Beyond Dirac Fermions

Abdulhakim Mohd Nazir[1] Tomás Lannon[2]

[1]University College Dublin

[2]Queen's University Belfast

Supervisor: Dr. Giandomenico Palumbo

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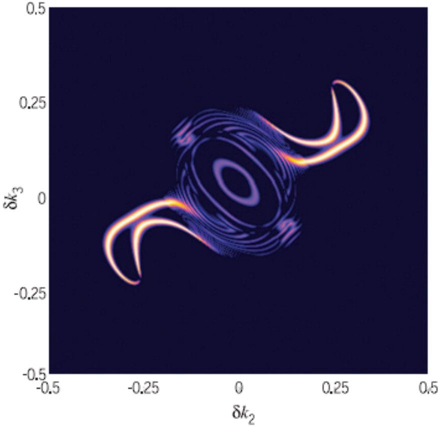
[B. Bradlyn et.al, Science 353, (2016)]



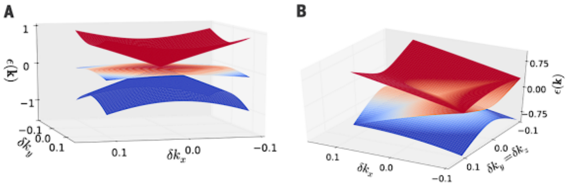
Motivation For Project

Dirac fermions play a fundamental role in both particle physics and condensed matter physics. Unconventional Dirac-like fermions can be used to construct new theoretical models.

- Emergent quasiparticle excitations $\Longrightarrow$ topological protection of quantum information $\Longrightarrow$ Potential use in topological quantum computing [1].
- Quasiparticles in conventional crystals $\Longrightarrow$ unconventional exotic fermion types.

Band topology $\Longrightarrow$ fascinating surface states, ARPES (angle-resolved photoemission spectroscopy) signatures and magnetotransport properties of many material candidates [B. Bradlyn et.al, Science 353, (2016)].

Figure 1: Energy dispersion near a threefold degeneracy (A and B) of exotic fermion types [2].



Dirac Theory in 3+1 Dimensions

The Dirac theory of fermions in 3+1 dimensions is described by the following Hamiltonian:

$$H = \psi^\dagger \mathcal{H} \psi \tag{1}$$

where $\psi = (\psi_1, \psi, \ldots, \psi_n)^T$ is a n-component spinor with $n = 4$ in 3+1 dimensions. The density Hamiltonian $\mathcal{H}$ can be written as the following:

$$\mathcal{H} = \alpha_i p_i + \alpha_0 m \tag{2}$$

where m is the Dirac mass, $p_i = (p_x, p_y, p_z)$ are the momenta and $\alpha_\mu = (\alpha_0, \alpha_x, \alpha_y, \alpha_z)$ are Dirac matrices that satisfy the Clifford algebra $Cl_{3,1}$:

$$\{\alpha_\mu, \alpha_\nu\} = 2\delta_{\mu\nu}\mathbb{1} \tag{3}$$

where $\delta_{\mu\nu}$ is the Kronecker delta and $\mathbb{1}$ is the identity matrix.

Dirac Theory in 3+1 Dimensions

One can prove that α_μ are even-order matrices $2^s \times 2^s$, where s is some positive integer. In 3+1 dimensions, the value of s is 2.

Upon diagonalisation of the density Hamiltonian $\mathcal{H}$, the following eigenvalues are obtained:

$$E_-^j = -\sqrt{p_x^2 + p_y^2 + p_z^2 + m^2}, \qquad E_+^j = \sqrt{p_x^2 + p_y^2 + p_z^2 + m^2} \tag{4}$$

where j is the degeneracy of the eigenvalues, which is equal to $j = 2^{s-1} = 2$ in 3+1 dimensions.

