Beyond Dirac Fermions

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Abstract

The dynamics of all but one (neutrino) of the fermions in the standard model are described by the Dirac equation. Other equations proposed by Majorana and Weyl extend the reach of the Dirac equation to describe particles, albeit yet to be observed, that may exist as emergent properties of condensed matter systems. In this investigation, we construct the Hamiltonian describing Dirac-like fermions in $3+1$ dimensions, whose wavefunctions satisfy $SO(3)$ symmetry, rather than the $SO(4)$ symmetry of Dirac. We determine these Hamiltonians for n-component spinors with $5 \leq n \leq 8$, and write down the matrix form of the spin projection operators for the spin-j representation of $SO(3)$. The Hamiltonians developed yield the same eigenvalue shape as those of Dirac, that is, $\pm\sqrt{p_1^2+p_2^2+p_3^2+m_1^2}$, however, we also obtain eigenvalues corresponding to a second, independent mass; $\pm\sqrt{p_1^2+p_2^2+p_3^2+m_2^2}$. For odd n, we account for the odd number of eigenvalues by demanding a zero eigenvalue, resulting in the theory not obeying Lorentz invariance. In one case ($n=8$), we obtained a Hamiltonian that produced this form of eigenvalue for three independent masses.

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1 Introduction

In the introductory section, we give a brief overview of Dirac's theory for describing spin-half particles including a discussion on the components of the Dirac Hamiltonian and Lagrangian, as well as the eigenvalues of the Hamiltonian. Some elements of the second quantisation of the Dirac field are discussed that solve some of the teething problems of Dirac's theory. Then, in Section 2, we give a description of the problem investigated in this report. We discuss the methods used to solve this problem in Section 3, and present our results in Section 4. Finally, a discussion on the consequences of our findings and a summary and conclusion will be provided in Sections 5 and 6 respectively.

Dirac's theory describing spin-1/2 fermions proved very successful, predicting the existence of the positron, discovered experimentally by Carl Anderson in 1932, and contributing to Dirac's Nobel prize the following year. Dirac's theory of fermions is capable of describing every fermion present in the standard model (with the exception of the nuetrino), and hence is an essential tool for the development and investigation of fundamental particle physics. Not only this, quasiparticles have been observed in low energy condensed matter systems, such as graphene, that obey fermionic statistics, thus qualifying as Dirac fermions.

The goal of this investigation was to go beyond the Dirac picture and build models for unconventional Dirac-like fermions. Some extended models already exist à la Majorana, who constructed a theory describing fermions with real wavefunctions meaning that they are their own antiparticle, and Weyl, whose theory describes massless fermions. Although no known fundamental particles are confirmed to obey either of these models, there is much optimism that they will be discovered in condensed matter systems as emergent quasiparticles. If successful, Majorana fermions are a serious contender for building qubits in topological quantum computing. Because of their topological nature, they are not sensitive to local perturbations causing decoherence and hence may solve the problem of quantum error correction at the hardware level – an exciting possibility!

1.1 Dirac Theory

The Dirac equation in $3+1$ dimensions is,

$$\left(i\gamma^\mu\partial_\mu - \frac{mc}{\hbar}\right)\psi = 0,$$

where the solutions ψ are spinors on 4 dimensions (as a result of working in spacetime) describing the wavefunction of an electron, and transform according to the $SO(4)$ group. Since the spinors are vectors in $\mathbb{R}^4$, the resulting γ^μ matrices are 4×4. Under this formalism, these "gamma" matrices (of which there are four) are defined by,

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

in the so-called Dirac basis. They satisfy the anti-commutation relation,

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} I_4,$$

and generate the Clifford algebra. The Hamiltonian density for a Dirac fermion with mass m can be defined in terms of the gamma matrices as,

$$\mathcal{H} = \gamma^i p_i + \gamma^0 m, \quad (i = 1, 2, 3)\,, \tag{1.1.1}$$

with Lagrangian density,

$$\mathcal{L} = -c\hbar\bar{\psi}\gamma^\mu\partial_\mu\psi - mc^2\bar{\psi}\psi. \tag{1.1.2}$$

Here ∂_μ represents the 4-dimensional derivative and $\bar{\psi}$ is the adjoint field to ψ. The spinors transform as:

$$\psi' = S(\Lambda)\psi,$$

where $S(\Lambda)$ is a representation of the Lorentz group. To ensure the Dirac equation remains covariant, we must enforce the following relation:

$$S(\Lambda)\gamma^\mu S^{-1}(\Lambda) = \left(\Lambda^{-1}\right)^\mu{}_\nu\,\gamma^\nu. \tag{1.1.3}$$

Defining the adjoint field as $\bar{\psi} = \psi^\dagger\gamma^0$, its Lorentz transformation follows as $\bar{\psi}' = \bar{\psi}S^{-1}(\Lambda)$. From these definitions, the quantity $\bar{\psi}\psi$ is a scalar invariant under the Lorentz group;

$$\bar{\psi}'\psi' = \psi^\dagger\Lambda^\mu{}_\nu{}^{-1}\gamma^0\Lambda^\mu{}_\nu\psi = \psi^\dagger\gamma^0\Lambda^\mu{}_\nu{}^{-1}\Lambda^\mu{}_\nu\psi = \psi^\dagger\gamma^0\psi = \bar{\psi}\psi, \tag{1.1.4}$$

and using the relation given by Equation 1.1.3,

$$\bar{\psi}'\gamma^\mu\psi' = \bar{\psi}S^{-1}(\Lambda)\gamma^\mu S(\Lambda)\psi = \bar{\psi}\Lambda^\mu{}_\nu\gamma^\nu\psi = \bar{\psi}\gamma^\mu\psi. \tag{1.1.5}$$

Since the final expressions in Equations 1.1.5 and 1.1.4 are exactly those (bar constants) that appear in the Lagrangian in Equation 1.1.2, $\mathcal{L}$ is also invariant under a Lorentz transformation.

