d}\right) = (c\tau+d)^k \exp\left(\frac{2\pi imc\nu}{c\tau+d}\right)\phi(\tau,\nu)$ *for all* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.
2. $\phi(\tau, \nu + \lambda\tau + \mu) = \exp\left(-2\pi im\left(\lambda^2\tau + 2\lambda\nu\right)\right)\phi(\tau,\nu)$ *for all* $\lambda, \mu \in \mathbb{Z}$.
3. ϕ *has a Fourier expansion of the form*

$$\phi(\tau,\nu) = \sum_{n=0}^{\infty} \sum_{r^2 \leq 4mn} c(n,r)\, q^n z^r$$

where $q = e^{2\pi i\tau}$ *and* $z = e^{2\pi i\nu}$.

3.2 Jacobi Theta Functions

Jacobi Theta functions are an example of these Jacobi forms and appear in string theory in the elliptic genus of K3.

Definition 9. *For $\tau \in \mathbb{H}$ and $\nu \in \mathbb{C}$ we define the Generalised Theta Function as*

$$\theta_{a,b}(\tau,\nu) = \sum_{\mathbb{Z}} \exp\left(\pi i\tau(n+a)^2\right)\exp\left(2\pi i(n+a)(\nu+b)\right)$$

From this, we define the Jacobi Theta functions as:

$$\begin{aligned}
\theta_1(\tau,\nu) &= -\theta_{\frac{1}{2},\frac{1}{2}}(\tau,\nu) \\
&= -i\sum_{\mathbb{Z}}(-1)^n q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} e^{\pi i(2n+1)\nu} \\
\theta_2(\tau,\nu) &= \theta_{\frac{1}{2},0}(\tau,\nu) \\
&= \sum_{\mathbb{Z}} q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} e^{\pi i(2n+1)\nu} \\
\theta_3(\tau,\nu) &= \theta_{0,0}(\tau,\nu) \\
&= \sum_{\mathbb{Z}} q^{\frac{1}{2}n^2} e^{2\pi i n\nu} \\
\theta_4(\tau,\nu) &= \theta_{0,\frac{1}{2}}(\tau,\nu) \\
&= \sum_{\mathbb{Z}}(-1)^n q^{\frac{1}{2}n^2} e^{2\pi i n\nu}
\end{aligned}$$

3.2.1 Modular Transformations of Theta Functions

We now consider modular transformations on the above theta functions. Firstly we consider the transformation $\tau \to \tau + 1$.

Lemma 3.2.1.

$$\begin{aligned}
\theta_1(\tau+1,\nu) &= e^{\frac{\pi i}{4}}\theta_1(\tau,\nu) \\
\theta_2(\tau+1,\nu) &= e^{\frac{\pi i}{4}}\theta_2(\tau,\nu) \\
\theta_3(\tau+1,\nu) &= \theta_4(\tau,\nu) \\
\theta_4(\tau+1,\nu) &= \theta_3(\tau,\nu)
\end{aligned}$$

Proof. Here we only show the identities for $\theta_1(\tau+1,\nu)$ and $\theta_3(\tau+1,\nu)$ as the proof of the other two identities follows similarly.

$$\begin{aligned}
\theta_1(\tau+1,\nu) &= -i\sum_{n\in\mathbb{Z}} e^{\pi i(\tau+1)\left(n+\frac{1}{2}\right)^2} e^{\pi i(2n+1)\nu} \\
&= e^{\pi i\left(n^2+n+\frac{1}{4}\right)}\theta_1(\tau,\nu)
\end{aligned}$$

Noting that $n^2+n \equiv 0 \mod 2\ \forall n \in \mathbb{Z}$ the we have the desired result.