Dirac Theory in 3+1 Dimensions

Properties of the matrices α_i and α_0:

1. $\mathcal{H}$ is Hermitian: $\hat{\mathcal{H}} = \mathcal{H}^\dagger \longrightarrow \alpha_i = \alpha_i^\dagger$ and $\alpha_0 = \alpha_0^\dagger$
2. α_i and α_0 are square matrices: $\alpha_i, \alpha_0 \in M_n(\mathbb{C})$
3. α_i and α_0 yield eigenvalues ± 1
4. α_i and α_0 are constant matrices$\longrightarrow$ no dependence on $\vec{r}$ and t.
5. Anti-commutation relations: $\{\alpha_i, \alpha_j\} = \{\alpha_i, \alpha_0\} = 0$
6. The trace of the matrices α_i and α_0 are zero: $Tr(\alpha_i) = 0$ and $Tr(\alpha_0) = 0$

$$\left(\alpha^2 = \alpha_0^2 = \mathbb{1}\right) \tag{5}$$

7. α_i and α_0 are even-order matrices: $2^s \times 2^s$, where s is some positive integer. This follows from conditions 2, 3 and 6.

Dirac Theory in 3+1 Dimensions

It is evident that in 3+1 dimensions, the lowest value of s is 2.

Recall that α_μ are $2^s \times 2^s$, where s is some positive integer.

Consider Pauli spin matrices (2×2):

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{6}$$

Satisfies all above properties but unable to construct α_0 for m in the 2×2 representation. Next available representation is $4 \times 4 \implies$ lowest value of s is 2 in 3+1 dimensions.

Thus α_μ are 4×4 matrices which are the Dirac matrices that satisfy the Clifford algebra $Cl_{3,1}$ $\implies \{\alpha_\mu, \alpha_\nu\} = 2\delta_{\mu\nu}\mathbb{1}$.

Dirac Theory in 3+1 Dimensions

One such 4×4 representation is also known as the Pauli-Dirac representation.

$$\alpha_x = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \underline{0} & \sigma_x \\ \sigma_x & \underline{0} \end{bmatrix} \quad \alpha_y = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \underline{0} & \sigma_y \\ \sigma_y & \underline{0} \end{bmatrix}$$

$$\alpha_z = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \underline{0} & \sigma_z \\ \sigma_z & \underline{0} \end{bmatrix} \quad \alpha_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \mathbb{1} & \underline{0} \\ \underline{0} & -\mathbb{1} \end{bmatrix} \tag{7}$$

where $\underline{0}$ is the 2×2 zero-matrix and $\mathbb{1}$ is the 2×2 identity matrix.

Dirac spinor $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ is a 4-component spinor in 3+1 dimensions.

The Model – Dirac-Like Theory in 3+1 Dimensions

The model for Dirac-like theory of fermions in 3+1 dimensions should be described by the following Hamiltonian:

$$H = \psi^\dagger \mathcal{H} \psi \tag{8}$$

where $\psi = (\psi_1, \psi, \ldots, \psi_n)^T$ is a n-component spinor with n some positive integer. The density Hamiltonian $\mathcal{H}$ can be written as the following:

$$\mathcal{H} = \beta_i p_i + \beta_4 m_1 + \beta_5 m_2 \tag{9}$$

where m_1 and m_2 are two independent masses, $p_i = (p_x, p_y, p_z)$ are the momenta and $\beta_\mu = (\beta_x, \beta_y, \beta_z, \beta_4, \beta_5)$ are $n \times n$ matrices, which do not satisfy the Clifford algebra $\mathrm{Cl}_{3,1}$ $\implies \{\beta_\mu, \beta_\nu\} \neq 2\delta_{\mu\nu}\mathbb{1}$.

The Model – Dirac-Like Theory in 3+1 Dimensions

The density Hamiltonian $\mathcal{H}$ constructed with these β_μ matrices should have eigenvalues:

$$E_0^j = 0, \quad E_{1,\pm}^k = \pm\sqrt{p_x^2 + p_y^2 + p_z^2 + m_1^2}, \quad E_{2,\pm}^l = \pm\sqrt{p_x^2 + p_y^2 + p_z^2 + m_2^2} \qquad (10)$$

where j, k, l is the degeneracy of each eigenvalue.

The presence of the zero-energy eigenvalue, $E_0^j = 0$, means that our Dirac-like model is not Lorentz invariant.

This non-Lorentz invariant property is representative of why our Dirac-like model cannot satisfy the Clifford algebra $\mathrm{Cl}_{3,1} \implies \{\beta_\mu, \beta_\nu\} \neq 2\delta_{\mu\nu}\mathbb{1}$.