When diagonalised, the Hamiltonian given by Equation (1.1.1) yields the eigenvalues,

$$E = \pm\sqrt{p_1^2 + p_2^2 + p_3^2 + m^2},$$

each with degeneracy 2. This is the standard expression for the energy of a relativistic particle. Despite Dirac's best efforts to remove the negative energy solutions of the relativistic, spinless Klein-Gordon equation by linearising its Hamiltonian, they re-emerge unharmed. This may have been OK for a quantum theory (the Dirac sea interpretation), but upon taking the non-relativistic limit of the solutions to the Dirac equation, the negative energy solutions remained – negative energy of a free particle is not allowed in Classical physics! In the Klein-Gordon model, the negative energy solutions could be ignored because the positive energy solutions formed a complete basis for all solutions – this is not the case in Dirac. It wasn't until 1949, when the Foldy-Wouthuysen transformation (FW) was developed, that we could separate the negative and positive solutions, and upon taking the non-relativistic limit, have the negative energies tend to zero. This FW transformation consisted of a unitary transformation of the Dirac spinors such that one pair of components ($U^{(1)}$ and $U^{(2)}$) corresponded to the positive energy eigenvalues, and the other pair ($U^{(3)}$ and $U^{(4)}$) corresponded to the negative eigenvalues;

$$U_{\text{FW}} = \begin{pmatrix} U^{(1)} \\ U^{(2)} \\ U^{(3)} \\ U^{(4)} \end{pmatrix}$$

Interpretation of the negative energy solutions outside the classical limit is still an issue however, whose genius solution we shall discuss now. Upon second quantisation of the Dirac field, $\psi(\mathbf{r}, t)$, it could be decomposed into a Fourier series given by:

$$\psi(\mathbf{r}, t) = \frac{1}{\sqrt{V}} \sum_{i=1}^{2} \int d^3\mathbf{p} \sqrt{\frac{mc^2}{E_{\mathbf{p}}}} \left[a_{\mathbf{p}}^{(i)}(t) U^{(i)}(\mathbf{p})\, e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{r}} + b_{\mathbf{p}}^{(i)}(t)^{\dagger} V^{(i)}(\mathbf{p})\, e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{r}} \right],$$

where the summation is over possible spins (up and down) and the integration is taken over all momenta of the electrons. There is a lot to unpack from this equation; the factor $1/\sqrt{V}$ is a normalisation constant over a quantisation volume V, $E_{\mathbf{p}}$ is the energy associated with momentum state $\mathbf{p}$, $U^{(i)}(\mathbf{p})$ and $V^{(i)}(\mathbf{p})$ are the i^{th} components of the FW transformed spinors[1] and the exponential terms are the space components of the plane wave solutions to the Dirac equation. However, the most important parts are the factors $a_{\mathbf{p}}^{(i)}(t)$ and $b_{\mathbf{p}}^{(i)}(t)^\dagger$, which are the Fourier amplitudes that have been promoted to the role of operators. Upon selection of an appropriate charge operator, Q, we find that the total charge in the volume V is,

$$Q(t) = q_e \sum_{i=1}^{2} \int d^3\mathbf{p} \Big[\underbrace{a_{\mathbf{p}}^{(i)}(t)^\dagger a_{\mathbf{p}}^{(i)}(t)}_{N_e} - \underbrace{b_{\mathbf{p}}^{(i)}(t)^\dagger b_{\mathbf{p}}^{(i)}(t)}_{N_p} + \mathbb{1} \Big], \tag{1.1.6}$$

where N_e and N_p are number operators. Notice that the N_p is preceded by a negative sign, meaning the b operators increase the total charge (since q_e is negative), while the a operators decrease the total charge. This means the b operators represent a particle with the same mass as the electron, *opposite* charge, and *positive* energy – these are positrons. This interpretation of the negative energy solutions as particles with opposite charge and positive energy, resolves the negative energy problem. Notice that there is an infinite zero point charge given by $\mathbb{1}$ under the integral in equation (1.1.6). This can be removed by choosing a different, but equivalent, definition for the charge operator.