Now for θ_3 we have:

$$\begin{aligned}\theta_3(\tau+1,\nu) &= \sum_{n\in\mathbb{Z}} e^{\pi i(\tau+1)n^2} e^{2\pi i n\nu} \\ &= \sum_{n\in\mathbb{Z}} e^{\pi i n^2} q^{\frac{1}{2}n^2} e^{2\pi i n\nu} \\ &= \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{1}{2}n^2} e^{2\pi i n\nu} \\ &= \theta_4(\tau,\nu)\end{aligned}$$

□

We now consider the transformations $\tau \to -\frac{1}{\tau}$ and $\nu \to \frac{\nu}{\tau}$. To do this we will need Poisson's Summation Formula:

Theorem 3.2.2. *Let $f : \mathbb{R} \to \mathbb{C}$ be a Schwartz function. Then*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$$

where $\hat{f}(s) = \int_{\mathbb{R}} e^{-2\pi i s x} f(x)\, dx$.

Theorem 3.2.3. *For $\tau \in \mathbb{H}$ and $\nu \in \mathbb{C}$ we have*

$$\theta_3\left(\frac{-1}{\tau}, \frac{-\nu}{\tau}\right) = \sqrt{-i\tau}\, e^{\pi i \frac{\nu^2}{\tau}} \theta_3(\tau,\nu)$$

Proof. Define $f_{\tau,\nu} : \mathbb{R} \to \mathbb{C}$ by $f_{\tau,\nu}(x) = e^{\pi i x^2 \tau + 2\pi i x \nu}$. So we have that

$$\theta_3(\tau,\nu) = \sum_{n\in\mathbb{Z}} f_{\tau,\nu}(n).$$

We also have that

$$\hat{f}_{\tau,\nu}(s) = \frac{1}{\sqrt{-i\tau}} e^{\pi i (s-\nu)^2 \left(\frac{-1}{\tau}\right)}.$$

So applying Poisson's Summation Formula we see that

$$\begin{aligned}
\theta_3(\tau,\nu) &= \sum_{n\in\mathbb{Z}} f_{\tau,\nu}(n) \\
&= \sum_{n\in\mathbb{Z}} \hat{f}_{\tau,\nu}(n) \\
&= \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{-i\tau}} e^{\pi i (n-\nu)^2\left(\frac{-1}{\tau}\right)} \\
&= \frac{1}{\sqrt{-i\tau}} \sum_{n\in\mathbb{Z}} e^{\pi i n^2\left(\frac{-1}{\tau}\right)} e^{2\pi i n\left(\frac{-\nu}{\tau}\right)} e^{\pi i\left(\frac{-\nu^2}{\tau}\right)} \\
&= \frac{1}{\sqrt{-i\tau}} e^{\pi i\left(\frac{-\nu^2}{\tau}\right)} \theta_3\left(\frac{-1}{\tau}, \frac{-\nu}{\tau}\right).
\end{aligned}$$

Rearranging this completes the proof. □

Theorem 3.2.4. *For $\tau \in \mathbb{H}$ and $\nu \in \mathbb{C}$ we have*

$$\theta_2\left(\frac{-1}{\tau}, \frac{-\nu}{\tau}\right) = \sqrt{-i\tau} e^{\pi i \frac{\nu^2}{\tau}} \theta_4(\tau,\nu)$$

3.2.2 Product Representations of Theta Functions

We first state the Jacobi Triple Product Identity which we make use of in this section.

Theorem 3.2.5. *For $x, y \in \mathbb{C}$ where $|x| < 1$ and $y \neq 0$ we have*

$$\prod_{m=1}^{\infty} \left(1 - x^{2m}\right)\left(1 + x^{2m-1}y^2\right)\left(1 + \frac{x^{2m-1}}{y^2}\right) = \sum_{n\in\mathbb{Z}} x^{n^2} y^{2n}$$