The Model – Dirac-Like Theory in 3+1 Dimensions

A set of eight linearly independent 3×3 Hermitian matrices with SU(3) symmetry. Used in the study of the strong interaction in particle physics. Using these particular four out of the set of eight:

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \lambda_2 = \begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{bmatrix}$$

$$\implies \mathcal{H} = \lambda_1 p_x + \lambda_2 p_y + \lambda_6 p_z + \lambda_7 m$$

Diagonalisation yields Dirac-like eigenvalues:

$$E_0^1 = 0, \qquad E_{1,-}^1 = -\sqrt{p_x^2 + p_y^2 + p_z^2 + m^2},$$
$$E_{1,+}^1 = \sqrt{p_x^2 + p_y^2 + p_z^2 + m^2}$$

These matrices are not compatible with the Clifford algebra $Cl_{3,1} \implies \{\lambda_i, \lambda_j\} \neq 2\delta_{ij}\mathbb{1}$.

The Model – Dirac-Like Theory in $3 + 1$ Dimensions

1. This model follows $SO(3)$ symmetry.

2. The Lie algebra for $SO(3)$ for dimension n is:

$$so(3) := \{X : n \times n \text{ matrix} \,|\, X^T + X = 0,\ \mathrm{Tr}(X) = 0\}$$

3. The generators (J_1, J_2, J_3) of this algebra obey:

$$[J_i, J_j] = J_k \delta_{ij}$$

4. Each value of n we consider corresponds to particles of different spin, j:

$$2j + 1 = n$$

The Model – Dirac-Like Theory in 3+1 Dimensions

The goals of this project are:

1. Construct $n \times n$ matrices β_μ for $5 \leq n \leq 8$ such that the corresponding density Hamiltonian $\mathcal{H}$ has the desired Dirac-like eigenvalues.

2. Plot the real spectra of energy dispersion relations $E(k)$ for each n.

3. Identify possible β_6 associated with a third independent mass term m_3 such that

$$\mathcal{H} = \beta_i p_i + \beta_4 m_1 + \beta_5 m_2 + \beta_6 m_3 \tag{11}$$

Results – $\mathcal{H}_{n=5} = \beta_i p_i + \beta_4 m_1 + \beta_5 m_2$

$$\mathcal{H}_{n=5} = \begin{bmatrix} 0 & p_x + im_1 & -m_2 + im_3 & 0 & ip_y - p_z \\ p_x - im_1 & 0 & 0 & ip_y - p_z & 0 \\ -m_2 - im_3 & 0 & 0 & im_4 + m_5 & 0 \\ 0 & -ip_y - p_z & -im_4 + m_5 & 0 & -p_x + im_1 \\ -ip_y - pz & 0 & 0 & -p_x - im_1 & 0 \end{bmatrix} \quad (12)$$

$$E = 0, \pm\sqrt{p_x^2 + p_y^2 + p_z^2 + {m_1}^2}, \pm\sqrt{p_x^2 + p_y^2 + p_z^2 + {m_1}^2 + {m_2}^2 + {m_3}^2 + {m_4}^2 + {m_5}^2} \quad (13)$$

Grouping with sufficient free parameters $\Longrightarrow {m_1}^2 + {m_2}^2 + {m_3}^2 + {m_4}^2 + {m_5}^2 = M^2$

$$\begin{aligned} E_0^1 = 0, \quad E_{1,-}^1 &= -\sqrt{p_x^2 + p_y^2 + p_z^2 + {m_1}^2}, \quad E_{1,+}^1 = \sqrt{p_x^2 + p_y^2 + p_z^2 + {m_1}^2} \\ E_{2,-}^1 &= -\sqrt{p_x^2 + p_y^2 + p_z^2 + M^2}, \quad E_{2,+}^1 = \sqrt{p_x^2 + p_y^2 + p_z^2 + M^2} \end{aligned} \quad (14)$$

Results – Non-Hermitian $\mathcal{H}_{n=5}$?