Another teething problem of Dirac's theory, was the Hamiltonian postulated in Equation (1.1.1) did not conserve the angular momentum of the particle it described. Consider the time evolution of the component L_1 of the angular momentum:

$$\frac{\partial L_1}{\partial t} \propto [H, L_1] = \left[\gamma^i p_i + \gamma^0 m, r_2 p_3 - r_3 p_2\right],$$

where we've used $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. We use the fact that $\gamma^0 m$ is a constant and the canonical commutator $[r_i, p_j] = i\hbar\delta_{ij}$ to obtain,

$$[H, L_1] = -i\left(\gamma^2 p_3 - \gamma^3 p_2\right) \neq 0.$$

This issue was resolved by introducing a new angular momentum operator $\mathbf{S} = \frac{\hbar}{2}\mathbf{\Sigma}$, where the matrix $\mathbf{\Sigma}$ is:

$$\Sigma_i = \begin{pmatrix} \sigma_i & 0_2 \\ 0_2 & \sigma_i \end{pmatrix}$$

and the σ_i are the Pauli matrices. Letting the total angluar momentum be $\mathbf{J} = \mathbf{L} + \mathbf{S}$, we obtain,

$$\frac{\partial J_1}{\partial t} \propto [H, J_1] = \cdots = 0,$$

[1]Strictly speaking, $U^{(i)}(\mathbf{p})$ are the FW tansformed spinors and: $V^{(1)}(\mathbf{p}) := -U^{(4)}(-\mathbf{p})$ and $V^{(2)}(\mathbf{p}) := U^{(3)}(-\mathbf{p})$

and the *total* angular momentum **J** is conserved. This required additional angular momentum component **S** was coined spin.

2 The Model

This project moves away from the Dirac picture and considers the Hamiltonian of the form,

$$H = \bar{\psi}\mathcal{H}\psi,$$

where ψ is an n-component spinor and n a positive integer. The form of the Hamiltonian density should be similar to that of the Dirac theory, i.e. linear in the mass and momentum terms;

$$\mathcal{H} = \beta_i p_i + \beta_4 m_1 + \beta_5 m_2, \qquad (i = 1, 2, 3), \tag{2.0.1}$$

where m_1 and m_2 are two independent masses and the p_i are the usual components of momentum. The β_ν terms in Equation (2.0.1) are $n \times n$ matrices, analogous to the γ^μ from Dirac's theory, except they won't satisfy a Clifford algebra due to the inclusion of the fifth matrix. Upon diagonalisation, we should expect the eigenvalues of the Hamiltonian to agree with the relativistic energy-momentum relation to obtain,

$$E_1^{(j)} = \pm\sqrt{p_1^2 + p_2^2 + p_3^2 + m_1^2}, \quad E_2^{(k)} = \pm\sqrt{p_1^2 + p_2^2 + p_3^2 + m_2^2}, \quad E_3^{(l)} = 0, \tag{2.0.2}$$

with appropriate degeneracy[2] (j, k, l) depending on the dimension of the Hamiltonian. It goes without saying of course that another constraint on the Hamiltonian is that its spectrum must be real, so as to describe an observable. We include the possibility of a zero eigenvalue because we will be working with Hamiltonian matrices of odd dimension. Each "square root" eigenvalue comes in pairs, so to allow for the possibility of having an odd number of eigenvalues, we include zero. The consequence of this is that the model will not be relativistically invariant, which is not necessarily an issue if we imagine to find these quasiparticle excitations in low-energy condensed matter systems.

The goal for this project was to identify the matrices β_ν for $5 \leq n \leq 8$ that resulted in the eigenvalues stated in Equation (2.0.2). We no longer consider the spinors as four dimensional vectors in $\mathbb{R}^{3+1}$, but rather elements of $\mathbb{R}^3$. We can make this change because we only expect to find these (quasi-)particles in low energy condensed matter systems, and thus they need not obey Lorentz symmetry. We still demand the relativistic energy eigenvalues however because this is what is obtained experimentally. The models investigated here then have rotations described by the $SO(3)$ group. The representation of this group will be different for each value of n due to higher dimensional matrices describing particles of different spin, and they will also be calculated in this investigation.

[2]The degeneracy given is the degeneracy of each the positive and negative versions of each eigenvalue.

3 Method

In the initial stages of the investigation, our attention was focused on examining the form of the characteristic polynomial that a matrix with given eigenvalues should have. Since the eigenvalues of a matrix $\mathcal{H}$ are the solutions to,

$$|\mathcal{H} - \lambda \mathbb{1}| = 0,$$

the characteristic equation must factorise to (for the $n = 5$ case),

$$\lambda \left(\lambda^2 - p_1^2 - p_2^2 - p_3^2 - m_1^2\right) \left(\lambda^2 - p_1^2 - p_2^2 - p_3^2 - m_2^2\right) = 0.$$

Upon expansion of this quintic expression to,

$$\lambda^5 + \lambda^3 \left(-2p_1^2 - 2p_2^2 - 2p_3^2 - m_1^2 - m_2^2\right) + \lambda\big(p_1^4 + p_2^4 + p_3^4 + 2p_1^2p_2^2 + 2p_1^2p_3^2 + 2p_2^2p_3^2 + p_1^2m_1^2 + p_2^2m_1^2 + p_3^2m_1^2 + p_1^2m_2^2 + p_2^2m_2^2 + p_3^2m_2^2 + m_1^2m_2^2\big),$$

the idea was to analyse the coefficients of each power of λ and, by trial and error, deduce the form of the matrix that would eliminate the even powers of λ. From this line of reasoning, we were able to obtain a matrix that gave the correct coefficient of λ^3, however the linear term proved too taxing to guess by hand (even for the smallest 5×5 case) and this approach was abandoned.

