The Investigation and Application of String Theory in Supersymmetric Black Holes

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Contents

Abstract

This report is for the investigation and application of string theory in supersymmetric black holes, specifically concerning the statistical entropy of the extremal black hole computed using string theory methods. As methods of string theory and general relativity are set out in this investigation, readers should expect to gain insight into various fundamental mathematical and physical principles in black holes in string theory which are foundational topics of current research.

Chapter 1

Introduction

At July 2023, during the writing of this report, the James Webb Space Telescope detected the most distant active supermassive black hole to date [1]. At this time there is much taking place in terms of the discovery of black holes. Thus, the study of black holes is a timely topic in theoretical physics research, which is why I chose to study it. My personal aim for this project is to gain knowledge of string theory and its applications to supersymmetric black holes. It is also my goal that interested readers will learn more about these topics and that the principles explored would be beneficial to both the reader and I in terms of future study. The topics discussed in this report ultimately relate to the unification of quantum field theory (QFT) and general relativity (GR) which is one of the most important goals of current research relating to black holes. String theory is related as, although incomplete, it proposes to combine QFT and GR into a single framework.

1.1 Quantisation

A suitable starting point for a discussion of string theory in supersymmetric black holes is quantisation, primarily the evolution of classical mechanics to quantum mechanics.

In classical mechanics, the state of a system is described in terms of canonical coordinates (q_i, p_i), where qi are the generalised coordinates and pi are the generalised momentum [5]. The Poisson bracket for two variables, $A(q_i, p_i, t)$ and $B(q_i, p_i, t)$ that are functions of the canonical coordinates and time, is given by

$$[A, B]_{PB} = \sum_{i}^{N} \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right). \tag{1.1}$$

The specific Poisson bracket between the position and momentum of a system is

$$[q_i, p_i]_{PB} = \delta_{ij} \tag{1.2}$$

where δ_{ij} is the Kronecker delta function of two variables. This function is defined to have a value of 1 if the two variables are equal and a value of zero otherwise:

$$\delta_{ij} = \begin{cases} 0 & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j \end{cases} \tag{1.3}$$

Classical mechanics describes the behavior of macroscopic systems and is deterministic; if the present state of a body is known, it is possible to predict its future state [2]. In contrast, quantum mechanics gives a microscopic description of particles. It is probabilistic by Heisenberg's uncertainty principle [3].

In quantum mechanics the variables A and B change to linear operators and, analogous to Poisson brackets, the commutator brackets between the two operators is

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}. \tag{1.4}$$

The specific commutator between the position and momentum of a quantum mechanical system is

$$[q_i, p_i] = i\delta_{ij}. \tag{1.5}$$

Thus, in the quantum mechanical limit, Poisson brackets in classical mechanics are proportional to the commutator brackets in quantum mechanics via the relation

$$[\hat{A}, \hat{B}]_{PB} = \frac{[\hat{A}, \hat{B}]}{i}. \tag{1.6}$$

1.2 Quantum field theory

Quantum field theory is a theoretical framework that combines classical field theory, quantum mechanics and special relativity. QFT provides the framework for the standard model, a theory of all the observed forces of nature [8]. This theory describes electromagnetism, the weak interaction responsible for radioactivity and the strong nuclear force that governs the structure of nuclei. No other current theory has been so universally successful [8].

1.2.1 The Klein-Gordon and Dirac Equation

Two fundamental equations in QFT are the Klein-Gordon equation and the Dirac equation. These equations describe the behavior of spin-0 and spin$-1/2$ particles respectively.

The Klein-Gordon equation (also Schrödinger's relativistic wave equation) is a second-order partial differential equation given by

$$(\Box + m^2)\phi = 0 \tag{1.7}$$

where ϕ is the wavefunction describing the spin-zero particle, and the operator $\Box$ is the d'Alembertian given by

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2. \tag{1.8}$$

At its publication, the Klein-Gordon equation was noted to be limited in that the field satisfying the Klein-Gordon equation describes a negative energy wave-function of a particle [9].

It is now known that in the QFT multi-particle description, this problem does not arise. Nevertheless, the proposed problem motivated Dirac to construct his relativistic first order differential equation which holds for spin $\frac{1}{2}$ particles, denoted ψ.

The Dirac equation is given by

$$(\gamma^\mu p_\mu - m)\psi = 0 \tag{1.9}$$

where γ^μ are the gamma matrices and p_μ is equal to the Kronecker delta function, δ_μ.

1.2.2 Canonical quantisation

Canonical quantisation is the process of quantising a field. Field theory is different from mechanics as it predicts how fields, rather than sets of point particles, interact with matter. A field is a physical quantity represented by a scalar, vector or tensor that has a value for each point in spacetime [11].

In order to express the state of a system in classical field theory, the coordinates of classical mechanics take the continuum limit [4]. Practically, this means that in equation 1.1, $N \to \infty$ and the discrete index i is set to continuous position coordinates, $\mathbf{x} = (x, y, z)$. The summation becomes an integration over the continuum position coordinate, and the coordinates q_i and p_i become $\psi(\mathbf{x})$ and $\pi(\mathbf{x})$ respectively.

The Poisson bracket for classical fields becomes

$$[A, B]_{PB} = \int \left(\frac{\partial A}{\partial \psi(\mathbf{x})} \frac{\partial B}{\partial \pi(\mathbf{x})} - \frac{\partial A}{\partial \pi(\mathbf{x})} \frac{\partial B}{\partial \psi(\mathbf{x})} \right) d\mathbf{x}. \tag{1.10}$$

A and B are now functions of the field coordinate $\psi(\mathbf{x})$ and the momentum density $\Pi(\mathbf{x})$. The momentum density is given by

$$\Pi(\mathbf{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \tag{1.11}$$

where $\mathcal{L}$ is the Lagrangian density, a measure of the distribution of the Lagrangian over space.

The Poisson brackets between field coordinates and momentum density in classical field theory are given by

$$[\psi(\mathbf{x}), \Pi(\mathbf{y})]_{PB} = \delta(\mathbf{x}\text{-}\mathbf{y}) \tag{1.12}$$

where $\delta(\mathbf{x}\text{-}\mathbf{y})$ is the Dirac delta function, analogous to the Kronecker delta function but used in the continuum limit.

Canonical quantisation is now applied to derive quantum field theory from classical field theory. For a quantum description, the fields ψ and Π are set to field operators $\hat{\psi}$ and $\hat{\Pi}$. By the same relation as equation 1.6, the Poisson bracket can be written in terms of the commutator bracket. In quantum field theory the commutator brackets between the field coordinates and momentum are given by

$$[\hat{\psi}(\mathbf{x}), \hat{\Pi}(\mathbf{y})] = i\delta(\mathbf{x}\text{-}\mathbf{y}). \tag{1.13}$$

1.3 General Relativity

General relativity is the geometric theory of gravity that generalizes Einstein's special relativity and Newton's law of universal gravitation, providing a unified description of gravity as a geometric property of spacetime [6].

QFT and GR are known as the pillars of modern theoretical physics [12]. Both are equally successful, however, intrinsically incompatible. Quantum gravity has the ability to unify QFT and GR, however, is currently not a functional theory. It is estimated to be most significant at the plank length scale, which is outside of direct experimental reach.

1.3.1 Black holes

A large part of quantum gravity research focuses on the black hole, as its thermodynamic properties are expected to be an important feature of quantum gravity theory [12]. Black holes are points in space that are so dense they create deep gravity sinks [21]. Gravity is so strong there that even light cannot escape [13]. A black hole is remarkably simple in that it is completely specified by three observable quantities, mass (M), spin (S) and charge (Q). A black hole is also remarkably complex in that it possesses a huge entropy. Entropy gives an account of the number of microstates of a system. Therefore, the entropy of a black hole signifies an incredibly complex microstructure. [16]

1.3.2 Black hole geometry

The simplest spherically symmetric black hole is static and uncharged and is described by Schwarzschild geometry. This geometry was derived by Karl Schwarzschild at the close of 1915, a few weeks after Albert Einstein published his fundamental paper on the Theory of General Relativity [10]. Schwarzchild geometry is described by a metric which is the static solution of the vacuum Einstein-Hilbert action, given by

$$S = \frac{1}{16\pi G} \int R\sqrt{g} d^4x - \frac{1}{16\pi} F^2_{\mu\nu} \sqrt{g} d^4x \tag{1.14}$$

where G is Newton's gravitational constant, R is the Ricci scalar of the metric $g_{\mu\nu}$, where $g = det(g_{\mu\nu})$. $F_{\mu\nu}$ is the electromagnetic field strength. In this convention, the indices μ and ν take values $0, 1, 2$ and 3.

The metric solution to the Einstein-Hilbert action is given by

$$ds^2 = g_{\mu\nu}dx^{\mu}dx_{\nu}. \tag{1.15}$$

which can also be written as

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega. \tag{1.16}$$

Here, t is the Schwarzschild time which represents the time recorded by a standard clock at rest at spatial infinity and r is the Schwarzschild radial coordinate. It does not measure the spatial distance from the orgin but defined so that the area of a sphere at r is $4\pi r^2$ [6]. Ω is the solid angle of this sphere [16].

1.3.3 The event horizon and the surface gravity

From equation 1.16, the metric appears to be singular at $r = 2GM$. Although there is not a real singularity here, many of the quantum properties of black holes come from the geometry in the region near the Schwarzschild radial coordinate, r which can be labelled r_H and is renamed the event horizon. The event horizon defines the boundary where the velocity needed to escape the black hole exceeds the speed of light [13]. The area of the event horizon is given by

$$A_H = 4\pi r_H^2. \tag{1.17}$$

Another important parameter for the study of black holes is the surface gravity. To define this parameter, notice that the Schwarzchild metric of equation 1.15 looks precisely like

$$ds^2 = -\rho^2\kappa^2 dt^2 + d\rho^2 \tag{1.18}$$

which is the metric in an alternative coordinate system called Rindler coordinates. The coordinate ρ measures the shortest radial distance of the black hole and κ, which is given by

$$\kappa = \frac{1}{4GM} \tag{1.19}$$

represents the surface gravity of the black hole. The surface gravity of the black hole is defined as the acceleration due to gravity experienced by a test particle which is very close to the surface of the black hole [14].