An unexpected result occurred. We managed to construct the following Hamiltonian $\mathcal{H}_{n=5}$

$$\mathcal{H}_{n=5} = \begin{bmatrix} 0 & p_x + im_1 & m_2 & 0 & ip_y - p_z \\ p_x - im_1 & 0 & 0 & ip_y - p_z & 0 \\ m_2 & 0 & 0 & m_1 & 0 \\ 0 & -ip_y - p_z & -m_1 & 0 & -p_x + im_1 \\ -ip_y - pz & 0 & 0 & -p_x - im_1 & 0 \end{bmatrix} \tag{15}$$

$$\begin{aligned} E_0^1 = 0, \quad E_{1,-}^1 &= -\sqrt{p_x^2 + p_y^2 + p_z^2 + {m_1}^2}, & E_{1,+}^1 &= \sqrt{p_x^2 + p_y^2 + p_z^2 + {m_1}^2} \\ E_{2,-}^1 &= -\sqrt{p_x^2 + p_y^2 + p_z^2 + {m_2}^2}, & E_{2,+}^1 &= \sqrt{p_x^2 + p_y^2 + p_z^2 + {m_2}^2} \end{aligned} \tag{16}$$

This is an interesting case to consider. The 5×5 matrix is clearly not Hermitian, yet the eigenvalues are real, describing $E(k)$. Every non-Hermitian Hamiltonian with a real spectrum is pseudo-Hermitian [3].

Results – Real Spectra of Pseudo-Hermitian $\mathcal{H}_{n=5}$

Using standard notation of crystal momentum k for energy dispersion relations $E(k)$:

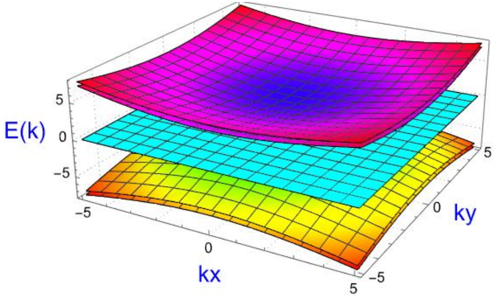


Figure 2: 5×5 real spectra of $E(k)$ $[m_1 \neq m_2]$

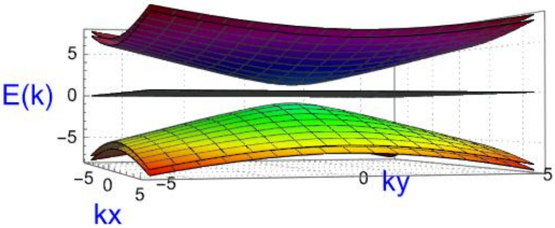


Figure 3: Side view of 5×5 real spectra of $E(k)$

Results – $\mathcal{H}_{n=6} = \beta_i p_i + \beta_4 m_1 + \beta_5 m_2$

1. Simply the direct sum of two 3×3 density Hamiltonians from Gell-Mann matrices $\lambda_{i=1,2,6,7}$.

2. This gives the beta matrices $\beta_\mu = (\beta_x, \beta_y, \beta_z, \beta_4, \beta_5)$ to be:

$$\beta^i = \lambda^i \oplus \lambda^i, \quad \beta^4 = \lambda^0 \oplus 0_3, \quad \beta^5 = 0_3 \oplus \lambda^0 \tag{17}$$

3. In matrix form, the density Hamiltonian $\mathcal{H}_{n=6}$ is:

$$\mathcal{H}_{n=6} = \begin{bmatrix} 0 & p_x + ip_y & 0 & 0 & 0 & 0 \\ p_x - ip_y & 0 & pz + im_1 & 0 & 0 & 0 \\ 0 & p_z - im_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_x + ip_y & 0 \\ 0 & 0 & 0 & p_x - ip_y & 0 & p_z + im_2 \\ 0 & 0 & 0 & 0 & p_z - im_2 & 0 \end{bmatrix} \tag{18}$$

4. Not compatible with the Clifford algebra $\mathrm{Cl}_{3,1} \implies \{\beta_\mu, \beta_\nu\} \neq 2\delta_{\mu\nu}\mathbb{1}$, due to the nature of the Gell-Mann matrices λ_i.