In the next stage of the project, we moved our focus to the fact that the matrices we wish to obtain should not satisfy a Clifford algebra. We were aware of the Gell-Mann matrices, λ_i, used in the study of the strong nuclear force and the quark model[3], and knew that they did not satisfy any Clifford algebra, so they seemed like a reasonable staring point. Taking the Hamiltonian density to be of the form,

$$\mathcal{H}_{n=3} = \lambda_1 p_1 + \lambda_2 p_2 + \lambda_6 p_3 + \lambda_7 m_1,$$

we could generate the correct eigenvalue shape;

$$\pm\sqrt{p_1^2 + p_2^2 + p_3^2 + m_1^2}.$$

Now the question remained on how we could use an increase the dimension (n) of this Hamiltonian density to include a second, independent, mass. The key word here is *independent*. It occurred to us that we could use a mixture of combinations of the 3×3 Gell-Mann Hamiltonian density and the 4×4 Dirac Hamiltonian density to create square matrices of

[3] The Gell-Mann matrices used in the investigation are given in Appendix A.

dimension $6, 7$ and 8. The particular way to combine these matrices was obvious; a tensor product wouldn't work as this operation creates cross terms and we want to avoid crossing m_1 and m_2, rather we would use a direct sum. A direct sum would leave the rows and columns of each matrix used independent from each other, thus allowing us to assign one Hamiltonian density m_1 and the other m_2, leaving them independent in the eigenvalues.

An issue with this approach of taking direct sums, was that in very few cases can the Hamiltonian of a system be split up so that terms involving one mass term are completely separate from those involving the other. To get around this, we considered generating a Hamiltonian that had eigenvalues akin to:

$$\sqrt{p_1^2 + p_2^2 + p_3^2 + m_1^2}, \quad \sqrt{p_1^2 + p_2^2 + p_3^2 + m_1^2 + m_2^2}.$$

While m_1 and m_2 both appear in the second expression, we can take $m_1^2 + m_2^2 = M^2$, and regard m_2 as a parameter which can be used to "tune" the value of M, thus M acts as if it were independent to m_1.

To determine the spin-j representation of $SO(3)$, the following method was employed which we will describe for the $n = 5$ case. The spin-j representation will have dimension $2j + 1$, so we must have:

$$2j + 1 = 5 \implies j = 2,$$

hence the 5×5 model describes a spin-2 particle. If we work in the z basis, the possible values for the spin (and thus the eigenvalues of the z projection of the spin operator, S_z) are,

$$m_j = 2\hbar, \hbar, 0, -\hbar, -2\hbar,$$

and the diagonalised matrix form of S_z is,

$$S_z = \hbar \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}. \tag{3.0.1}$$

For the remaining projection operators S_x and S_y, we exploit the ladder operators which are defined by,

$$S_+ = S_x + iS_y, \quad S_- = S_x - iS_y. \tag{3.0.2}$$

Adding and subtracting Equations (3.0.2) gives:

$$S_x = \frac{1}{2}\left(S_+ + S_-\right), \quad S_y = \frac{1}{2i}\left(S_+ - S_-\right). \tag{3.0.3}$$

For some state $|j, m_j\rangle$, applying each ladder operator gives,

$$S_\pm |j, m_j\rangle = \hbar\sqrt{j(j+1) - m_j\left(m_j \pm 1\right)}\, |j, m_j \pm 1\rangle. \tag{3.0.4}$$

To obtain the matrix form of S_x and S_y, we must first calculate the entries of the ladder operators. Following our definition of S_z in Equation (3.0.1), the ladder operator matrices are:

$$S_\pm = \hbar \begin{pmatrix} \langle 2,2| S_\pm |2,2\rangle & \dots & \langle 2,2| S_\pm |2,-2\rangle \\ \vdots & \ddots & \vdots \\ \langle 2,-2| S_\pm |2,2\rangle & \dots & \langle 2,-2| S_\pm |2,-2\rangle \end{pmatrix}.$$

Each of the bra-ket sandwiches were evaluated using the definitions in Equation (3.0.4), for example we take,

$$\langle 2,1| S_+ |2,0\rangle = \langle 2,1| \left(\hbar\sqrt{2(2+1) - 0(0+1)}\right) |2,1\rangle = \sqrt{6}\,\langle 2,1|2,1\rangle = \sqrt{6},$$

where the last equality holds because $|2,1\rangle$ is normalised hence $\langle 2,1|2,1\rangle = 1$. For other elements, we follow,

$$\langle 2,0| S_- |2,-1\rangle = a\,\langle 2,0|2,-2\rangle = 0,$$

where a is an irrelevant constant and the last equality holds because $|2,0\rangle$ and $|2,-2\rangle$ are orthogonal. This reasoning is used for each element of $S_\pm$ and we obtain:

$$S_+ = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad S_- = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}.$$

Finally, using Equations (3.0.3), we arrive at the spin-2 representation for $SO(3)$:

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}, \quad S_y = \frac{1}{2i}\begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ -2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & -\sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & -\sqrt{6} & 0 & 2 \\ 0 & 0 & 0 & -2 & 0 \end{pmatrix},$$

$$S_z = \hbar\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

The same approach was used to obtain the spin-j representation of $SO(3)$ for each investigated value of n. All the generators, S_i, for each spin-j representation should satisfy the commutation relations,

$$[S_\alpha, S_\beta] = \delta_{\alpha\beta} S_\gamma,$$

where $\alpha, \beta, \gamma = x, y, z$.