1.3.4 Black hole mechanics

As already discussed, an important property of black holes for the proposed unification of QFT and GR is that the laws of black hole mechanics closely resemble the laws of thermodynamics.

In thermodynamics, the zeroth law states that the temperature of a body at thermal equilibrium is constant throughout the body. For a stationary black hole, the surface gravity κ is constant at the event horizon. In correspondence with the zeroth law, this is like saying that a stationary black hole is at thermal equilibrium [16].

The first law of thermodynamics concerns the conservation of energy equation:

$$dE = TdS + \mu dQ + \Omega dJ \tag{1.20}$$

where E is the energy of the system and T is the temperature. Q is the charge of the system, which has an associated chemical potential μ and J is the spin with an associated chemical potential Ω. For black holes, the equation which corresponds to the first law is

$$dM = \frac{\kappa}{8\pi G} dA + \mu dQ + \Omega dJ. \tag{1.21}$$

This equation relates the mass, rotation and charge of a black hole to its entropy. Thus, the entropy of a black hole, S is related to the area, A of the event horizon, previously defined as A_H [25].

By the first law of thermodynamics, if the black hole has energy E and entropy S, it must also have temperature. If the black hole has temperature, like any hot body it must radiate. Classically this is impossible, however, Hawking showed that, after including quantum effects, it is possible for the black hole to radiate [16]. This theoretical thermal blackbody radiation released outside of a black hole's event horizon is called Hawking radiation [7].

The second law of thermodynamics states that in a physical process, the total entropy never decreases so that $\Delta S \geq 0$. Black holes obey a version of this law. A black hole possesses an entropy that is proportional to the area of the event horizon. In fact it can be shown that the net area A of the event horizon in any process never decreases so that $\Delta A \geq 0$ [15]. More specifically, the black hole entropy is given by

$$S_{BH} = \frac{A}{4} \tag{1.22}$$

where natural units are used and S_{BH} is the Bekenstein-Hawking entropy. The Bekenstein-Hawking entropy of a black hole never decreases.

The advantage of black hole mechanics is that it provides a starting point to understanding the complex interactions of black holes.

1.4 String theory

As already discussed, QFT and GR are not unified, and although still incomplete, string theory proposes to be the all-encomposing theory of the universe, unifying all the forces of nature, including gravity into a single framework [24].

At a fundamental level, string theory says that elementary matter does not consist of point particles as in conventional QFT, but of one-dimensional loops of string (of zero thickness) [17]. The low energy limit of string theory gives rise to gravity coupled to other fields. As a result, string theories typically have black hole solutions [19].

1.4.1 Black hole entropy in string theory

String theory is consistent with the black hole entropy prediction since strings have a variety of string states, so have entropy like the black hole. The entropy in string theory is proportional to the mass of the string, unlike the black hole entropy which is proportional to the area of the event horizon. It is, however, possible to shrink a black hole's size and reduce its mass while keeping its entropy constant.

When the string is reduced to the size of a parameter called the string length scale l_s and the black hole has the minimum mass possible that is compatible with its charge and angular momentum, it is called an extremal black hole [16].

The entropy for an extremal black hole is indistinguishable from the string entropy [21]. Thus, extremal black hole entropy can be calculated from string theory.

Why is entropy so important for black holes in string theory? The significance of entropy is that it contains information about the microstructure of a black hole, therefore gives valuable and quantifiable information about it [19].

1.4.2 Supersymmetric black holes in string theory

Extremal black holes are often supersymmetric. Supersymmetry is an important extension to the standard model. Supersymmetry is a special type of symmetry that maps bosons to fermions and fermions to bosons [27].

For any bosonic or fermionic state of zero energy, there may be an arbitrary number $n_B^{E=0}$ of bosonic states and an arbitrary number $n_F^{E=0}$ of fermionic states. The difference between these two numbers $n_B^{E=0} - n_F^{E=0}$ is important as it provides information about the supersymmetry of the system [27]. If this difference is zero, the supersymmetry is broken, but if non-zero, the supersymmetry remains unbroken. The quantity $n_B^{E=0} - n_F^{E=0}$ is related to the trace of the operator $(-1)^F$ so that bosonic states contribute $+1$ to the trace and fermionic states contribute -1 to the trace [27], and

$$Tr(-1)^F = n_B^{E=0} - n_F^{E=0}. \tag{1.23}$$

From the black hole to string theory correlation, string theory is shown to have the ability to create a framework for studying the classical and quantum properties of black holes [19].

1.5 Outline

The foundational concepts surrounding black holes in string theory have been set out in the introduction. Chapter 2 focuses on Partition functions in string theory. This will include calculations of the partition functions on the torus for both bosonic and fermionic cases. The bosonic case will begin with an introduction to action principles, including the Nambu-Goto and Polyakov action. The equations of motion and oscillator modes for the string will then lead to the partition function calculations. Following much of the same methodology, the fermionic partition functions on the torus will be calculated.

In Chapter 3 the concepts of modular forms, elliptical functions, and Jacobi forms are introduced. Specific Jacobi functions known as Jacobi theta functions are defined and used in the computation of the elliptic genus of K3, which is also defined.

In Chapter 4 the elliptic genus of K3 calculation is used to quantify the degeneracy of black hole microstates and Chapter 5 concludes the discussion.

The ultimate aim of this report is to use string theory to investigate the properties of supersymmetric black holes. In doing this, it will provide an anchor for various foundational concepts in theoretical physics and scope to further study of the concepts presented in the report.

Chapter 2

Partition Functions in String Theory

Partition functions are a tool derived from statistical mechanics where they are used to obtain expression for temperature and chemical potential of a system. Here they are applied to string theory as a method of obtaining expressions for these quantities in terms of the black hole.

2.1 Bosonic partition functions

The bosonic string is the simplest string theory model, so is a good place to start. Although the bosonic string itself is not a viable physical model, since most particles are fermions, it requires the same structures and formalism required for the analysis of more realistic models [22].

The bosonic string is a one dimensional object that sweeps out a two dimensional surface as it moves through spacetime. This two dimensional surface is called the worldsheet. The points on the worldsheet are described by a function of two coordinates $X(\tau, \sigma)$, where τ is the time-like coordinate and σ is the space-like coordinate [18].

2.1.1 Action Principles

The dynamics of a closed bosonic string can be described by the action. A general expression for the action is

$$\mathcal{S} = \int_M \mathcal{L} \, d\tau d\sigma \tag{2.1}$$

where M denotes the worldsheet. $\mathcal{L}$ is the Lagrangian density and the Lagrangian associated with the Lagrangian density is given by

$$L = \int_M \mathcal{L} \, d\sigma. \tag{2.2}$$

The simplest invariant action in bosonic string theory is the Nambu-Goto action which is proportional to the proper area of the worldsheet. In order to express the Lagrangian in terms of $X(\tau, \sigma)$, the Nambu-Goto action is used where the induced metric h_α is introduced:

$$h_\alpha = \partial_\alpha X \partial^\alpha X. \tag{2.3}$$

The indices α run over the values (τ,σ) and the Lagrangian associated with this induced metric is

$$L_{NG} = -\frac{1}{2\pi\alpha'} \int_0^{2\pi} (-h)^{1/2} d\sigma \tag{2.4}$$

which is defined as the Nambu-Goto Lagrangian [18]. Here $\det h_\alpha$ is denoted as h and the parameter α' is the Regge slope. The tension in the string is equal to the mass per unit length of the string [24] and is related to the Regge slope by

$$T = \frac{1}{2\pi\alpha'}. \tag{2.5}$$

The action S_{NG} corresponding to the Nambu- Goto Lagrangian is given by

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int d\tau d\sigma (-h)^{1/2} \tag{2.6}$$

The Nambu-Goto action can be simplified by introducing an independent worldsheet metric $\gamma_{ab}(\tau, \sigma)$. With this metric the action becomes

$$S_P[X, \gamma] = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma (-\det\gamma_{ab})^{1/2} \gamma^{ab} \partial_\alpha X \partial^\alpha X \tag{2.7}$$

This S_P is called the Polyakov action. Varying the S_P with respect to the worldsheet metric γ_{ab} such that $\delta\gamma = \gamma + \delta\gamma$ shows the classical equivalence between the Polyakov action and the Nambu-Goto action:

$$\delta_\gamma S_P = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma \delta((-\gamma)^{1/2}\gamma^{ab})\partial_\alpha X\partial^\alpha X \tag{2.8}$$

where detγ_{ab} is denoted as γ and, via the chain rule,

$$\delta_\gamma S_P = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma[(-\gamma)^{1/2}\delta(\gamma^{ab})\partial_\alpha X\partial^\alpha X - \delta((-\gamma)^{1/2})\gamma^{ab}\partial_\alpha X\partial^\alpha X]. \tag{2.9}$$

It must be noted that the variation of the determinant of a general two-by-two matrix A is given by a general form of Jacobi's formula

$$\delta(detA) = (\det A)\mathrm{Tr}(\mathrm{Adj}(A)\delta A). \tag{2.10}$$

If A is invertible, which is the case for the Polyakov action, then $\mathrm{Adj}(A) = A^{-1}$.