Results – $\mathcal{H}_{n=6} = \beta_i p_i + \beta_4 m_1 + \beta_5 m_2$

$$\mathcal{H}_{n=6} = \begin{bmatrix} 0 & p_x + im_1 & m_2 + im_3 & m_4 + im_5 & 0 & ip_y - p_z \\ p_x - im_1 & 0 & 0 & 0 & ip_y - p_z & 0 \\ m_2 - im_3 & 0 & 0 & 0 & 0 & 0 \\ m_4 - im_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & -ip_y - p_z & 0 & 0 & 0 & -p_x + im_1 \\ -ip_y - p_z & 0 & 0 & 0 & -p_x - im_1 & 0 \end{bmatrix} \tag{19}$$

Grouping with sufficient free parameters $\implies {m_1}^2 + {m_2}^2 + {m_3}^2 + {m_4}^2 + {m_5}^2 = M^2$

$$E_0^2 = 0, \quad E_{1,-}^1 = -\sqrt{p_x^2 + p_y^2 + p_z^2 + {m_1}^2}, \quad E_{1,+}^1 = \sqrt{p_x^2 + p_y^2 + p_z^2 + {m_1}^2}$$
$$E_{2,-}^1 = -\sqrt{p_x^2 + p_y^2 + p_z^2 + M^2}, \quad E_{2,+}^1 = \sqrt{p_x^2 + p_y^2 + p_z^2 + M^2} \tag{20}$$

where m_1 and M can act as independent mass terms, satisfying our Dirac-like model.

Results – $\mathcal{H}_{n=6} = \beta_i p_i + \beta_4 m_1 + \beta_5 m_2$

$$E_0^2 = 0, \quad E_{1,-}^1 = -\sqrt{p_x^2 + p_y^2 + p_z^2 + {m_1}^2}, \quad E_{1,+}^1 = \sqrt{p_x^2 + p_y^2 + p_z^2 + {m_1}^2}$$
$$E_{2,-}^1 = -\sqrt{p_x^2 + p_y^2 + p_z^2 + {m_2}^2}, \quad E_{2,+}^1 = \sqrt{p_x^2 + p_y^2 + p_z^2 + {m_2}^2} \tag{21}$$

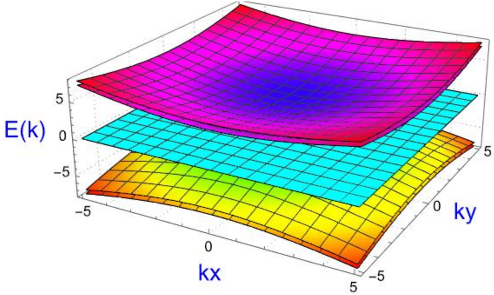


Figure 4: 6×6 real spectra of $E(k)$ $[m_1 \neq m_2]$

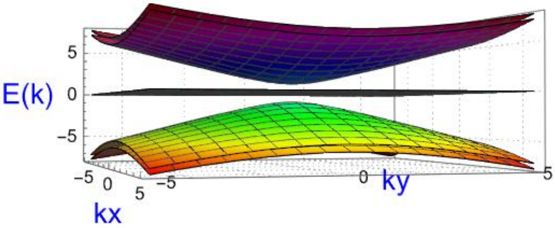


Figure 5: Side view of 6×6 real spectra of $E(k)$

Results – $\mathcal{H}_{n=7} = \beta_i p_i + \beta_4 m_1 + \beta_5 m_2$

1. We take the direct sum of a 3×3 Gell-Mann Hamiltonian with a 4×4 Dirac Hamiltonian.

2. This gives the beta matrices $\beta_\mu = (\beta_x, \beta_y, \beta_z, \beta_4, \beta_5)$ to be:

$$\beta^i = \lambda^i \oplus \gamma^i, \quad \beta^4 = \lambda^4 \oplus 0_4, \quad \beta^5 = 0_3 \oplus \gamma^0 \tag{22}$$

3. In matrix form, the Hamiltonian $\mathcal{H}_{n=7}$ is:

$$\mathcal{H}_{n=7} = \begin{bmatrix} 0 & p_x + ip_y & 0 & 0 & 0 & 0 & 0 \\ p_x - ip_y & 0 & p_z + im_1 & 0 & 0 & 0 & 0 \\ 0 & p_z - im_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_2 & 0 & p_z & p_x - ip_y \\ 0 & 0 & 0 & 0 & m_2 & p_x + ip_y & -p_z \\ 0 & 0 & 0 & p_z & p_x - ip_y & -m_2 & 0 \\ 0 & 0 & 0 & p_x + ip_y & -p_z & 0 & -m_2 \end{bmatrix} \tag{23}$$

