Matrix Models, Lattice Simulations and M-Theory

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August 10, 2023

Institiúid Ard-Léinn Bhaile Átha Cliath | Dublin Institute for Advanced Studies

Acknowledgements

We are immensely grateful to the Dublin Institute for Advanced Studies for this opportunity and the support necessary to complete it. In particular we would like to thank our supervisor, Prof. O'Connor, for endless patience and advice in all aspects of both our work and our academic interests. Thank you as well to Dr. Atri Dey and Dr. Saki Koizumi for their coordination of the internships and help throughout, and to our fellow interns for their conversation and discussions throughout the project.

Introduction

- Under Denjoe's supervision, we worked on creating an algorithm in Python to simulate the dynamics of the bosonic BFFS model.
- We will present our work chronologically; We began by developing a hybrid monte carlo algorithm for a simple harmonic oscillator, followed by a quantum harmonic oscillator, and finally the 11-dimensional bosonic BFSS model.

Background - M-theory

- M-theory is a unification of the five consistent versions of superstring theory into one model through S-duality and T-duality, as well as combining 11-D supergravity as a separate limiting case.
- It is considered a speculative candidate framework for a unification of all fundamental forces into a theory of everything.
- Several open questions include testable results, details of the compactification of the required extra dimensions, and finding a complete non-perturbative formulation.
- Mystery, magic, membrane, matrix?

BFSS Model

Ten dimensional action derived via dimensional reduction of the 10-D supersymmetric Yang-Mills theory

$$S_M = \frac{1}{g^2} \int dt \mathrm{Tr} \left\{ \frac{1}{2} (\mathcal{D}_0 X^i) + \frac{1}{4} [X^i, X^j]^2 - \frac{i}{2} \Psi^T C_{10} \Gamma^0 D_0 \Psi + \frac{1}{2} \Psi^T C_{10} \Gamma^i [X^i, \Psi] \right\} \tag{1}$$

- A matrix model proposed by Banks, Fischler, Schencker, and Susskind naturally arising from type IIA string theory in the low-energy limit.
- In this context it forms an effective description of a collection of D0-branes.
- They showed that in this limit the model is described by 11D-supergravity.

BFSS Model

Ten dimensional action derived via dimensional reduction of the 10-D supersymmetric Yang-Mills theory

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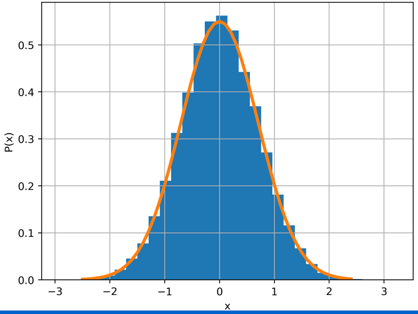
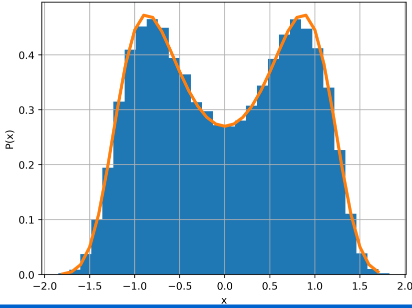
- It is conjectured that in the limit as the matrix size, i.e. the number of branes, goes to infinity the model is equivalent to full uncompactified M-theory.
- This project studied the above action for a system of branes, derived in the literature, by discretisation onto the lattice and simulation via Hybrid Monte Carlo to investigate the dynamics of the system.

Hamiltonian Monte Marlo

- Markov chain Monte Carlo algorithm
- Uses Hamiltonian dynamics to generate a list of samples that converge to a target distribution

$$f(x) = x^2 \to P(x) = e^{-x^2} \qquad f(x) = x^4 - x^2 \to P(x) = e^{-(x^4 - x^2)} \tag{3}$$





Hamiltonian Monte Carlo

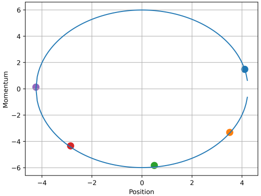
Hamilton's Equations

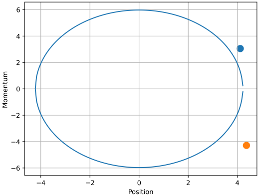
$$\frac{\partial p_i}{\partial t} = -\frac{\partial H}{\partial x_i} \qquad \frac{\partial x_i}{\partial t} = \frac{\partial H}{\partial p_i} \tag{4}$$

Define the Hamiltonian

$$H = \frac{p^2}{2} + S(x), \qquad S(x) = -\ln f(x) \leftrightarrow f(x) = e^{-S(x)} \tag{5}$$