Applying this to the variation of the determinant of γ_{ab} it is given that

$$\delta\gamma = \delta det(\gamma_{ab}) = \gamma(\gamma^{ab}\delta\gamma_{ab}). \tag{2.11}$$

Further using the fact that for covariant and contravariant tensors

$$\gamma^{ab}\delta\gamma_{ab}\gamma = -\gamma_{ab}\delta\gamma^{ab}\gamma, \tag{2.12}$$

$\delta(-\gamma)^{1/2}$ is computed as follows

$$\delta(-\gamma^{1/2}) = -\frac{1}{2}\frac{\delta\gamma}{(-\gamma)^{1/2}} = -\frac{1}{2}\frac{(-\gamma)\delta\gamma^{ab}\gamma_{ab}}{(-\gamma)^{1/2}} = -\frac{1}{2}(-\gamma)^{1/2}\delta\gamma^{ab}\gamma_{ab}. \tag{2.13}$$

Using the above information, the variation of S_P with respect to the worldsheet metric from 2.9 becomes

$$\delta_\gamma S_P = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma(-\gamma)^{1/2}\delta(\gamma^{ab})[\partial_\alpha X\partial^\alpha X - \frac{1}{2}\gamma_{ab}(\gamma_{cd}\partial_c X\partial_d X)] \tag{2.14}$$

which gives

$$\partial_\alpha X\partial^\alpha X - \frac{1}{2}\gamma_{ab}(\gamma_{cd}\partial_c X\partial_d X) = 0 \tag{2.15}$$

where it is recognised that $\partial_\alpha X \partial^\alpha X$ is equivalent to h_{ab} so that

$$h_{ab} = \frac{1}{2}\gamma_{ab}(\gamma_{cd}\partial_c X \partial_d X). \tag{2.16}$$

Taking the square root of minus the determinant of both sides of equation 2.16 gives

$$\frac{1}{2}(-\gamma)^{1/2}\gamma^{ab}\delta_\alpha X \delta^\alpha X = (-h)^{1/2}. \tag{2.17}$$

Since γ_α is proportional to the induced metric h_α, it can be eliminated from the Polyakov action. Thus, it is shown to be equivalent to the Nambu-Goto action, and

$$S_p[X,\gamma] = -\frac{1}{2\pi\alpha'}\int d\tau d\sigma (-h)^{1/2} \tag{2.18}$$

$$= S_{NG}[X]. \tag{2.19}$$

Substitution of the induced metric into the action equivalents gives

$$\begin{aligned} S &= -\frac{1}{2\pi\alpha'}\int d\tau d\sigma(-\partial_\alpha X \partial^\alpha X)^{1/2} \\ &= \frac{1}{4\pi\alpha'}\int d\tau d\sigma(\partial_\alpha X \partial^\alpha X) \end{aligned} \tag{2.20}$$

which is the simplest invariant action in bosonic string theory.

2.1.2 The Lagrangian and Hamiltonian equations

From the action, the Lagrangian for a closed bosonic string [23] is given by

$$L = \frac{1}{4\pi\alpha'}\int_0^{2\pi} \partial_\alpha X \partial^\alpha X \, d\sigma, \tag{2.21}$$

The associated Lagrangian density is

$$\mathcal{L} = \frac{1}{4\pi\alpha'}\partial_\alpha X \partial^\alpha X. \tag{2.22}$$

Since the string is closed, the boundary conditions on sigma are periodic

$$X(\tau,\sigma) = X(\tau,\sigma+2\pi) \tag{2.23}$$

and limits of integration are 0 and 2π.

In Minkowski space, the Lagrangian can be written as

$$L = \frac{1}{4\pi\alpha'}\int_0^{2\pi} (\dot{X}^2 - X'^2)\, d\sigma \tag{2.24}$$

where the Lagrangian density is

$$\mathcal{L} = \frac{1}{4\pi\alpha'}(\dot{X}^2 - X'^2). \tag{2.25}$$

$\dot{X}$ is the derivative of X with respect to τ and X' is the derivative of X with respect to σ.

The Hamiltonian associated with the Lagrangian density is

$$\begin{aligned}\mathcal{H} &= \frac{\partial\mathcal{L}}{\partial\dot{X}}\dot{X} - \mathcal{L}\\ &= \frac{1}{4\pi\alpha'}\int_0^{2\pi}(2\dot{X}^2 - \dot{X}^2 + X'^2)d\sigma\\ &= \frac{1}{4\pi\alpha'}\int_0^{2\pi}(\dot{X}^2 + X'^2)d\sigma \end{aligned} \tag{2.26}$$

2.1.3 Equations of motion

In Minkowski space, the Euler-Lagrangian equation can be written in the following form

$$\partial_\tau\left(\frac{\partial\mathcal{L}}{\partial_\sigma\dot{X}}\right) + \partial_\sigma\left(\frac{\partial\mathcal{L}}{\partial_\sigma\dot{X}}\right) - \frac{\partial\mathcal{L}}{\partial X} = 0 \tag{2.27}$$

where the Lagrangian density is given by equation 2.25 so that by the Euler-Lagrangian equation, the equation of motion is

$$2\ddot{X} - \ddot{X} - X'' = 0 \tag{2.28a}$$

$$\ddot{X} - X'' = 0. \tag{2.28b}$$

Or in Euclidean space, the Euler-Lagrangian equation

$$\partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial^\alpha X}\right) - \frac{\partial\mathcal{L}}{\partial X} = 0 \tag{2.29}$$

gives from equation 2.22, the equation of motion to be

$$\partial_\alpha \partial^\alpha X = 0. \tag{2.30}$$

Using the method of separation of variables, X can be expressed as $X = a(\tau)b(\sigma)$. From this the equation of motion 2.28b becomes

$$\ddot{a}b - b''a = 0 \tag{2.31a}$$

and

$$\frac{\ddot{a}}{a} = \frac{b''}{b}. \tag{2.31b}$$

Since equation 2.31b must be true for all (τ, σ), it must be equal to some arbitrary constant, chosen to be k^2. The solutions of the equation of motion for non-zero k are then

$$a = e^{\pm ik\sigma} \tag{2.32a}$$

$$b = e^{-ik\tau} \tag{2.32b}$$

which upon substitution can be shown to satisfy equation 2.31b. Linear solutions for a and b result from the the special case that $k = 0$, these are given by

$$a = \beta\sigma + \gamma \tag{2.33a}$$

$$b = \zeta\tau + x. \tag{2.33b}$$

where β, γ, ζ and x are chosen constants.

2.1.4 Oscillator modes

Using these linear solutions defined in equations 2.33a and 2.33b, and the periodic boundary conditions defined in equation 2.23, it can be shown that the harmonic function $X(\tau, \sigma)$ takes the general form

$$X(\tau, \sigma) = x + \alpha' p\tau + \beta\sigma \tag{2.34}$$

where x is the centre of mass position of the string and p is the centre of mass momentum carried by the string.[1] The harmonic function is also expressed in terms of the oscillator modes

$$X(\tau,\sigma) = X_L + X_R + \alpha' p\tau + x \tag{2.35}$$

where X_L and X_R are the left moving and right moving solutions respectively which correspond to the functions $\tau - \sigma$ and $\tau + \sigma$.

The general solution to the equation of motion is given by

$$X(\tau,\sigma) = i\sqrt{\frac{\alpha'}{2}}\sum_{k\neq 0}\frac{\alpha_k}{k}e^{-ik\tau+ik\sigma} + i\sqrt{\frac{\alpha'}{2}}\sum_{k\neq 0}\frac{\tilde{\alpha_k}}{k}e^{-k\tau-ik\sigma} + \alpha' p\tau + x \tag{2.36}$$

where the α_k and $\tilde{\alpha_k}$ represent the two independent sets of oscillators corresponding to the left-moving and right-moving waves along the string.

2.1.5 The Hamiltonian in terms of oscillator modes

To write the Hamiltonian $\mathcal{H}$ in terms of oscillator mode sectors α_k and $\tilde{\alpha_k}$, the commutation between $X(\sigma)$ and $\pi(\sigma')$ is used, where Π is the canonical momentum density given by

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{X}} = \frac{1}{2\pi\alpha'}\dot{X} \tag{2.37}$$

and the commutation relation between the position and the momentum density is given by

$$[X(\sigma), \Pi(\sigma')] = i\delta(\sigma - \sigma'). \tag{2.38}$$

where $\delta(\sigma - \sigma')$ is the Dirac delta function for periodic boundary conditions. Taking equal times and setting $\tau = 0$,

$$[X, \dot{X}] = 2\pi i\alpha'\delta(\sigma - \sigma'). \tag{2.39}$$

[1]The periodic boundary conditions of equation 2.23 imply that the momentum modes are quantised to integers.

The derivative of X with respect to τ at $\tau = 0$ is

$$\begin{aligned}\dot{X} &= i\sqrt{\frac{\alpha'}{2}}\sum_k(-ik)\frac{\alpha_k}{k}e^{-ik\tau+ik\sigma} + i\sqrt{\frac{\alpha'}{2}}\sum_k(-ik)\frac{\tilde{\alpha_k}}{k}e^{-ik\tau-ik\sigma} + \alpha' p \\ &= \sqrt{\frac{\alpha'}{2}}\sum_k \alpha_k e^{-ik\tau+ik\sigma} + \sqrt{\frac{\alpha'}{2}}\sum_k \tilde{\alpha_k} e^{-ik\tau-ik\sigma} + \alpha' p \end{aligned} \tag{2.40}$$

Thus, at $\tau = 0$,

$$X = i\sqrt{\frac{\alpha'}{2}}\sum_k \frac{1}{k}\left(\alpha_k e^{ik\sigma} + \tilde{\alpha_k} e^{-ik\sigma}\right) + \alpha' p + x \tag{2.41}$$

and

$$\dot{X} = \sqrt{\frac{\alpha'}{2}}\sum_k \left(\alpha_k e^{ik\sigma} + \tilde{\alpha_k} e^{-ik\sigma}\right) + \alpha' p. \tag{2.42}$$

The Fourier transformation of $X(\tau, \sigma)$ allows expressions to be found for α_k and $\tilde{\alpha_k}$. A system of equations are formed by multiplying both sides of equations 2.41 and 2.42 by $e^{-ik'\sigma}$ and integrating over sigma.