Theorem 3.2.6. *For $\tau \in \mathbb{H}$ and $\nu \in \mathbb{C}$ we have the following product representations for the Theta functions:*

$$\begin{aligned}
\theta_1(\tau,\nu) &= q^{\frac{1}{8}} z^{\frac{1}{2}} \prod_{m=1}^{\infty} (1 - q^m)(1 - q^m z)\left(1 - \frac{q^{m-1}}{z}\right) \\
\theta_2(\tau,\nu) &= q^{\frac{1}{8}} z^{\frac{1}{2}} \prod_{m=1}^{\infty} (1 - q^m)(1 + q^m z)\left(1 + \frac{q^{m-1}}{z}\right) \\
\theta_3(\tau,\nu) &= \prod_{m=1}^{\infty} (1 - q^m)\left(1 + q^{\frac{2m-1}{2}} z\right)\left(1 + \frac{q^{\frac{2m-1}{2}}}{z}\right) \\
\theta_4(\tau,\nu) &= \prod_{m=1}^{\infty} (1 - q^m)\left(1 - q^{\frac{2m-1}{2}} z\right)\left(1 - \frac{q^{\frac{2m-1}{2}}}{z}\right)
\end{aligned}$$

where $q = e^{2\pi i\tau}$ and $z = e^{2\pi i\nu}$.

Proof. We only prove the identities for θ_3 and θ_2 as the remaining two follow similarly. First, consider the expansion for θ_3. Writing $z = e^{2\pi i\nu}$ and using the Jacobi Triple Product Identity we get

$$\begin{aligned}\theta_3(\tau,\nu) &= \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2} z^n \\ &= \prod_{m=1}^{\infty}(1-q^m)\left(1+q^{\frac{2m-1}{2}}z\right)\left(1+\frac{q^{\frac{2m-1}{2}}}{z}\right)\end{aligned}$$

Now for θ_2 we have

$$\begin{aligned}\theta_2(\tau,\nu) &= \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} e^{\pi i(2n+1)\nu} \\ &= q^{\frac{1}{8}} z^{\frac{1}{2}} \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}\left(n^2+n\right)} e^{2\pi i n\nu} \\ &= q^{\frac{1}{8}} z^{\frac{1}{2}} \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2}\left(zq^{\frac{1}{2}}\right)^n\end{aligned}$$

So applying the Jacobi Triple Product identity as before we get

$$\begin{aligned}\theta_2(\tau,\nu) = q^{\frac{1}{8}} z^{\frac{1}{2}} &\prod_{m=1}^{\infty}(1-q^m)\left(1+q^{\frac{2m-1}{2}}q^{\frac{1}{2}}z\right)\left(1+\frac{q^{\frac{2m-1}{2}}}{q^{\frac{1}{2}}z}\right) \\ = q^{\frac{1}{8}} z^{\frac{1}{2}} &\prod_{m=1}^{\infty}(1-q^m)(1+q^m z)\left(1+\frac{q^{m-1}}{z}\right)\end{aligned}$$

□

Chapter 4

Siegel Modular Forms and Black Holes

4.1 Siegel Modular Forms

Siegel modular forms generalise the concept of modular forms by extending the upper half-plane to the more general Siegel upper half-space. They play an important role in the computation of black hole degeneracies, which are determined by the Fourier coefficients of the inverse of a particular Siegel modular form. In this section, we discuss Siegel modular forms and use them to calculate these degeneracies.

Definition 10. *The Siegel upper half-space, $\mathbb{H}_2$, is defined as*

$$\mathbb{H}_2 = \left\{ \Omega = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}) : \mathrm{Im}(\tau), \mathrm{Im}(\sigma), \det(\mathrm{Im}(\Omega)) > 0 \right\}.$$

where here $\mathrm{GL}_2(\mathbb{C})$ denotes the ring of 2×2 matrices with complex entries.

We now define the symplectic group, Γ_2, which will act on $\mathbb{H}_2$, as the full modular group acted on the $\mathbb{H}$.