4 Results

4.1 $n = 8$

We begin by presenting the results for the 8×8 Hamiltonian model. In this case, we were able to obtain the desired results for three independent masses. The form of the Hamiltonian density is:

$$\mathcal{H}_{n=8} = \beta_i p_i + m_1\beta_4 + m_2\beta_5 + m_3\beta_6 + m_4\beta_7 + m_5\beta_8,$$

which in matrix form is,

$$\mathcal{H}_{n=8} = \begin{pmatrix} 0 & p_1 + im_1 & 0 & m_2 + im_3 & 0 & 0 & 0 & ip_2 - p_3 \\ p_1 - im_1 & 0 & 0 & 0 & 0 & 0 & ip_2 - p_3 & 0 \\ 0 & 0 & 0 & 0 & p_1 + im_2 & ip_2 - p_3 & 0 & 0 \\ m_2 - im_3 & 0 & 0 & 0 & 0 & 0 & m_4 + im_5 & 0 \\ 0 & 0 & p_1 - im_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -ip_2 - p_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -ip_2 - p_3 & 0 & m_4 - im_5 & 0 & 0 & 0 & -p_1 + im_1 \\ -ip_2 - p_3 & 0 & 0 & 0 & 0 & 0 & -p_1 - im_1 & 0 \end{pmatrix},$$

which yielded the eigenvalues,

$$E_1^{(1)} = \pm\sqrt{p_1^2 + p_2^2 + p_3^2 + m_1^2}, \quad E_2^{(1)} = \pm\sqrt{p_1^2 + p_2^2 + p_3^2 + m_2^2}, \quad E_3^{(2)} = 0,$$
$$E_4^{(1)} = \pm\sqrt{p_1^2 + p_2^2 + p_3^2 + m_1^2 + m_2^2 + m_3^2 + m_4^2 + m_5^2}.$$

We can use the parameters m_3, m_4, m_5 to let,

$$m_1^2 + m_2^2 + m_3^2 + m_4^2 + m_5^2 = M^2, \tag{4.1.1}$$

such that M acts independently from m_1 and m_2. Of course, we don't require three parameters to achieve this, simply including only one, m_3 would suffice. Below in Figure 1 shows the energy spectrum plotted, with a view of each eigenvalue where $m_1 \neq m_2 \neq M$.

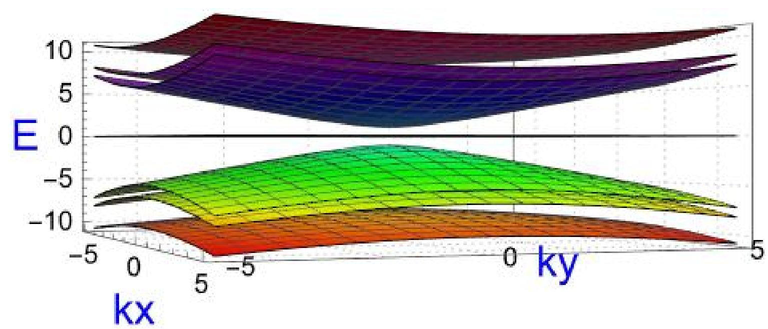

Figure 1: Side view of the 8×8 spectrum of the energy dispersion relations $E(k)$ where $m_1 \neq m_2 \neq M$. In the figure, kx and ky are the components p_1 and p_2 of the momentum.

To obtain the spin-j representation of $SO(3)$ for the 8×8 Hamiltonian, we must have,

$$2j + 1 = 8 \implies j = \frac{7}{2}.$$

Hence, this $n = 8$ model represents a spin-7/2 particle. Using the process highlighted in

Section 3, the matrices describing the spin-7/2 representation of $SO(3)$ were found to be,

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{7} & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{7} & 0 & 2\sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\sqrt{3} & 0 & \sqrt{15} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{15} & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & \sqrt{15} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{15} & 0 & 2\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\sqrt{3} & 0 & \sqrt{7} \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{7} & 0 \end{pmatrix},$$

$$S_y = \frac{\hbar}{2i}\begin{pmatrix} 0 & \sqrt{7} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{7} & 0 & 2\sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & -2\sqrt{3} & 0 & \sqrt{15} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{15} & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & \sqrt{15} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{15} & 0 & 2\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & -2\sqrt{3} & 0 & \sqrt{7} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{7} & 0 \end{pmatrix},$$

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -7 \end{pmatrix}.$$

4.2 $n = 7$

For the 7×7 case,we take the direct sum of the Dirac gamma matrices with the Gell-Mann lambda matrices. This gave a possible set of beta matrices to be,

$$\beta_i = \gamma^i \oplus \lambda^i, \quad \beta_4 = \gamma^0 \oplus 0_3, \quad \beta_5 = 0_4 \oplus \lambda^0. \tag{4.2.1}$$