The equation for X is given by

$$\begin{aligned}\int_0^{2\pi} X e^{-ik'\sigma} d\sigma &= i\sqrt{\frac{\alpha'}{2}}\sum_k \frac{1}{k}\int_0^{2\pi}\left(\alpha_k e^{ik\sigma} + \tilde{\alpha_k} e^{-ik\sigma}\right)e^{-ik'\sigma} \\ &= i\sqrt{\frac{\alpha'}{2}}\sum_k \frac{1}{k}\int_0^{2\pi}\left(\alpha_k e^{i(k-k')\sigma} + \tilde{\alpha_k} e^{-i(k-k')\sigma}\right)d\sigma \\ &= i\sqrt{\frac{\alpha'}{2}}\sum_k \frac{2\pi}{k}(\alpha_k e^{i(k-k')\sigma} + \tilde{\alpha_k} e^{-i(k-k')\sigma}) \\ &= i\sqrt{\frac{\alpha'}{2}}\sum_k \frac{2\pi}{k}(\alpha_k \delta_{k,k'} + \tilde{\alpha_k}\delta_{k,-k'}) \\ &= \frac{2\pi i}{k'}\sqrt{\frac{\alpha'}{2}}(\alpha_{k'} + \tilde{\alpha}_{-k'}) \end{aligned} \tag{2.43}$$

where the Kronecker delta function was used. Similarly, the $\dot{X}$ equation gives

$$\int_0^{2\pi} \dot{X} e^{-ik'\sigma} d\sigma = 2\pi\sqrt{\frac{\alpha'}{2}}(\alpha_{k'} + \tilde{\alpha}_{-k'}) \tag{2.44a}$$

This gives the following system of equations, where for simplicity k' is replaced with k

$$\alpha_k - \tilde{\alpha_k} = \frac{k}{2\pi i\sqrt{\frac{\alpha'}{2}}} \int_0^{2\pi} X e^{-ik\sigma} d\sigma \tag{2.45a}$$

$$\alpha_k + \tilde{\alpha_k} = \frac{1}{2\pi i\sqrt{\frac{\alpha'}{2}}} \int_0^{2\pi} \dot{X} e^{-ik\sigma} d\sigma. \tag{2.45b}$$

These equations are solved simultaneously to find the oscillation modes

$$\alpha_k = \frac{1}{4\pi i\sqrt{\frac{\alpha'}{2}}}\left(k\int_0^{2\pi} X e^{-ik\sigma} d\sigma + i\int_0^{2\pi} \dot{X} e^{-ik\sigma} d\sigma\right) \tag{2.46a}$$

$$\tilde{\alpha_k} = \frac{1}{4\pi i\sqrt{\frac{\alpha'}{2}}}\left(-k\int_0^{2\pi} X e^{ik\sigma} d\sigma + i\int_0^{2\pi} \dot{X} e^{ik\sigma} d\sigma\right) \tag{2.46b}$$

The commutation is then given by

$$[\alpha_k, \alpha_{k'}] = \alpha_k\alpha_{k'} - \alpha_{k'}\alpha_k \tag{2.47}$$

which is equivalent to

$$\begin{aligned}
[\alpha_k, \alpha_{k'}] &= -\frac{1}{8\pi^2\alpha'}\int_0^{2\pi}\int_0^{2\pi}\Big[kX(\sigma) + i\dot{X}(\sigma), k'X(\sigma') + i\dot{X}(\sigma')\Big]e^{-ik\sigma - ik'\sigma'} d\sigma d\sigma' \\
&= -\frac{i}{8\pi^2\alpha'}\int_0^{2\pi}\int_0^{2\pi}\left(k\Big[X(\sigma), \dot{X}(\sigma')\Big] + k'\Big[\dot{X}(\sigma), \dot{X}(\sigma')\Big]\right)e^{-ik\sigma - ik'\sigma'} d\sigma d\sigma' \\
&= -i\frac{k - k'}{8\pi^2\alpha'}\int_0^{2\pi}\int_0^{2\pi} 2\pi i\alpha'\delta(\sigma - \sigma')e^{-ik\sigma - ik'\sigma'} d\sigma d\sigma' \\
&= \frac{k - k'}{4\pi}\int_0^{2\pi} e^{i(-k + k')\sigma'} d\sigma \\
&= \frac{k - k'}{4\pi} 2\pi\delta_{k,-k} \\
&= k\delta_{k,-k}
\end{aligned} \tag{2.48}$$

Therefore,

$$[\alpha_k, \alpha_{k'}] = k \tag{2.49}$$

where the commutation relation $[X(\sigma), X(\sigma')] = [\dot{X}(\sigma), \dot{X}(\sigma')] = 0$ was used. Using much the same method, it can be shown that

$$[\tilde{\alpha_k}, \tilde{\alpha_{k'}}] = k \tag{2.50}$$

and that any commutators of α and $\tilde{\alpha}$ are equal to zero.

2.1.6 Periodic partition function calculations

The partition functions of a closed bosonic string are an important mathematical method leading to the investigation of modular invariance. Partition functions are built from complex quantities that are functions of τ for left moving oscillation modes and of $\bar{\tau}$ for right moving oscillation modes.

The full partition is the product of the left moving part of the string and the corresponding right moving part. The most general form of the Partition function for a closed bosonic string is given by a modification of the standard formula for path integrals in quantum mechanics

$$Z = \mathrm{Tr}[\mathrm{e}^{-\beta_\mathrm{L}\mathcal{H}_\mathrm{L}-\beta_\mathrm{R}\mathcal{H}_\mathrm{R}}] \tag{2.51}$$

where $\mathcal{H}_L$ and $\mathcal{H}_R$ correspond to the energy levels of the left and right moving oscillations. $\beta_L = 2\pi i\tau$ and $\beta_R = -2\pi i\bar{\tau}$ where the torus moduli parameter $\bar{\tau}$ is complex conjugate of τ. From the periodic boundary conditions $\tau = 1$ such that thermodynamic β is defined as $\beta_L = \beta_R = \beta$ where $\beta = \frac{1}{T}$.

Depending on the theory it is possible to have periodic and anti-periodic boundary conditions. The periodic case has already been defined in equation 2.23. In this case, k runs over integers from $-\infty$ to $+\infty$, except zero.

For the periodic case, the left and right moving mode solutions can be written in the form

$$\mathrm{Tr}[\mathrm{e}^{-\beta_\mathrm{L}\mathcal{H}_\mathrm{L}}] = \mathrm{e}^{\frac{\beta_\mathrm{L}}{24}} = \mathrm{e}^{-\frac{\beta_\mathrm{L}}{24}} \prod_{\mathrm{k}=1}^{\infty} \frac{1}{1-\mathrm{e}^{\beta_\mathrm{L}\mathrm{k}}}$$

$$\mathrm{Tr}[\mathrm{e}^{-\beta_\mathrm{R}\mathcal{H}_\mathrm{R}}] = \mathrm{e}^{-\frac{\beta_\mathrm{R}}{24}} \prod_{\mathrm{k}=1}^{\infty} \frac{1}{1-\mathrm{e}^{\beta_\mathrm{R}\mathrm{k}}}$$

where zeta function regularization

$$\sum_{k=1}^{\infty} k = -\frac{1}{12} \tag{2.52}$$

is used. In terms of parameters τ and $\bar{\tau}$, the total partition function can be written as

$$Z = e^{-\frac{2\pi i\tau}{24}} \prod_{k=1}^{\infty} \frac{1}{1-e^{2\pi i\tau k}} \; e^{\frac{2\pi i\bar{\tau}}{24}} \prod_{k=1}^{\infty} \frac{1}{1-e^{-2\pi i\bar{\tau}k}}$$

$$Z = \frac{1}{e^{\frac{2\pi i\tau}{24}}} \prod_{k=1}^{\infty} \frac{1}{1-e^{2\pi i\tau k}} \; \frac{1}{e^{-\frac{2\pi i\bar{\tau}}{24}}} \prod_{k=1}^{\infty} \frac{1}{1-e^{-2\pi i\bar{\tau}k}}$$

The partition function can then be written in terms of q and $\bar{q}$ where $q = e^{2\pi i\tau}$ and $\bar{q} = e^{-2\pi i\bar{\tau}}$. Thus,

$$Z = \mathrm{Tr}[\mathrm{e}^{-\beta_\mathrm{L}\mathcal{H}_\mathrm{L}-\beta_\mathrm{R}\mathcal{H}_\mathrm{R}}] = \frac{1}{\mathrm{q}^{\frac{1}{24}}\prod_{\mathrm{k=1}}^{\infty}(1-\mathrm{q^k})} \frac{1}{\bar{\mathrm{q}}^{\frac{1}{24}}\prod_{\mathrm{k=1}}^{\infty}(1-\bar{\mathrm{q}}^\mathrm{k})}$$

The Dedekind η function is a function of q given by

$$\eta(q) = \frac{1}{q^{\frac{1}{24}}\Pi_{k=1}^{\infty}(1-q^k)} \tag{2.53}$$

The partition function can be written in terms of the Dedekind η function as

$$Z = \mathrm{Tr}[\mathrm{e}^{-\beta_\mathrm{L}\mathcal{H}_\mathrm{L}-\beta_\mathrm{R}\mathcal{H}_\mathrm{R}}] = \frac{1}{\eta(\mathrm{q})}\frac{1}{\eta(\bar{\mathrm{q}})} = \frac{1}{|\eta(\mathrm{q})|^2} \tag{2.54}$$

2.1.7 Anti-periodic partition function calculations

The anti-periodic boundary conditions are given by

$$X(\tau,\sigma) = -X(\sigma+2\pi) \tag{2.55}$$

In this case, k runs over half-integers from $-\infty$ to $+\infty$.