Solve for some later time





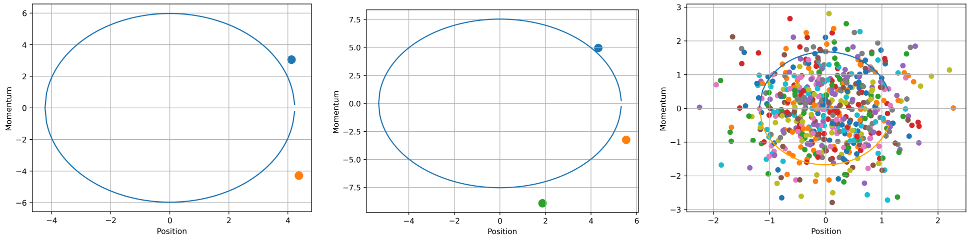
Hamiltonian Monte Carlo - Iterate

Metropolis Check

Accept or reject based on the probability of the new point:

$$e^{-\Delta H} > \mathrm{Random}(0,1) \qquad \Delta H = H_{i+1} - H_i$$

Iterate through the phase space



Hamiltonian Monte Carlo - Uses

Compute observables

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 e^{-f(x)} dx \qquad \Longrightarrow \qquad \langle x^2 \rangle = \frac{1}{N_{samples}} \sum_i^N x^2$$

Model real systems

- Harmonic Oscillator

$$H = K(p) + S(x) = \frac{p^2}{2m} + \frac{1}{2} k x^2 \tag{6}$$

- Apply to actions in higher dimensions:

$$H = K(p_x, p_y, p_z) + S(x, y, z) \tag{7}$$

Hamiltonian Monte Carlo - Uses

Compute observables

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 e^{-f(x)} dx \qquad \Longrightarrow \qquad \langle x^2 \rangle = \frac{1}{N_{samples}} \sum_i^N x^2$$

Model real systems

- Add more degrees of freedom: $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$
- Add **more** degrees of freedom: $X = \left[\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}, \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}, \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}, \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \right]$

Path Integral Formulation of QM

Classical Probability

$$P_{ac} = \sum_b P_{ab} P_{bc} \qquad P_{abc} \equiv P_{ab} P_{bc} \tag{8}$$

Quantum Mechanics

$$\phi_{ac} = \sum_b \phi_{ab} \phi_{bc} \qquad \phi_{abc} \equiv \phi_{ab} \phi_{bc}, \; \phi \in \mathbb{C} \tag{9}$$

Quantum Probability

$$P_{ac}{}^{q} = |\phi_{ac}|^2 \tag{10}$$

Path Integral Formulation of QM

Generalization $A \to C \to K$

$$P_{ack}{}^{q} = |\sum_b \sum_d \dots \sum_j \phi_{abcd\dots k}|^2 \qquad \phi_{abcd\dots k} \equiv \phi_{ab}\phi_{bc}\phi_{cd}\cdots\phi_{jk} \tag{11}$$

- Take measurements of x at equal time intervals $t_{i+1} = t_i + \epsilon$
- x_i corresponds to t_i
- $x_1, x_2, x_3, \dots$ form a continuous path $x(t)$ as $\epsilon \to 0$

Quantum probability amplitude of path in region 'R' ($\Phi \in \mathbb{C}$)

$$\dots \int_{a_i}^{b_i} \int_{a_{i+1}}^{b_{i+1}} \dots \Phi(\dots x_i, x_{i+1}, \dots) \dots dx_i dx_{i+1} \dots = \int_R \Phi(\dots x_i, x_{i+1}, \dots) \dots dx_i dx_{i+1} \dots \tag{12}$$

Postulate I

- **Postulate I:** The probability a particle has a path '$x(t)$' in a region 'R' of spacetime is given by: $|\phi(R)|^2$, where:

$$\phi(R) = \lim_{\epsilon \to 0} \int_R \Phi(\ldots x_i, x_{i+1}, \ldots)\ldots dx_i dx_{i+1} \ldots = \int_R [dx]\Phi[x(t)]$$

Postulate II

- **Postulate II:** The magnitude of the contributions of Φ is equal for all paths, but the phase is the classical action. i.e.:

$$\Phi[x(t)] \propto \exp\left(\frac{i}{\hbar} S[x(t)]\right)$$

$$S[x(t)] = \int \mathcal{L}(\dot{x}(t), x(t))dt$$

The Action

- Returning to the discrete world of 'x_i's we look at the action between two points:

$$S(x_{i+1}, x_i) = \min \int_{t_i}^{t_{i+1}} \mathcal{L}(\dot{x}(t), x(t))dt \tag{13}$$

- and then the total action is easy to find:

$$S = \sum_i S(x_{i+1}, x_i) \tag{14}$$

The Action

- Combining the sum expression of the action with the two postulates yields:

$$\phi(R) = \lim_{\epsilon \to 0} \int_R \exp\left[\frac{i}{\hbar} \sum_i S(x_{i+1}, x_i)\right] \ldots \frac{dx_i}{A} \frac{dx_{i+1}}{A} \ldots \tag{15}$$