Again the Hamiltonian density will have the form:

$$\mathcal{H}_{n=7} = \beta_i p_i + m_1 \beta_4 + m_2 \beta_5,$$

which, for the beta matrices defined in Equations (4.2.1), has eigenvalues,

$$E_1^{(4)} = \pm\sqrt{p_1^2 + p_2^2 + p_3^2 + m_1^2}, \quad E_2^{(2)} = \pm\sqrt{p_1^2 + p_2^2 + p_3^2 + m_2^2}, \quad E_3^{(1)} = 0.$$

The corresponding Hamiltonian density in matrix form is:

$$\mathcal{H}_{n=7} = \begin{pmatrix} m_1 & 0 & p_3 & p_1 - ip_2 & 0 & 0 & 0 \\ 0 & m_1 & p_1 + ip_2 & -p_3 & 0 & 0 & 0 \\ -p_3 & p_1 - ip_2 & -m_1 & 0 & 0 & 0 & 0 \\ p_1 + ip_2 & p_3 & 0 & -m_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p_1 - ip_2 & 0 \\ 0 & 0 & 0 & 0 & p_1 + ip_2 & 0 & p_3 - im_2 \\ 0 & 0 & 0 & 0 & 0 & p_3 + im_2 & 0 \end{pmatrix}$$

The commutation relations satisfied by the 7×7 beta matrices given in Equation (4.2.1) were found to be:

$$[\beta_i, \beta_i] = 0, \quad [\beta_4, \beta_5] = 0,$$
$$[\beta_1, \beta_2] = -2i\,(\sigma_3 \oplus \sigma_3) \oplus -2i\lambda_3, \quad [\beta_1, \beta_3] = 2i\,(\sigma_2 \oplus \sigma_2) \oplus i\lambda_5, \quad [\beta_2, \beta_3] = -2i\,(\sigma_1 \oplus \sigma_1) \oplus i\lambda_4,$$
$$[\beta_1, \beta_5] = -i\,(0_4 \oplus \lambda_4)\,, \quad [\beta_2, \beta_5] = i\,(0_4 \oplus \lambda_5)\,, \quad [\beta_3, \beta_5] = 2i\,(0_5 \oplus \sigma_2)\,,$$
$$[\beta_i, \beta_4] = -2\begin{pmatrix} 0_2 & \sigma_i \\ \sigma_i & 0_2 \end{pmatrix} \oplus 0_3,$$

where the final commutator is a block matrix and $i = 1, 2, 3$.

Using an alternative approach, we obtained a second possible Hamiltonian which gave the desired eigenvalues after paramaterisation of additional mass terms. In matrix form, this Hamiltonian density is,

$$\mathcal{H}_{n=7} = \begin{pmatrix} 0 & p_1 + im_1 & 0 & m_2 + im_3 & 0 & 0 & ip_2 - p_3 \\ p_x - im_1 & 0 & 0 & 0 & 0 & ip_2 - p_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ m_2 - im_3 & 0 & 0 & 0 & 0 & m_4 + im_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -ip_2 - p_3 & 0 & m_4 - im_5 & 0 & 0 & p_1 + im_1 \\ -ip_2 - p_3 & 0 & 0 & 0 & 0 & -p_1 - im_1 & 0 \end{pmatrix}$$

which gave the eigenvalues,

$$E_1^{(1)} = \pm\sqrt{p_1^2 + p_2^2 + p_3^2 + m_1^2}, \quad E_2^{(1)} = \pm\sqrt{p_1^2 + p_2^2 + p_3^2 + M^2}, \quad E_3^{(3)} = 0.$$

where we've let $M^2 = m_1^2 + m_2^2 + m_3^2 + m_4^2 + m_5^2$. Figure 2 below shows the energy spectrum for each of the 7×7 Hamiltonians developed.

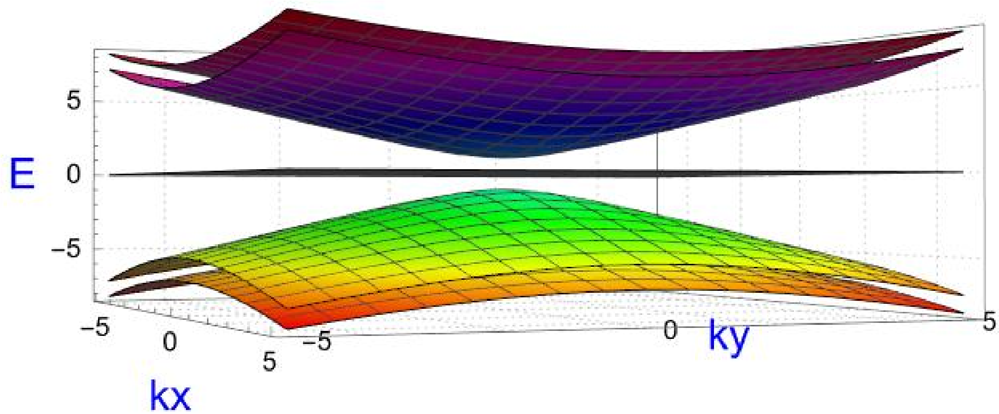

Figure 2: Side of the 7×7 real spectra of energy dispersion relations $E(k)$. In the figure, kx and ky are the components p_1 and p_2 of the momentum.