The quantisation conditions for the anti-periodic case are given by

$$[\alpha_{k+1/2}, \alpha{-k-1/2}] = k+1/2 \tag{2.56}$$

$$[\alpha_{k-1/2}, \alpha{-k+1/2}] = k-1/2 \tag{2.57}$$

This commutator sum is -1/24 [18] so that the partition function for the anti-periodic case is given by

$$\mathrm{Tr}[\mathrm{e}^{-\beta_{\mathrm{L}}\mathcal{H}_{\mathrm{L}}}] = \mathrm{e}^{\frac{\beta_{\mathrm{L}}}{48}} \prod_{\mathrm{k}}^{\infty} \frac{1}{1-\mathrm{e}^{-\beta_{\mathrm{L}}(\mathrm{k}-1/2)}} \tag{2.58}$$

$$\mathrm{Tr}[\mathrm{e}^{-\beta_{\mathrm{R}}\mathcal{H}_{\mathrm{R}}}] = \mathrm{e}^{\frac{\beta_{\mathrm{R}}}{48}} \prod_{\mathrm{k}}^{\infty} \frac{1}{1-\mathrm{e}^{-\beta_{\mathrm{R}}(\mathrm{k}-1/2)}} \tag{2.59}$$

where the total partition function is

$$Z = \mathrm{Tr}[\mathrm{e}^{-\beta_{\mathrm{L}}\mathcal{H}_{\mathrm{L}}-\beta_{\mathrm{R}}\mathcal{H}_{\mathrm{R}}}] = \mathrm{e}^{\frac{\beta_{\mathrm{L}}}{48}} \prod_{\mathrm{k}}^{\infty} \frac{1}{1-\mathrm{e}^{-\beta_{\mathrm{L}}(\mathrm{k}-1/2)}} \mathrm{e}^{\frac{\beta_{\mathrm{R}}}{48}} \prod_{\mathrm{k}}^{\infty} \frac{1}{1-\mathrm{e}^{-\beta_{\mathrm{R}}(\mathrm{k}-1/2)}} \tag{2.60}$$

2.2 Fermionic partition functions on the torus

2.2.1 The Lagrangian, action and Hamiltonian

The Lagrangian for a single fermion is given by

$$L = \frac{1}{4\pi}\int_0^{2\pi} (\bar{\psi}\partial_\omega\psi + \tilde{\psi}\partial_{\bar{\omega}}\bar{\tilde{\psi}})d\sigma \tag{2.61}$$

so that the action is given by

$$S = \frac{1}{4\pi}\int\int_0^{2\pi} (\bar{\psi}\partial_\omega\psi + \tilde{\psi}\partial_{\bar{\omega}}\bar{\tilde{\psi}})d\sigma d\tau \tag{2.62}$$

where $\omega = \tau + i\sigma$, $\bar{\omega} = \tau - i\sigma$,

$$\partial_\omega = \partial_\tau + i\partial_\sigma \tag{2.63a}$$

$$\partial_{\bar{\omega}} = \partial_\tau - i\partial_\sigma. \tag{2.63b}$$

The action may then be written as

$$S = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} (\bar{\psi}(\partial_\tau + i\partial_\sigma)\psi + \tilde{\psi}(\partial_\tau - i\partial_\sigma)\bar{\tilde{\psi}})d\sigma d\tau \tag{2.64a}$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} (\bar{\psi}\partial_\tau\psi + i\bar{\psi}\partial_\sigma\psi + \tilde{\psi}\partial_\tau\bar{\tilde{\psi}} - i\tilde{\psi}\partial_\sigma\bar{\tilde{\psi}})d\sigma d\tau \tag{2.64b}$$

Introducing the Jacobian J which is given by

$$J = \begin{vmatrix} \frac{\partial_\omega}{\partial_\tau} & \frac{\partial_\omega}{\partial_\sigma} \\ \frac{\partial_{\bar{\omega}}}{\partial_\tau} & \frac{\partial_{\bar{\omega}}}{\partial_\sigma} \end{vmatrix} = \begin{vmatrix} 1 & i \\ 1 & -i \end{vmatrix} = -2i = \frac{2}{i} \tag{2.65}$$

The action in terms of the Jacobian is

$$S = \frac{1}{8\pi} \int \int_0^{2\pi} (\bar{\psi}\partial_\tau \psi i - \bar{\psi}\partial_\sigma \psi + \tilde{\psi}\partial_\tau \bar{\tilde{\psi}} i + \tilde{\psi}\partial_\sigma \bar{\tilde{\psi}}) J d\sigma d\tau \tag{2.66}$$

So that the Lagrangian density is

$$\mathcal{L} = \frac{1}{8\pi}(\bar{\psi}\partial_\tau \psi i - \bar{\psi}\partial_\sigma \psi + \tilde{\psi}\partial_\tau \bar{\tilde{\psi}} i + \tilde{\psi}\partial_\sigma \bar{\tilde{\psi}})\frac{2}{i} \tag{2.67a}$$

$$= \frac{1}{4\pi i}(\bar{\psi}\partial_\tau \psi i - \bar{\psi}\partial_\sigma \psi + \tilde{\psi}\partial_\tau \bar{\tilde{\psi}} i + \tilde{\psi}\partial_\sigma \bar{\tilde{\psi}}). \tag{2.67b}$$

In terms of ∂_ω and $\partial_{\bar{\omega}}$

$$\mathcal{L} = \frac{1}{4\pi i}(\bar{\psi}\partial_\omega \psi + \tilde{\psi}\partial_{\bar{\omega}} \bar{\tilde{\psi}}) \tag{2.68}$$

The Hamiltonian in terms of the Lagrangian density is given by

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial(\partial_\tau \psi)}\partial_\tau \psi + \frac{\partial \mathcal{L}}{\partial(\partial_\tau \tilde{\psi})}\partial_\tau \tilde{\psi} - \mathcal{L} \tag{2.69}$$

From equation 2.67b this becomes

$$\mathcal{H} = \frac{1}{4\pi i}(\bar{\psi}\partial_\tau \psi i + \bar{\tilde{\psi}}\partial_\tau \tilde{\psi} - (\bar{\psi}\partial_\tau \psi i - \bar{\psi}\partial_\sigma \psi + \tilde{\psi}\partial_\tau \bar{\tilde{\psi}} i + \tilde{\psi}\partial_\sigma \bar{\tilde{\psi}})) \tag{2.70}$$

Thus,

$$\mathcal{H} = \frac{1}{4\pi i}(\bar{\psi}\partial_\sigma \psi - \tilde{\psi}\partial_\sigma \bar{\tilde{\psi}}). \tag{2.71}$$

2.2.2 Equations of motion

In mode expansion, the term holomorphic is used to describe the left-moving oscillatory mode, represented by ψ. The term antiholomorphic is used to describe the right-moving oscillatory mode which is represented by $\tilde{\psi}$.

The Euler-Lagrangian equation for the holomorphic case

$$\partial_\omega \left(\frac{\partial \mathcal{L}}{\partial_\omega \bar{\psi}} \right) + \partial_{\bar{\omega}} \left(\frac{\partial \mathcal{L}}{\partial_\omega \bar{\psi}} \right) - \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = 0 \tag{2.72}$$

gives

$$\frac{1}{4\pi}(0 + 0 + \partial_\omega \psi) = 0 \tag{2.73}$$

and

$$\partial_\omega \psi = 0. \tag{2.74}$$

so that the equation of motion is

$$\partial_\tau \psi + i\partial_\sigma \psi = 0. \tag{2.75}$$

Similarly for the antiholomorphic case

$$\partial_\omega \left(\frac{\partial \mathcal{L}}{\partial_\omega \bar{\bar{\tilde{\psi}}}} \right) + \partial_{\bar{\omega}} \left(\frac{\partial \mathcal{L}}{\partial_\omega \bar{\bar{\tilde{\psi}}}} \right) - \frac{\partial \mathcal{L}}{\partial \bar{\bar{\tilde{\psi}}}} = 0 \tag{2.76}$$

so that the second equation of motion is

$$\partial_\tau \tilde{\psi} - i\partial_\sigma \tilde{\psi} = 0. \tag{2.77}$$

2.2.3 Solutions to the equations of motion

Using the Lagrangian density 2.67*b*, the canonical momenta are given by

$$\Pi(\psi) = \frac{\partial \mathcal{L}}{\partial(\partial_\tau \psi)} = \frac{1}{4\pi i}\psi. \tag{2.78}$$

and

$$\Pi(\tilde{\psi}) = \frac{\partial \mathcal{L}}{\partial(\partial_\tau \tilde{\psi})} = \frac{1}{4\pi i}\psi. \tag{2.79}$$

where it is noted that in two dimensions $\bar{\psi} = \psi$ and $\bar{\tilde{\psi}} = \tilde{\psi}$.

Using the method of separation of variables, ψ can be expressed as $\psi = a(\tau)b(\sigma)$ so that

$$\partial_\tau \psi = \partial_\tau(a(\tau)b(\sigma)) = \dot{a}b \tag{2.80}$$

and

$$\partial_\sigma \psi = \partial_\sigma (a(\tau) b(\sigma)) = ab' \tag{2.81}$$

Hence, the equation of motion for the holomorphic case is

$$\partial_\tau \psi + i\partial_\sigma \psi = \dot{a}b + ib'a = 0 \tag{2.82}$$

So that division by ab gives

$$\frac{\dot{a}}{a} = -i\frac{b'}{b} \tag{2.83}$$

which must be true for all values of τ and σ. Since τ and σ are independent variables, the only way for the two sides to be equal for all τ and σ is if they are equal to a constant. For convenience this constant is set equal to -k so that $\dot{a} = -ka$ and $b' = -ik\sigma$.

The solutions to these equations are

$$\dot{a} = -k\alpha_k e^{-k\tau} = -ka \tag{2.84}$$

and

$$b' = -ik\beta_k e^{-ik\sigma} = -ikb \tag{2.85}$$

The equation of motion for a particular value of k is

$$\psi(\tau, \sigma) = a(\tau) b(\sigma) = \psi_k e^{-k\tau - ik\sigma} \tag{2.86}$$

where $\psi_k = \alpha_k \beta_k$.

Recalling the periodic boundary condition given by

$$\psi(\tau, \sigma + 2\pi) = \psi(\tau, \sigma) \tag{2.87}$$

lead to

$$\psi_k e^{-k\tau - ik\sigma - 2\pi ik} = \psi_k e^{-k\tau - ik\sigma} \iff e^{-2\pi ik} = 1. \tag{2.88}$$

which only holds true if k is an integer.