Results – $\mathcal{H}_{n=7} = \beta_i p_i + \beta_4 m_1 + \beta_5 m_2$

$$E_0^1 = 0, \quad E_{1,-}^1 = -\sqrt{p_x^2 + p_y^2 + p_z^2 + {m_1}^2}, \quad E_{1,+}^1 = \sqrt{p_x^2 + p_y^2 + p_z^2 + {m_1}^2}$$
$$E_{2,-}^2 = -\sqrt{p_x^2 + p_y^2 + p_z^2 + {m_2}^2}, \quad E_{2,+}^2 = \sqrt{p_x^2 + p_y^2 + p_z^2 + {m_2}^2} \tag{24}$$

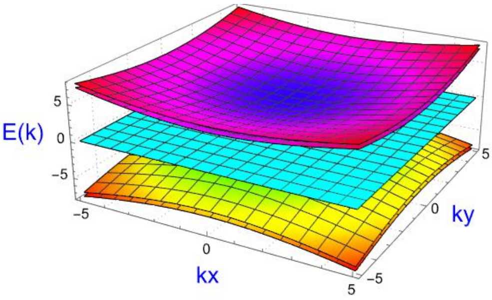


Figure 6: 7×7 real spectra of $E(k)$ $[m_1 \neq m_2]$

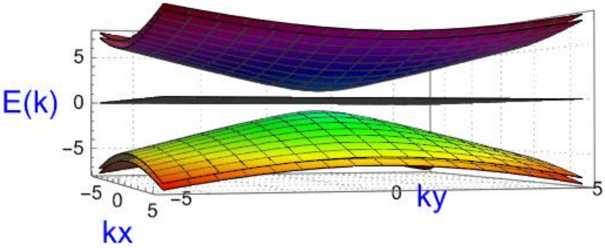


Figure 7: Side view of 7×7 real spectra of $E(k)$

Results – $\mathcal{H}_{n=7} = \beta_i p_i + \beta_4 m_1 + \beta_5 m_2$

$$\mathcal{H}_{n=7} = \begin{bmatrix} 0 & p_x + im_1 & 0 & m_2 + im_3 & 0 & 0 & ip_y - p_z \\ p_x - im_1 & 0 & 0 & 0 & 0 & ip_y - p_z & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ m_2 - im_3 & 0 & 0 & 0 & 0 & m_4 + im_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -ip_y - p_z & 0 & m_4 - im_5 & 0 & 0 & -p_x + im_1 \\ -ip_y - p_z & 0 & 0 & 0 & 0 & -p_x - im_1 & 0 \end{bmatrix} \quad (25)$$

Grouping with sufficient free parameters $\implies {m_1}^2 + {m_2}^2 + {m_3}^2 + {m_4}^2 + {m_5}^2 = M^2$

$$E_0^3 = 0, \quad E_{1,-}^1 = -\sqrt{p_x^2 + p_y^2 + p_z^2 + {m_1}^2}, \quad E_{1,+}^1 = \sqrt{p_x^2 + p_y^2 + p_z^2 + {m_1}^2}$$
$$E_{2,-}^1 = -\sqrt{p_x^2 + p_y^2 + p_z^2 + M^2}, \quad E_{2,+}^1 = \sqrt{p_x^2 + p_y^2 + p_z^2 + M^2} \quad (26)$$

where m_1 and M can act as independent mass terms, satisfying our Dirac-like model.

Results – Possible β_6 of m_3 for $\mathcal{H}_{n=8}$

The 8×8 case is perhaps the most interesting case to study for $5 \leq n \leq 8$. So far, this is the only case where we were able to build a matrix $\mathcal{H}$ that incorporated a β_6 for a third independent mass term, m_3, such that

$$\mathcal{H}_{n=8} = \beta_i p_i + \beta_4 m_1 + \beta_5 m_2 + \beta_6 m_3 :$$

$$\begin{bmatrix} 0 & p_x + im_1 & 0 & m_2 + im_3 & 0 & 0 & 0 & ip_y - p_z \\ p_x - im_1 & 0 & 0 & 0 & 0 & 0 & ip_y - p_z & 0 \\ 0 & 0 & 0 & 0 & p_x + im_2 & ip_y - p_z & 0 & 0 \\ m_2 - im_3 & 0 & 0 & 0 & 0 & 0 & m_4 + im_5 & 0 \\ 0 & 0 & p_x - im_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -ip_y - p_z & 0 & 0 & 0 & 0 & 0 \\ 0 & -ip_y - p_z & 0 & m_4 - im_5 & 0 & 0 & 0 & -p_x + im_1 \\ -ip_y - p_z & 0 & 0 & 0 & 0 & 0 & -p_x - im_1 & 0 \end{bmatrix} \tag{27}$$