Definition 11. *The symplectic group, Γ_2, is given by*

$$\Gamma_2 = \left\{ \gamma \in \mathrm{GL}_4(\mathbb{Z}) : \gamma^T J \gamma = J \right\}$$

where $J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$ *and* I_2 *is the* 2×2 *identity matrix.*

For a matrix $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_2$ *we define the action of* Γ_2 *on* $\Omega \in \mathbb{H}_2$ *by* $(A\Omega + B)(C\Omega + D)^{-1}$. *It can be shown that* $C\Omega + D$ *is non-singular and that this operation is well-defined.*

Now we can define a Siegel modular form as:

Definition 12. *A holomorphic function* $F : \mathbb{H}_2 \to \mathbb{C}$ *is a Siegel modular form of weight* k *if*

$$F\left((A\Omega + B)(C\Omega + D)^{-1}\right) = \det(C\Omega + D)^k F(\Omega)$$

for $\Omega \in \mathbb{H}_2$ *and* $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_2$.

4.2 Black Holes and Modular Forms

We now apply modular forms and related concepts to the study of black hole entropy. Here we will determine the Fourier Coefficients of the inverse of the Igusa Cusp form of weight 10, a Siegel Modular form, which corresponds to black hole degeneracies of extremal supersymmetric black holes.

We begin by giving the formula for the Elliptic Genus of K3, Z_{K3}, from [3], the Fourier coefficients of which we will use in the calculation of the black hole degeneracies.

$$Z_{K3}(\tau, z) = 8\left(\frac{\theta_2^2(\tau, z)}{\theta_2^2(\tau, 0)} + \frac{\theta_3^2(\tau, z)}{\theta_3^2(\tau, 0)} + \frac{\theta_4^2(\tau, z)}{\theta_4^2(\tau, 0)}\right)$$

This is an example of a Jacobi form of weight 0 and index 1 discussed in chapter 3.

Taking $q = e^{2\pi i\tau}$ and $y = e^{2\pi i z}$, Z_{K3} has an Fourier expansion

$$Z_{K3}(\tau, z) = \sum_{n, j \in \mathbb{Z}} c\left(4n - j^2\right) q^n y^j$$

Using the Fourier coefficients of Z_{K3} one can define the Igusa cusp form of weight 10, Φ_{10} in [4] as

$$\Phi_{10}(\rho, \sigma, v) = e^{2\pi i(\rho + \sigma + v)} \prod_{\substack{k, l, j \in \mathbb{Z} \\ k, l \geq 0, j < 0 \text{ for } k = l = 0}} \left(1 - e^{2\pi i(k\sigma + l\rho + jv)}\right)^{c\left(4kl - j^2\right)}$$

Writing $x = e^{2\pi i\sigma}$, $y = e^{2\pi i\rho}$ and $w = e^{2\pi i v}$ we have

$$\Phi_{10}(\rho,\sigma,v) = xyw \prod_{\substack{k,l,j\in\mathbb{Z} \\ k,l\geq 0, j<0 \text{ for } k=l=0}} \left(1 - x^k y^l w^j\right)^{c\left(4kl-j^2\right)}$$

Expanding the inverse of this Φ_{10} we get

$$\frac{1}{\Phi_{10}(\rho,\sigma,v)} = \sum_{m,n,p} g(m,n,p)\, x^m y^n w^p$$

In [4] Sen shows the degeneracy, $d(P,Q)$, associated with electric charge Q and magnetic charge P is given by the integral

$$d(P,Q) = (-1)^{Q\cdot P+1} \int_{\mathcal{C}} e^{-\pi i\left(\rho Q^2+\sigma P^2+2vQ\cdot P\right)} \frac{1}{\Phi_{10}(\rho,\sigma,v)}\, \mathrm{d}\rho\, \mathrm{d}\sigma\, \mathrm{d}v$$