The general solution to the equation of motion is given by equation 2.86 summed over all the integers k

$$\psi = \sqrt{-i}\sum_{k\in\mathbb{Z}} \psi_k e^{-k\tau - ik\sigma} \tag{2.89}$$

where a normalisation constant of $\sqrt{-i}$ has been chosen for later convenience.

The oscillation modes at equal times are considered so that τ is set equal to zero and

$$\psi = \sqrt{-i}\sum_{k\in\mathbb{Z}} \psi_k e^{-ik\sigma}. \tag{2.90}$$

From equation 2.80 and equation 2.81 the equation of motion for the antiholomorphic case is

$$\partial_\tau \tilde{\psi} - i\partial_\sigma \tilde{\psi} = \dot{a}b - ib'a = 0 \tag{2.91}$$

Following the same method as for the holomorphic case, the equation of motion for a particular value of k is

$$\psi(\tau,\sigma) = a(\tau)b(\sigma) = \psi_k e^{-k\tau + ik\sigma} \tag{2.92}$$

which gives

$$\tilde{\psi} = \sqrt{-i}\sum_{k\in\mathbb{Z}} \tilde{\psi}_k e^{ik\sigma} \tag{2.93}$$

where constraints at equal times are again applied.

2.2.4 Oscillation modes

To find the an expression for the oscillation mode ψ_k in terms of ψ, both sides of equation 2.90 are multiplied by $e^{ik'\sigma}$ and the result is integrated over sigma between 0 and 2π:

$$\begin{aligned}\int_0^{2\pi} \psi e^{ik'\sigma} d\sigma &= \int_0^{2\pi} \sqrt{-i}\sum_{k\in\mathbb{Z}} \psi_k e^{-ik\sigma} e^{ik'\sigma} d\sigma \\ &= \sqrt{-i}\sum_{k\in\mathbb{Z}} \psi_k \int_0^{2\pi} e^{i(k'-k)\sigma} d\sigma \end{aligned} \tag{2.94}$$

If $k = k'$

$$\begin{aligned}\int_0^{2\pi} \psi e^{ik'\sigma} d\sigma &= \sqrt{-i} \sum_{k \in \mathbb{Z}} \psi_k \int_0^{2\pi} d\sigma \\ &= 2\pi\sqrt{-i} \sum_{k \in \mathbb{Z}} \psi_k (1) \end{aligned} \tag{2.95}$$

if $k \neq k'$

$$\begin{aligned}\int_0^{2\pi} \psi e^{ik'\sigma} d\sigma &= \sqrt{-i} \sum_{k \in \mathbb{Z}} \psi_k \left. \frac{e^{i(k'-k)\sigma}}{i(k'-k)} \right|_0^{2\pi} \\ &= \sqrt{-i} \sum_{k \in \mathbb{Z}} \psi_k \frac{e^{2\pi(k'-k)\sigma} - 1}{i(k'-k)} \\ &= 2\pi\sqrt{-i} \sum_{k \in \mathbb{Z}} \psi_k (0) \end{aligned} \tag{2.96}$$

Thus,

$$\begin{aligned}\int_0^{2\pi} \psi e^{ik'\sigma} d\sigma &= 2\pi\sqrt{-i} \sum_{k \in \mathbb{Z}} \psi_k \begin{cases} 1 & k = k' \\ 0 & k \neq k' \end{cases} \\ &= 2\pi\sqrt{-i} \sum_{k \in \mathbb{Z}} \psi_k \delta_{k,k'} \\ &= 2\pi\sqrt{-i}\psi k' \end{aligned} \tag{2.97}$$

This equation holds for any integer k' and, for simplicity k' is labelled k. The resulting equation for ψ_k is given by

$$\psi_k = \frac{1}{2\pi\sqrt{-i}} \int_0^{2\pi} \psi e^{ik\sigma} d\sigma \tag{2.98}$$

To find an expression for the oscillation mode $\tilde{\psi}_k$ in terms of ψ, both sides of equation 2.93 are multiplied by $e^{-ik'\sigma}$ and the result is integrated over sigma between 0 and 2π. This gives

$$\int_0^{2\pi} \tilde{\psi} e^{-ik'\sigma} d\sigma = 2\pi\sqrt{-i}\, \tilde{\psi} k' \tag{2.99}$$

where for simplicity k' is labelled k so that the resulting equation for $\tilde{\psi}_k$ is given by

$$\tilde{\psi}_k = \frac{1}{2\pi\sqrt{-i}} \int_0^{2\pi} \tilde{\psi} e^{-ik\sigma} d\sigma. \tag{2.100}$$

Using the quantisation rules given by the anticommutators

$$\begin{aligned}
\{\psi(\sigma), \psi(\sigma')\} &= -2\pi i \delta(\sigma - \sigma') \\
&= \left\{ \frac{1}{2\pi\sqrt{-i}} \int_0^{2\pi} \psi(\sigma) e^{ik\sigma} d\sigma, \frac{1}{2\pi\sqrt{-i}} \int_0^{2\pi} \psi(\sigma') e^{ik'\sigma'} d\sigma' \right\} \\
&= \frac{i}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \{\psi(\sigma), \psi(\sigma')\} e^{ik\sigma + ik'\sigma'} d\sigma d\sigma' \\
&= \frac{i}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} (-2\pi i \delta(\sigma - \sigma')) e^{ik\sigma + ik'\sigma'} d\sigma d\sigma' \\
&= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \delta(\sigma - \sigma') e^{ik\sigma + ik'\sigma'} d\sigma d\sigma' \\
&= \frac{1}{2\pi} \int_0^{2\pi} e^{i(k+k')\sigma} d\sigma \\
&= \frac{1}{2\pi} 2\pi e^{i(k+k')\sigma} \\
&= \delta_{k,-k'}
\end{aligned} \tag{2.101}$$

which applies for any k and k'. The remaining anticommutators are computed similarly and are given by

$$\{\tilde{\psi}_k, \tilde{\psi}_{-k}\} = 1, \tag{2.102}$$

$$\{\tilde{\psi}_k, \tilde{\psi}'_k\} = \delta_{k,-k'} \tag{2.103}$$

and the rest of the anticommutators are equal to zero. These quantisation rules can be used to compute the partition function.

2.2.5 Ramond and NS sectors

For Fermionic Fields, Lorentz invariance allows for periodic and anti-periodic boundary conditions. These boundary conditions have associated sectors. The periodic conditions correspond to the Ramond (R) sector and the anti-periodic conditions correspond to the Neveu-Schwarz (NS) sector.

As in the bosonic case, the partition functions for the fermionic case is written using the trace function. Consider the fermionic partition functions on the Ramond sector given by

$$Z = \mathrm{Tr}[(-1)^{F_L+F_R} e^{-\beta_L \mathcal{H}_L - \beta_R \mathcal{H}_R}]. \tag{2.104}$$

where $(-1)^F = 1$ if the state is unoccupied by an oscillator and $(-1)^F = -1$ if the state is occupied by an oscillator.

The general form of the partition function is given by

$$Z = \mathrm{Tr}[(-1)^{\mathrm{F}} \mathrm{e}^{-\beta\mathcal{H}}] = \mathrm{e}^{-\frac{\beta}{24}} \prod_{\mathrm{k}=1}^{\infty} (1 - \mathrm{e}^{-\beta \mathrm{k}}) \tag{2.105}$$

where the value of k runs over the integers and

$$Z = \mathrm{Tr}[(-1)^{\mathrm{F_L+F_R}} \mathrm{e}^{-\beta_\mathrm{L}\mathcal{H}_\mathrm{L} - \beta_\mathrm{R}\mathcal{H}_\mathrm{R}}] = \mathrm{e}^{-\frac{\beta_\mathrm{L}}{24}} \prod_{\mathrm{k}=1}^{\infty} (1 - \mathrm{e}^{-\beta_\mathrm{L}\mathrm{k}}) \mathrm{e}^{-\frac{\beta_\mathrm{R}}{24}} \prod_{\mathrm{k}=1}^{\infty} (1 - \mathrm{e}^{-\beta_\mathrm{R}\mathrm{k}}) \tag{2.106}$$

where β_L and β_R are defined as before so that

$$Z = e^{-\frac{2\pi i \tau}{24}} \prod_{k=1}^{\infty} (1 - e^{-2\pi i \tau k}) e^{\frac{2\pi i \bar{\tau}}{24}} \prod_{k=1}^{\infty} (1 - e^{2\pi i \bar{\tau} k}). \tag{2.107}$$

In terms of q and $\bar{q}$

$$Z = \frac{1}{q^{\frac{1}{24}} \prod_{k=1}^{\infty} (1 - q^k)} \frac{1}{\bar{q}^{\frac{1}{24}} \prod_{k=1}^{\infty} (1 - \bar{q}^k)} \tag{2.108}$$

and in terms of the Dedekind η function

$$\eta(q) = \frac{1}{q^{\frac{1}{24}} \prod_{k=1}^{\infty} (1 - q^k)}, \tag{2.109}$$

the fermionic partition function in the Ramond sector is given by

$$Z_R = \eta(q)\eta(\bar{q}). \tag{2.110}$$

which is the inverse of the bosonic periodic case, where it has been assumed in the solution that no zero modes are present.

For the NS sector with anti-periodic boundary conditions, the value of the range of k must be changed. Here k runs over half integers, so will be replaced with $k - 1/2$ where k $\in \mathbb{Z}$.

Similarly, the partition function is given by

$$Z = \mathrm{Tr}[(-1)^{\mathrm{F_L}+\mathrm{F_R}}\mathrm{e}^{-\beta_\mathrm{L}\mathcal{H}_\mathrm{L}-\beta_\mathrm{R}\mathcal{H}_\mathrm{R}}] \tag{2.111}$$

$$Z = e^{-\frac{\beta_L}{48}}\prod_{k=1}^{\infty}(1-e^{-\beta_L(k-1/2)})e^{-\frac{\beta_R}{48}}\prod_{k=1}^{\infty}(1-e^{-\beta_R(k-1/2)}) \tag{2.112}$$

Chapter 3

Elliptic genus and twisted elliptic genus of K3

The elliptic genus of $K3$ is calculated via Jacobi forms, derived by adding a chemical potential to the partition functions of chapter 2. Jacobi forms combine the concept of both modular forms and elliptical functions.