Results – Possible β_6 for m_3: $\mathcal{H}_{n=8} = \beta_i p_i + \beta_4 m_1 + \beta_5 m_2 + \beta_6 m_3$

$$E = 0,\ 0,\ \pm\sqrt{p_x^2 + p_y^2 + p_z^2 + {m_1}^2},\ \pm\sqrt{p_x^2 + p_y^2 + p_z^2 + {m_2}^2}$$
$$\pm\sqrt{p_x^2 + p_y^2 + p_z^2 + {m_1}^2 + {m_2}^2 + {m_3}^2 + {m_4}^2 + {m_5}^2} \tag{28}$$

Grouping with sufficient free parameters $\implies {m_1}^2 + {m_2}^2 + {m_3}^2 + {m_4}^2 + {m_5}^2 = M^2$

$$E_0^2 = 0, \quad E_{1,-}^1 = -\sqrt{p_x^2 + p_y^2 + p_z^2 + {m_1}^2}, \quad E_{1,+}^1 = \sqrt{p_x^2 + p_y^2 + p_z^2 + {m_1}^2}$$
$$E_{2,-}^1 = -\sqrt{p_x^2 + p_y^2 + p_z^2 + {m_2}^2}, \quad E_{2,+}^1 = \sqrt{p_x^2 + p_y^2 + p_z^2 + {m_2}^2}$$
$$E_{3,-}^1 = -\sqrt{p_x^2 + p_y^2 + p_z^2 + M^2}, \quad E_{3,+}^1 = \sqrt{p_x^2 + p_y^2 + p_z^2 + M^2} \tag{29}$$

where m_1, m_2 and M can act as independent mass terms, satisfying our Dirac-like model.

The inclusion of a third independent mass term, $m_3 = M$, resulted in an additional positive and negative dispersion $E(k)$.

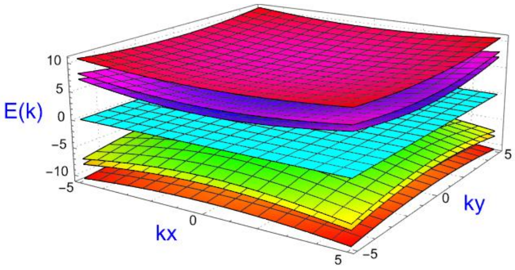


Figure 9: 8×8 real spectra of $E(k)$ $[m_1 \neq m_2 \neq m_3]$

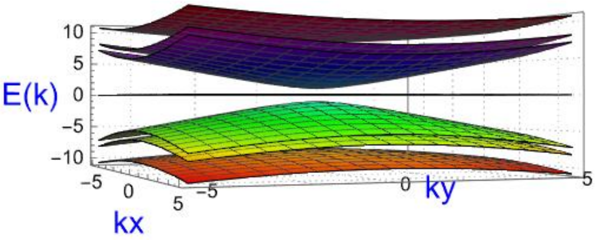


Figure 10: Side view of 8×8 real spectra of $E(k)$

Summary and Conclusion

1. Found $n \times n$ density Hamiltonians $\mathcal{H}$ with appropriate eigenvalues for each value of $5 \leq n \leq 8$.
2. Additionally, obtained matrix β_6 associated with a third mass term m_3 for $n = 8$.
3. Hamiltonians with zero-energy eigenvalues are not Lorentz invariant (exotic fermion types not constrained by full Poincaré symmetry) [2].
4. Only suitable for describing low-energy systems e.g. condensed matter systems [2].

Further Study

1. Anti-commutators for β matrices, general theory, precise band model applications etc.
2. Dirac-like model in $\mathcal{PT}$ symmetric QM [3]:

$$\mathcal{P}\psi(x) = \psi(-x), \quad \mathcal{T}\psi(x) = \psi(x)^*.$$

3. Restricting momentum components to $2+1$ or $1+1$ dimensions.[1][4]

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