Using the Fourier expansion for $\frac{1}{\phi_{10}(\rho,\sigma,v)}$ we get

$$\begin{aligned} d(P,Q) &= (-1)^{Q\cdot P+1} \sum_{m,n,p} g(m,n,p) \int_{\mathcal{C}} e^{-\pi i\left(\rho Q^2+\sigma P^2+2vQ\cdot P\right)} e^{2\pi i(\sigma m+\rho n+vp)}\, \mathrm{d}\rho\, \mathrm{d}\sigma\, \mathrm{d}v \\ &= (-1)^{Q\cdot P+1} g\left(\frac{Q^2}{2}, \frac{P^2}{2}, Q\cdot P\right) \end{aligned}$$

Evaluating these degeneracies in Table 4.1 we have:

$(Q^2,P^2)\backslash P\cdot Q$	-2	-1	0	1	2	3	4
(2,2)	-209304	130329	50064	25353	648	327	0
(2,4)	-2023536	1598376	1127472	561576	50064	8376	-648
(4,4)	-16620544	28698000	32861184	18458000	3859456	561576	12800
(2,6)	-15493728	16844421	16491600	8533821	1127472	130329	-15600
(4,6)	-53249700	474507528	632078672	392427528	110910300	18458000	1127472
(6,6)	2857656828	11890608225	16193130552	11232685725	4173501828	920577636	110910300

Table 4.1: Values of $d(P,Q)$

These results match the results from [4].

Chapter 5

Numerical Analysis of the Ramanujan τ Function

Finally, we performed some numerical experiments regarding the Ramanujan τ function and related functions. We have considered a more general form of Lehmer's conjecture regarding Δ^m and we determined the proportions of positive and negative Fourier coefficients of Δ^m up to 10^6 coefficients. We also found surprising behaviour of integers k for which $\tau(k)$ exceeds a bound given for prime k.

Definition 13. *For $n \in \mathbb{N}$ the Ramanujan Tau Function $\tau(n)$ is defined as the n^{th} Fourier coefficient of the Discriminant Function Δ.*
That is $\tau(n)$ satisfies

$$\sum_{n=1}^{\infty} \tau(n) q^n = \Delta(\tau) \coloneqq q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

where $q = e^{2\pi i \tau}$.

Many arithmetical properties of the $\tau(n)$ were observed but not proven by Ramanujan [5]:

1. τ is a multiplicative function.
2. $\tau\left(p^{r+1}\right) = \tau(p)\tau(p^r) - p^{11}\tau\left(p^{r-1}\right)$ for p a prime and $r > 0$
3. $|\tau(p)| \leq 2p^{11/2}$ for all primes p.

The first two of these conjectures were proven in 1917 by Mordell [12], while the third was only proven in 1974 by Delinge in which he proved a more general bound for all k: $|\tau(k)| \leq \sigma_0(k) k^{11/2}$ [13, 14].

5.1 An Extension of Lehmer's Conjecture

Another famous, and as yet unproven, conjecture on the Ramanujan Tau Function from Lehmer [6] is that $\tau(n) \neq 0$ for any n.

In this report, we extend this question further and consider zeros in the Fourier series of functions of the form $\Delta^n(\tau)$ for integers n. Using computational methods we check that these Fourier coefficients are non-vanishing and give the following:

Definition 14. *For $m, n \in \mathbb{N}$ we define $\tau_m(n)$ as the n^{th} Fourier coefficient of Δ^m. That is $\tau_m(n)$ satisfies*

$$\sum_{n=1}^{\infty} \tau_m(n) q^n = \Delta(\tau)^m$$

We also prove the following results numerically.

Theorem 5.1.1. *For $1 \leq m \leq 20$ $\tau_m(n)$ is non-vanishing for $n \leq 10^6$.*

Theorem 5.1.2. *For $21 \leq m \leq 100$ $\tau_m(n)$ is non-vanishing for $n \leq 10^5$.*

5.2 Proportion of Positive Values of τ_m

We also calculated the proportions of positive and negative values of $\tau_m(n)$ for $n < 10^6$ and $m \leq 20$ and have included these in Table 5.1.

m	Proportion of Positive Values of $\tau_m(n)$
1	0.500047
2	0.500052
3	0.499943
4	0.500247
5	0.50002
6	0.500083
7	0.500246
8	0.499866
9	0.499926
10	0.500318
11	0.50033
12	0.499666
13	0.500543
14	0.500263
15	0.499835
16	0.499852
17	0.499602
18	0.500494
19	0.499288
20	0.500149

Table 5.1: The proportion of positive Fourier coefficients of Δ^m for $m \leq 20$

From this table we see that for $1 \le m \le 20$ approximately half of the Fourier coefficients of Δ^m are positive.