3.1 The modular group and modular forms

The definition of the modular group follows from the transformation

$$\tau \to \frac{a\tau + b}{c\tau + d}. \tag{3.1}$$

This is called a unimodular transformation if a,b,c and d are integers and if $ad - bc = 1$ [29]. The set of all unimodular transformations form a group called the modular group which is denoted Γ.

The modular group Γ is represented by a two-by-two integer matrix

$$\mathcal{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.2}$$

where det$\mathcal{A} = 1$. The modular group Γ is generated by two matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.3}$$

From this, a modular form can be defined. A function f is a modular form if it satisfies 3 specific conditions [29]. The first is that f is meromorphic in the upper half complex plane H. That is, f is of the form

$$f(z) = \frac{g(z)}{h(z)} \tag{3.4}$$

where $g(z)$ and $h(z)$ are complex functions that are analytical at the point $i\infty$, and at all finite points of the upper half of the complex plane H. The second condition is that $f(\mathcal{A}\tau) = F(\mathcal{A})f(\tau)$ for every $\mathcal{A}$ representing the modular group Γ and $F(\mathcal{A}) \in \mathbb{C}$ is an analytic function given for each (A). The final condition is that the Laurent series of f, which is a representation of complex f as as power series, should have the form

$$f(\tau) = \sum_{n=-m}^{\infty} a(n)e^{2\pi i n\tau}. \tag{3.5}$$

3.2 Elliptic functions

An elliptic function is defined as a doubly periodic function with periods $2\omega_1$ and $2\omega_2$ such that

$$f(z + 2\omega_1) = f(z + 2\omega_2) = f(z) \tag{3.6}$$

which is analytic and has no singularities except in the finite part of the complex plane [29], where they are called poles. Elliptic functions obey a number of properties involving the cell which is a parallelogram region in the complex plane in which the function f is single-valued [29].

To be an elliptic function, the number of roots (or 'zeros') in a cell must be finite, the number of poles (complex singularities) in a cell should also be finite. An elliptic function with no poles in a cell is a constant (by Liouville's first theorem [29]).

3.3 Jacobi forms and Jacobi theta functions

As has been highlighted, adding a chemical potential to the partition functions of bosons and fermions on the torus studied in Chapter 2 leads to Jacobi forms. Jacobi forms are a mixture of modular forms and elliptic functions. Jacobi forms have two variables and as such have two integers associated with them. The first of these integers is the weight k which describes the transformation properties of the Jacobi form with respect to the modular group. The second integer is the index m which describes the transformation properties with respect to the elliptic variable [28].

A Jacobi form of weight k and index m is formally defined as a function $\phi(\tau, z)$ of two complex variables where the variable τ is on the upper half plane such that

$$\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e^{\frac{2\pi imcz^2}{c\tau+d}}\phi(\tau,z) \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \tag{3.7a}$$

and

$$\phi(\tau, z+\lambda\tau+\mu) = e^{-2\pi im(\lambda^2\tau+2\lambda z)}\phi(\tau,z) \text{ for all integers } \lambda \text{ and } \mu. \tag{3.7b}$$

where ϕ has a Fourier expansion

$$\phi(\tau,z) = \sum_{n\geq 0}\sum_{r^2\leq 4mn} C(n,r)e^{2\pi(n\tau+rz)}, \tag{3.7c}$$

Jacobi theta functions are derived from the generalised theta function, which for $\tau \in \mathbb{H}$ and $\nu \in \mathbb{C}$ is given by

$$\theta_{a,b}(\tau,\nu) = \sum_{\mathbb{Z}} e^{\pi i\tau(n+a)^2}e^{2\pi i(n+a)(\nu+b)} \tag{3.8}$$

From this, the Jacobi theta functions are defined as

$$\begin{aligned}\theta_1(\tau,\nu) &= -\theta_{\frac{1}{2},\frac{1}{2}}(\tau,\nu) \\ &= -i\sum_{\mathbb{Z}}(-1)^n q^{\frac{1}{2}(n+\frac{1}{2})^2}e^{\pi i(2n+1)\nu}\end{aligned} \tag{3.9a}$$

$$\theta_2(\tau,\nu) = \theta_{\frac{1}{2},0}(\tau,\nu)$$

$$= \sum_{\mathbb{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2} e^{\pi i(2n+1)\nu} \tag{3.9b}$$

$$\theta_3(\tau,\nu) = \theta_{0,0}(\tau,\nu)$$

$$= \sum_{\mathbb{Z}} q^{\frac{1}{2}n^2} e^{2\pi i n\nu} \tag{3.9c}$$

$$\theta_4(\tau,\nu) = \theta_{0,\frac{1}{2}}(\tau,\nu)$$

$$= \sum_{\mathbb{Z}} (-1)^n q^{\frac{1}{2}n^2} e^{2\pi i n\nu}. \tag{3.9d}$$

The Jacobi theta function equations are used in the calculation of the Elliptic genus of $K3$.

3.4 Elliptic Genus of K3

A genus is an integer representing the maximum number of cuttings along non-intersecting closed simple curves without a resultant manifold being disconnected [31]. Intuitively, a genus is the number of 'holes' in a surface [32]. For example, a sphere has a genus of zero whilst a torus has a genus of 1.

In mathematics, a $K3$ surface is a four dimensional manifold. It is a topological space in which each point has neighbourhood one-to-one mapping to the four dimensional Euclidean space $\mathbb{R}^4$. Thus, $K3$ can be described as a $\mathbb{Z}_2$ orbifold (which is a generalised manifold [34]) of the four dimensional torus T^4 [16].

The computation of the elliptic genus of $K3$ on the torus T^4 required the compactification of 2 complex bosons and 2 complex fermions on the torus T^4. For this reason, the elliptic genus of $K3$ can be written in terms of the Jacobi theta functions defined in Section 3.3 and derived from the fermionic and bosonic partition functions of Chapter 2.

The elliptic genus of K3 is given by the following trace

$$Z_{K3}(\tau,z) = \frac{1}{2} \sum_{\substack{a,b=0 \\ (a,b)\neq(0,0)}} \mathrm{Tr}_{\mathrm{RR}^{\mathrm{a}}_{\mathrm{g}}} \left[(-1)^{\mathrm{F}\mathrm{K}_3\mathrm{v}+\bar{\mathrm{F}}_{\mathrm{K}_3}} \mathrm{g}^{\mathrm{b}} \mathrm{e}^{2\pi\mathrm{i}\nu\mathrm{J}\mathrm{K}_3} \mathrm{q}^{\mathrm{L}_0-\mathrm{c}/24} \bar{\mathrm{q}}^{\bar{\mathrm{L}}_0-\bar{\mathrm{c}}/24} \right] \tag{3.10}$$

where $(c,\bar{c})$ are the central charges of the system and $c=6$ and $\bar{c}=6$ for $K3$. L_0 and $\bar{L}_0$ are the left and right Hamiltonian's with zero-point energy subtracted out. For fermions, J_{k3} represents the angular momentum and can take values of ± 1 [23].

To compute the elliptic genus of $K3$, the Jacobi theta functions are defined in terms of $z = e^{2\pi i\nu}$. The formula for the elliptic genus of $K3$ becomes

$$\chi(q, z) = 8\left(\frac{\theta_2^2(q, z)}{\theta_2^2(q, 1)} + \frac{\theta_3^2(q, z)}{\theta_3^2(q, 1)} + \frac{\theta_4^2(q, z)}{\theta_4^2(q, 1)}\right) \tag{3.11}$$

where the first term arises from the untwisted projected partition function, the second from the twisted unprojected partition function and the third from the twisted projected partition function [16].

$A(q, z)$ is a Jacobi form of index $m = 1$ and weight $k = 0$ and is given by

$$A(q, z) = \left(\frac{\theta_2^2(q, z)}{\theta_2^2(q, 1)} + \frac{\theta_3^2(q, z)}{\theta_3^2(q, 1)} + \frac{\theta_4^2(q, z)}{\theta_4^2(q, 1)}\right) \tag{3.12}$$

so that the elliptic genus of $K3$ is equal to $8A(q, z)$.

Following computation of the Fourier series expansion of the elliptic genus of $K3$ using the Jacobi theta functions, the first three terms in the elliptic genus of $K3$ are explicitly are given by

$$\chi(q, z) = 20 + \frac{2}{z} + 2z + ... \tag{3.13}$$

The evaluation of the elliptic genus of $K3$ is an important result for the calculation of the degeneracy of black hole microstates investigated in Chapter 4.

Chapter 4

Degeneracy of Black Hole Microstates

As already discussed in Chapter 1, the entropy of the black hole is known as Bekenstein-Hawking entropy and is given by

$$S_{BH} = \frac{A_H}{4} \tag{4.1}$$

where A_H is the area of the event horizon. An important question to consider relates to how this entropy may be understood from a statistical viewpoint. The statistical definition of entropy was developed by Boltzmann who showed that the definition of statistical entropy is equivalent to the definition of thermodynamic entropy within a constant factor, known as the Boltzmann constant. Specifically, entropy is a logarithmic measure of the number of microstates of a system:

$$S_{stat} = k_B \ln \Omega \tag{4.2}$$

In the same way, can the entropy of a black hole be understood from a statistical viewpoint as a logarithm of the number of states associated with the black hole [19]?