Again, to determine the spin representation of $SO(3)$ for the $n = 7$ model, we must have,

$$2j + 1 = 7 \implies j = 3.$$

Hence the 7×7 Hamiltonian model describes a spin-3 particle. Using the procedure highlighted in Section 3, the spin projection matrices were found to be,

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{6} & 0 & 0 & 0 & 0 & 0 \\ \sqrt{6} & 0 & \sqrt{10} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{10} & 0 & 2\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 2\sqrt{10} & 0 & 2\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 2\sqrt{3} & 0 & \sqrt{10} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{10} & 0 & \sqrt{6} \\ 0 & 0 & 0 & 0 & 0 & \sqrt{6} & 0 \end{pmatrix},$$

$$S_y = \frac{\hbar}{2i}\begin{pmatrix} 0 & \sqrt{6} & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{6} & 0 & \sqrt{10} & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{10} & 0 & 2\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & -2\sqrt{10} & 0 & 2\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & -2\sqrt{3} & 0 & \sqrt{10} & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{10} & 0 & \sqrt{6} \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{6} & 0 \end{pmatrix},$$

$$S_z = \hbar\begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 \end{pmatrix}.$$

4.3 $n = 6$

The Hamiltonian density in obtained for $n = 6$ had the form,

$$\mathcal{H}_{n=6} = \beta_i p_i + m_1\beta_4 + m_2\beta_5 + m_3\beta_6 + m_4\beta_7 + m_5\beta_8,$$

and in matrix form is,

$$\begin{pmatrix} 0 & p_x + im_1 & m_2 + im_3 & m_4 + im_5 & 0 & ip_y - p_z \\ p_x - im_1 & 0 & 0 & 0 & ip_y - p_z & 0 \\ m_2 - im_3 & 0 & 0 & 0 & 0 & 0 \\ m_4 - im_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & -ip_y - p_z & 0 & 0 & 0 & -p_x + im_1 \\ -ip_y - p_z & 0 & 0 & 0 & -p_x - im_1 & 0 \end{pmatrix},$$

which yielded the eigenvalues (after the same parameterisation in Equation (4.1.1)),

$$E_1^{(1)} = \pm\sqrt{p_1^2 + p_2^2 + p_3^2 + m_1^2}, \quad E_2^{(1)} = \pm\sqrt{p_1^2 + p_2^2 + p_3^2 + M^2}, \quad E_3^{(2)} = 0.$$

The generators for the spin-j representation of $SO(3)$ had dimension $n = 6$, giving the spin for this model to be,

$$2j + 1 = 6 \implies j = \frac{5}{2}.$$

The matrix form of these generators, calculated using the method outlined in Section 3, were,

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{5} & 0 & 0 & 0 & 0 \\ \sqrt{5} & 0 & 2\sqrt{2} & 0 & 0 & 0 \\ 0 & 2\sqrt{2} & 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 2\sqrt{2} & 0 \\ 0 & 0 & 0 & 2\sqrt{2} & 0 & \sqrt{5} \\ 0 & 0 & 0 & 0 & \sqrt{5} & 0 \end{pmatrix},$$

$$S_y = \frac{\hbar}{2i}\begin{pmatrix} 0 & \sqrt{5} & 0 & 0 & 0 & 0 \\ -\sqrt{5} & 0 & 2\sqrt{2} & 0 & 0 & 0 \\ 0 & -2\sqrt{2} & 0 & 3 & 0 & 0 \\ 0 & 0 & -3 & 0 & 2\sqrt{2} & 0 \\ 0 & 0 & 0 & -2\sqrt{2} & 0 & \sqrt{5} \\ 0 & 0 & 0 & 0 & -\sqrt{5} & 0 \end{pmatrix},$$

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 \end{pmatrix}.$$

4.4 $n = 5$

For the final case considered, we discovered a 5×5 Hamiltonian density that is non-Hermitian, which gave the desired eigenvalues. It is,

$$H_{n=5} = \begin{pmatrix} 0 & p_x + im_1 & m_2 & 0 & ip_y - p_z \\ p_x - im_1 & 0 & 0 & ip_y - p_z & 0 \\ m_2 & 0 & 0 & m_1 & 0 \\ 0 & -ip_y - p_z & -m_1 & 0 & -p_x + im_1 \\ -ip_y - pz & 0 & 0 & -p_x - im_1 & 0 \end{pmatrix} \tag{4.4.1}$$

The non-Hermitian components are the m_1 terms in positions $(4, 3)$ and $(3, 4)$. Despite this, the spectrum of this Hamiltonian density is real and its elements are,

$$E_1^1 = \pm\sqrt{p_1^2 + p_2^2 + p_3^2 + m_1^2}, \quad E_2^1 = \pm\sqrt{p_1^2 + p_2^2 + p_3^2 + m_2^2}, \quad E_3^1 = 0.$$

This real spectrum of a non-Hermitian operator is a consequence of the one-directional implication of the real eigenvalues of a Hermitian operator;

$$H = H^\dagger \implies \text{Spectrum of } H \text{ is real},$$

meaning that there exists operators with a real spectrum that are not Hermitian. This is the core of the field of study: pseudo-Hermitian quantum mechanics. The beta matrices for this Hamiltonian density are,

$$\beta_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \beta_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\beta_4 = \begin{pmatrix} 0 & i & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & i \\ 0 & 0 & 0 & -i & 0 \end{pmatrix}, \quad \beta_5 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The spin-2 representation of $SO(3)$ matrices were calculated in Section 3.