5.3 Bounds on τ

As mentioned above, Ramanujan conjectured that for p a prime $|\tau(p)| \le 2p^{11/2}$ [5]. While a more general bound exists $\forall k$, we considered integer values of k for which $|\tau(k)|$ exceeds this bound $2k^{11/2}$ and found surprising behaviour of these k.

Definition 15. *We use $k(n)$ to denote the n^{th} k such that $|\tau(k)| > 2k^{11/2}$*

Using numerical methods, all 83054 such values of $k(n) < 10^7$, and give the first 20 in Table 5.2.

n	$k(n)$
1	799
2	1751
3	2987
4	3149
5	3713
6	4841
7	5321
8	6157
9	6283
10	6901
11	7003
12	7849
13	8137
14	8143
15	8777
16	8789
17	9071
18	9077
19	10523
20	10609

Table 5.2: The first 20 values of $k(n)$

We first considered the proportion of the values of $k(n)$ and found they were not distributed evenly i.e. $\frac{1}{2}$ divisible by 2, $\frac{1}{3}$ divisible by 3 etc. For example, only 12.6% of the values of $k(n)$ were divisible by 2 while 18.1% were divisible by 47, the highest proportion we found for a prime. The proportion of $k(n)$ divisible by the first 20 primes is given in Table 5.3.

p	Proportion of $k(n) : p \mid k(n)$	$\frac{1}{p}$
2	0.126243167	0.5
3	0.039384015	0.333333333
5	0.027765069	0.2
7	0.00569509	0.142857143
11	0.067401931	0.090909091
13	0.000854866	0.076923077
17	0.121691911	0.058823529
19	0.034917042	0.052631579
23	0.000734462	0.043478261
29	0.066486864	0.034482759
31	0.000108363	0.032258065
37	2.41×10^{-5}	0.027027027
41	1.20×10^{-5}	0.024390244
43	8.43×10^{-5}	0.023255814
47	0.181400053	0.021276596
53	1.20×10^{-5}	0.018867925
59	0.007501144	0.016949153
61	0.015495942	0.016393443
67	0.068786573	0.014925373
71	0.000493655	0.014084507
73	2.41×10^{-5}	0.01369863
79	0.057564958	0.012658228
83	0.002215426	0.012048193
89	0	0.011235955
97	0.002901727	0.010309278
101	0.001312399	0.00990099
103	0.114696463	0.009708738
107	0.000156525	0.009345794
109	0	0.009174312
113	0	0.008849558
127	0.000397332	0.007874016
131	0.038288343	0.007633588
137	0	0.00729927
139	0.002925807	0.007194245
149	0.01878296	0.006711409
151	0.001312399	0.006622517
157	0.007561346	0.006369427
163	0	0.006134969
167	0.046343343	0.005988024
173	0	0.005780347

Table 5.3: The proportion of $k(n)$ divisible by p and the values of $\frac{1}{p}$

We also determined the proportions of values of $k(n)$ in each congruence class modulo primes. We found that for a prime p the proportion of values of $k(n) \equiv 1, 2, \ldots, p-1 \mod p$ was approximately equal with a different proportion of values of $k(n) \equiv 0 \mod p$.

For example taking the case of $p = 3$ we have 3.9% of values of $k(n) \equiv 0 \mod 3$ while 48.0% and 48.0% of values of $k(n) \equiv 1$ and $2 \mod 3$ respectively. These proportions for primes less than 20 are given in Table 5.4.