For a special class of black holes in string theory, known as $\mathcal{N} = 4$ supersymmetric BPS black holes in type II string theory, the answer to this question is complete [19]. This class of black holes do not Hawking radiate since they have zero temperature and are usually stable. They are also invariant under certain supersymmetry transformations. The stability and supersymmetry properties of this class of black holes allows a certain amount of control over its microscopic configuration. This allows the calculation of the degeneracy of the black hole. In string theory, in the limit where the $\mathcal{N} = 4$ supersymmetric BPS black hole is large,

$$S_{BH} = S_{stat} \tag{4.3}$$

where S_{stat} is the statistical entropy for black holes and is given by

$$S_{stat} = -\ln B_6. \tag{4.4}$$

The index $-B_6$ associated with the specific class of $\mathcal{N} = 4$ supersymmetric BPS black holes in type II string theory is considered [35]. This index is equal to the degeneracy of the BPS black hole microstates which is given by $d(Q, P)$ so that

$$S_{stat} = \ln d(Q, P) \tag{4.5}$$

where Q and P are the electric and magnetic charge vectors respectively. This degeneracy gives an understanding of the Bekenskein-Hawking entropy from a microscopic perspective.

The degeneracy of the black hole microstates is given by the integral

$$d(Q, P) = (-1)^{Q.P+1} \int_C e^{-\pi i(\rho Q^2 + \sigma \rho^2 + 2\nu Q.P)} \frac{1}{\tilde{\Phi}(\rho, \sigma, \nu)} d\rho d\sigma d\nu \tag{4.6}$$

where the contour C is defined over a three dimensional subspace of the three complex dimensional space labelled by $\rho = \rho_1 + i\rho_2$, $\sigma = \sigma_1 + i\sigma_2$ and $\nu = \nu_1 + i\nu_2$ [19].

The degeneracy calculation $d(Q, P)$ is extracted from the Fourier coefficients of $1/\tilde{\Phi}$. Thus, the expansion of $1/\tilde{\Phi}$ should be applied in powers of $e^{2\pi i\rho}$ and $e^{2\pi i\sigma}$, then each term expanded in powers of $e^{-2\pi i\nu}$. This is best done using the product representation of $\tilde{\Phi}$ given by a particular Siegel modular form [35], which is a generalisation of the conventional elliptic modular form.

It is given by

$$\tilde{\Phi}(\rho,\sigma,\nu) = e^{-2\pi i(\rho+\sigma/N+\nu)} \times \prod_{b=0}^{1}\prod_{r=0}^{N-1} \prod_{\substack{k\in\mathbb{Z}+\frac{r}{N}, l\in\mathbb{Z}, j\in 2\mathbb{Z}+b \\ k,l\geq 0 for k=l=0}} (1-e^{2\pi i(k\sigma+l\rho+j\nu)})^{-\sum_{s=0}^{N-1} e^{-2\pi i s l/N} c^{(r,s)}(4kl-j^2)} \tag{4.7}$$

where the coefficients $c^{r,s}$ are the Fourier coefficients of $F^{r,s}(\tau,z)$ defined as

$$F^{(0,0)}(\tau,z) = \frac{8}{N}A(\tau,z), \tag{4.8a}$$

$$F^{(0,s)}(\tau,z) = \frac{8}{N(N+1)}A(\tau,z) - \frac{2}{N+1}B(\tau,z)E_N(z) \quad \text{for } 1\leq s\leq (N-1) \tag{4.8b}$$

and

$$F^{(r,rk)}(\tau,z) = \frac{8}{N(N+1)}A(\tau,z) + \frac{2}{N+1}E_N(\frac{\tau+k}{N})B(\tau,z) \quad \text{for } 1\leq r\leq (N-1), 0\leq k\leq (N-1). \tag{4.8c}$$

The $\tilde{\Phi}$ written in terms of $\tilde{\alpha}$ and $\tilde{\beta}$ is given by

$$\frac{1}{\tilde{\Phi}(\rho,\sigma,\nu)} = e^{-2\pi i(\tilde{\alpha}\rho+\tilde{\beta}\sigma+\nu)} \times \prod_{b=0}^{1}\prod_{r=0}^{N-1} \prod_{\substack{k\in\mathbb{Z}+\frac{r}{N}, l\in\mathbb{Z}, j\in 2\mathbb{Z}+b \\ k,l\geq 0 for k=l=0}} (1-e^{2\pi i(k\sigma+l\rho+j\nu)})^{-\sum_{s=0}^{N-1} e^{-2\pi i s l/N} c^{(r,s)}(4kl-j^2)} \tag{4.9}$$

where $\tilde{\alpha}=1$ and $\tilde{\beta}=\frac{1}{N}$. For the $N=1$ model, $s=r=0$. Thus equation 4.9 becomes

$$\frac{1}{\tilde{\Phi}(\rho,\sigma,\nu)} = e^{-2\pi i(\rho+\sigma+\nu)} \prod_{\substack{k,l,j\in\mathbb{Z} \\ k,l\geq 0 for k=l=0}} (1-e^{2\pi i(k\sigma+l\rho+j\nu)})^{-\sum_{s=0}^{N-1} c^{(0,0)}(4kl-j^2)}. \tag{4.10}$$

The coefficients for $c^{(0,0)}$ are defined as

$$F^{(0,0)}(\tau,z) = \sum_{n,j\in\mathbb{Z}} c^{(0,0)}(4n-j^2)q^n z^j \tag{4.11}$$

$$= 8A(\tau,z) \tag{4.12}$$

which are the coefficients of the elliptic genus of $K3$, computed already in Chapter 3.

Using the Fourier expansion

$$\frac{1}{\tilde{\Phi}(\rho,\sigma,\nu)} = \sum_{m,n,p} g(m,n,p) e^{2\pi i(m\rho+n\sigma+p\nu)}, \tag{4.13}$$

the degeneracy of the black hole microstates in equation 4.6 is given by

$$\begin{aligned} d(Q,P) &= (-1)^{Q.P+1} \sum_{m,n,p} g(m,n,p) \int_C e^{2\pi i(m\rho+n\sigma+p\nu)} e^{-\pi i(\rho Q^2+\sigma\rho^2+2\nu Q.P)} d\rho d\sigma d\nu \\ &= (-1)^{Q.P+1} \sum_{m,n,p} g(m,n,p) \int_C e^{2\pi i((m-Q^2/2)\rho+(n-P^2/2)\sigma+(p-Q.P/2)\nu})d\rho d\sigma d\nu \\ &= (-1)^{Q.P+1} \sum_{m,n,p} g(m,n,p) \int_C e^{2\pi i(m\rho+n\sigma+p\nu)} e^{-\pi i(\rho Q^2+\sigma\rho^2+2\nu Q.P)} d\rho d\sigma d\nu \\ &= (-1)^{Q.P+1} \sum_{m,n,p} g(\frac{Q^2}{2},\frac{P^2}{2},Q.P) \end{aligned} \tag{4.14}$$

Defining $x = e^{2\pi i\rho}$, $y = e^{2\pi i\sigma}$ and $w = e^{2\pi i\nu}$ equation 4.13 gives

$$\frac{1}{\tilde{\Phi}(\rho,\sigma,\nu)} = \sum_{m,n,p} g(m,n,p) x^m y^n w^p \tag{4.15a}$$

$$= (xyw)^{-1} \prod_{\substack{k,l,j\in\mathbb{Z} \\ k,l\geq 0 for k=l=0}} (1-x^k y^l w^j)^{-c_b^{(0,0)}(4kl-j^2)} \tag{4.15b}$$

The expansion of equation 4.15b around $x = 0$, $y = 0$ and $w = \infty$ determines the coefficient of $x^m y^n w^p$, hence determines g(m,n,p). To do this computationally, three separate cases should be considered. First all cases are included, except where $k = 0$, then all cases are included, except where $l = 0$. The remaining special case of $k = l = 0$ is considered, where $j < 0$.

The resulting computed coefficient expansions give the degeneracy of the black hole microstates for different values of Q^2, P^2 and $Q.P$ which are given in Table 4.1 below.

$(Q^2,P^2)\backslash Q.P$	-2	0	1	2	3	4
(2,2)	-209304	50064	25353	648	327	0
(2,4)	-2023536	1127472	561576	50064	8376	-648
(4,4)	-16620544	32861184	18458000	3859456	561576	12800
(2,6)	-15493728	16491600	8533821	1127472	130329	-15600
(4,6)	-53249700	632078672	392427528	110910300	18458000	1127472
(2,8)	-97722288	185738352	100390104	16491600	1598376	-209304
(4,8)	945811584	9337042944	6216536784	2073849984	392427528	32861184
(6,6)	2857656828	16193130552	11232685725	4173501828	920577636	110910300
(6,8)	91631080464	315614079072	233641003920	100673013264	26563753008	4173501828

Table 4.1: Computed results for the the degeneracy of $\mathcal{N} = 4$ supersymmetric BPS black holes in type II string theory for specific values of Q^2, P^2 and $Q.P$.

The results of Table 4.1 signify an important result in the relation between string theory and black holes. The entropy of supersymmetric $\mathcal{N} = 4$ BPS black holes in type II string theory can be represented as a logarithm of the counted microstates of the black hole. This is in line with both statistical mechanics and classical thermodynamics.

Chapter 5

Conclusions

The investigation and application of string theory in supersymmetric black holes has been studied. The central results of this work is found in table 4.1 where the entropy of supersymmetric $\mathcal{N} = 4$ supersymmetric black holes in type II string theory was computed. This result followed from the calculation of the elliptic genus of $K3$, derived from the partition functions for the fermionic string. From the outset, this report presented the fundamental principles involved in string theory and supersymmetric black holes. String theory was described and its application to black holes was discussed, then applied practically.

At the conclusion of this report, the reader should have an understanding not only of black holes and string theory, but of action principles, partition functions, oscillator modes, Jacobi theta functions, degeneracy of black hole microstates and many other mathematical concepts. The knowledge and experience I have gained from writing this report cannot be understated. My favourite part of this project was the study of black hole mechanics and its relation to thermodynamics. In the future, I would like to complete a comprehensive study of black holes thermodynamics in general relativity and gain understanding of how these laws can constrain the physics, even beyond general relativity.

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