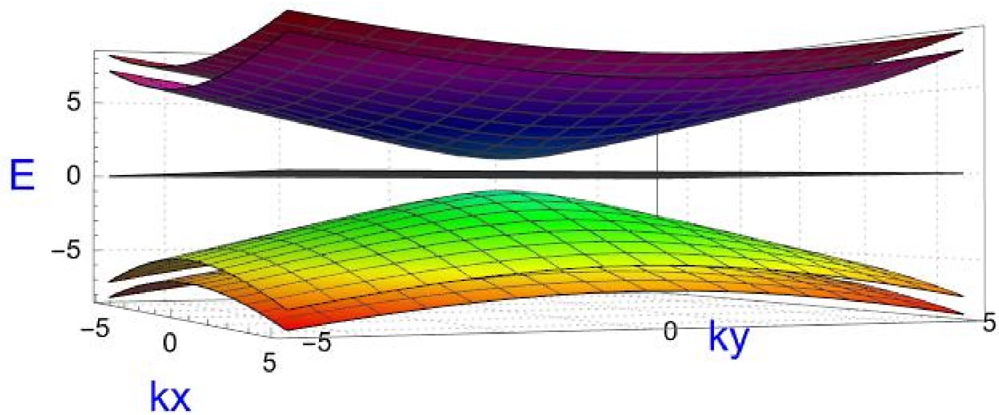

Figure 3: Side view of the 5×5 real spectra of energy dispersion relations $E(k)$ where $m_1 \neq M$. In the figure, kx and ky are the components p_1 and p_2 of the momentum.

5 Discussion

It is interesting to note that any Hamiltonian densities obtained in this investigation with at least one zero eigenvalue are not consistent with special relativity. If in frame A, we measure the energy of the system as 0, then,

$$E_A = c\sqrt{|p|^2 + m^2c^2} = 0. \tag{5.0.1}$$

There are only two possibilities for non-tachyonic[4] fields;

1. The momentum and mass are both zero, or,
2. $c = 0$ in frame A.

Suppose now we perform a Lorentz transformation into a new frame A', where the energy is not zero, and measure the energy of the system again:

$$E_{A'} = c\sqrt{|p'|^2 + m^2c^2} \neq 0 \tag{5.0.2}$$

If we selected possibility 1 above, we would not have a system to describe since the momentum and mass are both zero – this is just the trivial system. We are left with selecting possibility 2, which proves troublesome. According to the postulate of relativity, "the speed of light

[4]If the field were tachyonic, the square of the imaginary mass *may* cancel exactly with the momentum term to give zero energy.

is the same in every frame of reference", $c = 0$ in frame A' as well, resulting in $E_{A'} = 0$, contradicting the non-equality in Equation (5.0.2). The models provided in this investigation containing 0 as an eigenvalue then, cannot describe relativistic particles.

Further investigation in this topic could include looking into similar Hamiltonians, but for which the particle's momentum is restricted to, say, two dimensions ($p_3 = 0$). Much of the search for Majorana fermions is in the regime of 1 + 1-dimensional systems [1][2], leading to a more diverse non-Abelian statistical nature − the source of its potential for use in topological quantum computing. In three spatial dimensions, there are only two available particle exachange statistics available; fermionic and bosonic. In the fermionic case, swapping two particles results in each of their wavefunctions picking up a negative sign, in the bosonic case, swapping two particles leaves their wavefunctions unchanged. In two spatial dimensions, the order in which these swaps are performed matters, i.e. the operators that perform the swap do not commute (non-Abelian). It could be found that restricting the spatial dimensions of a system of several independent masses leads to a system with more interesting particle exchange properties.

Another possible avenue for further investigation pertains to the non-Hermitian Hamiltonian found for the $n = 5$ case that yielded real eigenvalues. This fascinating branch of quantum mechanics is called pseudo-Hermitian QM which replaces the Hermitian condition on the Hamiltonian with the condition that it must have exact parity and time ($\mathcal{PT}$) symmetry, i.e. for a wavefunction $\psi(x)$, the operators $\mathcal{P}$ and $\mathcal{T}$ act as:

$$\mathcal{P}\psi(x) := \psi(-x), \quad \mathcal{T}\psi(x) := \psi(x)^*.$$

This new condition means that the Hamiltonian will have a complete set of $\mathcal{PT}$ invariant eigenvectors thus ensuring its spectrum is real [3].

6 Conclusion

We were able to construct Hamiltonian densities for each value of n stated, that yielded the desired eigenvalues describing the energy of a particle consisting of two independent masses. In addition, we successfully built a Hamiltonian density that described a particle containing three independent masses for the $n = 8$ case. Furthermore, for each spin-j representation of $SO(3)$, we were able to construct the generators of this group and determine the spin of the particle in question (j). This investigation prompted study in many directions including group theory, condensed matter physics and quantum field theory, and has deepened my understanding and curiosity for each of these fields.

7 Acknowledgements

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A Gell-Mann Matrices

Out of the set of 8 Gell-Mann matrices, the four that were used in this report are:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}.$$

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