$m\backslash p$	2	3	5	7	11	13	17	19
0	0.126243167	0.039371975	0.027765069	0.00569509	0.067413972	0.000854866	0.121691911	0.034917042
1	0.873756833	0.48024177	0.243383823	0.165928191	0.093421148	0.083716618	0.055518097	0.053110025
2		0.480386255	0.241818576	0.165157608	0.092385677	0.083536013	0.054362222	0.053314711
3			0.242360392	0.166819178	0.093698076	0.082970116	0.054097334	0.052520047
4			0.244672141	0.166722855	0.092975654	0.083511932	0.054904038	0.054590989
5				0.164290702	0.093445228	0.082584824	0.055060563	0.052664532
6				0.165386375	0.093421148	0.083283165	0.055096684	0.054326101
7					0.093902762	0.083463771	0.05427794	0.053868567
8					0.092963614	0.08276543	0.05430202	0.053796325
9					0.09383052	0.082994197	0.055530137	0.053555518
10					0.092542201	0.083981506	0.054398343	0.054061213
11						0.082693188	0.054651191	0.055337491
12						0.083644376	0.055999711	0.054049173
13							0.055554218	0.052881258
14							0.055229128	0.052977581
15							0.054759554	0.052327401
16							0.054566908	0.053760204
17								0.054037132
18								0.053904689

Table 5.4: The proportions of $k(n)$ such that $k(n) \equiv m \mod p$

Finally, we determined the rolling cumulative proportion of them divisible by various primes. From Figure 5.1 we see that these proportions increase/decrease very slowly.

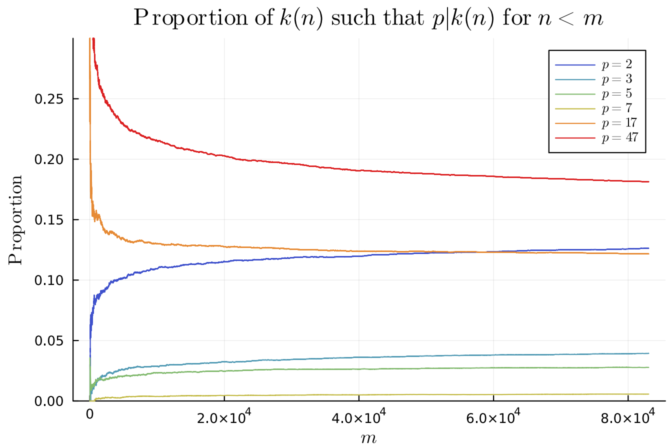

Figure 5.1: Plot of the rolling cumulaitve proportion of $k(n)$ divisible by p for various primes p

Chapter 6

Conclusion

In chapter 2 of this report, we considered some properties and examples of modular forms. We discussed their transformations under the matrices S and T. We derived the Fourier series for the Eisenstein series and discussed their importance in the study of modular forms. Jacobi forms were examined in chapter 3, as were the properties of the Jacobi Theta functions. We considered the behaviour of the Jacobi Theta functions under the modular transformations and applied the Jacobi triple product identity to find their product expansions. We determined the degeneracies of supersymmetric extremal black holes using Siegel modular forms in chapter 4. Finally, we performed numerical analysis of the Ramanujan τ function in chapter 5 and considered the proportion of positive Fourier coefficients of Δ^m, the strange behaviour of the integers $k(n)$ described in section 5.2 and considered an extension of Lehmer's conjecture to the Fourier coefficients of powers of the modular discriminant Δ.

There are some aspects of this report worth exploring further.

1. The applications of modular forms and their generalisation is an exciting area of active research at the interface of pure mathematics and physics worth further consideration.

2. The surprising behaviour of the values of $k(n)$ given in section 5.2 is worth further consideration both analytically and numerically.

3. Lehmer's conjecture, first described in 1947, has yet to be proven. As such more work on this and on the non-vanishing of the Fourier coefficients of Δ^m is needed.

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