Splitting the Past from the Future

- We split the region R into two parts R' and R''. We call some point $t_k \in \{t_i\}$ the present and organize the following:

$$\{R'\}_t < t' < t_k,$$
$$\{R''\}_t > t'' > t_k$$

Splitting the Past from the Future

- We now split the action sum into all points previous to t_k and after t_k. (remembering t_i corresponds to x_i)

$$S = \sum_{i=k}^{\infty} S(x_{i+1}, x_i) + \sum_{-\infty}^{k-1} S(x_{i+1}, x_i) \tag{16}$$

- allowing us to factor the exponential into the left and right factors:

$$\exp\left[\frac{i}{\hbar}\sum_{i=k}^{\infty} S(x_{i+1}, x_i)\right] \cdot \exp\left[\frac{i}{\hbar}\sum_{-\infty}^{k-1} S(x_{i+1}, x_i)\right] \tag{17}$$

A familiar form...

- Now the integration for x_i variables for $i > k$ can be performed on the left factor, and likewise for $i < k$ on the right factor, yielding:

$$\phi(R', R'') = \int \chi^*(x,t)\psi(x,t)dx \tag{18}$$

- where

$$\psi(x_k, t) = \lim_{\epsilon \to 0} \int_{R'} \exp\left[\frac{i}{\hbar}\sum_{-\infty}^{k-1} S(x_{i+1}, x_i)\right]\frac{dx_{k-1}}{A}\frac{dx_{k-2}}{A}\ldots \tag{19}$$

- and

$$\chi^*(x_k, t) = \lim_{\epsilon \to 0} \int_{R''} \exp\left[\frac{i}{\hbar}\sum_{i=k}^{\infty} S(x_{i+1}, x_i)\right]\frac{1}{A}\frac{dx_{k+1}}{A}\frac{dx_{k+2}}{A}\ldots \tag{20}$$

Bosonic BFSS Action

- Reducing the 10-dimensional supersymmetric Yang-Mills theory to one dimension and performing a Wick rotation to Euclidean time, the bosonic part of the action is:

$$S_b = \int_0^\beta \mathrm{tr}\left\{\frac{1}{2}(\mathcal{D}_t X^i)^2 - \frac{1}{4}[X^i, X^j]^2\right\} dt$$

- $\beta = \frac{1}{k_b T}$ is the reciprocal of the thermodynamic temperature of the system, X^i are NxN dimensional matrices with N degrees of freedom, and i, j running from 1 to 9 are the dimension. $\mathcal{D}_t$ is the covariant derivative which allows us to account for changes in the gauge field. These gauge fields are representations of the fundamental forces and encode information about the dynamics of the D0-branes.

Time Discretisation

- In order to simulate this system, we discretise time to Λ sites with a lattice spacing $a = \beta/\Lambda$. We impose the periodic boundary condition $t_\Lambda = t_0$ whilst $t_n = an, (n = 0, 1, 2, ..., \Lambda - 1)$.

$$\partial_t X_n^i \to \frac{X_{n+1}^i - X_n^i}{a} .$$

- In order to maintain gauge invariance, the gauge field at t_{n+1} must be transported to t_n using unitary matrices ($UU^\dagger = \mathbb{1}$) that are transporter fields.:

$$\mathcal{D}_t X_n^i \to \frac{U_{n,n+1} X_{n+1}^i U_{n+1,n} - X_n^i}{a} ,$$

$$\boxed{S_b = \sum_{n=0}^{\Lambda-1} \mathrm{tr} \left\{ -\frac{1}{a} X_n^i U_{n,n+1} X_{n+1}^i U_{n,n+1}^\dagger + \frac{1}{a} (X_n^i)^2 - \frac{a}{4} [X_n^i, X_n^j]^2 \right\}} .$$

Transporter Fields

- At each lattice site, there is a local U(N) (unitary) symmetry. This fact can be used to write the action S_b in a more simple form.

$$
\begin{aligned}
{X'}^i_0 &= X^i_0 \, , \\
{X'}^i_1 &= U_{0,1} \, X^i_1 \, U^\dagger_{0,1} \, , \\
&\dots \\
{X'}^i_{\Lambda-1} &= (U_{0,1} U_{1,2} \dots U_{\Lambda-2,\Lambda-1}) \, X^i_{\Lambda-1} \, (U_{0,1} U_{1,2} \dots U_{\Lambda-2,\Lambda-1})^\dagger
\end{aligned}
$$

- Let $\mathcal{W} = (U_{0,1} U_{1,2} \dots U_{\Lambda-2,\Lambda-1} U_{\Lambda-1,0})$

Discrete Bosonic Action

$$S_b = -\frac{1}{a}\mathrm{tr}\left\{\sum_{n=0}^{\Lambda-2} X'^i_n X'^i_{n+1} + X'^i_{\Lambda-1}\mathcal{W}X'^i_0\mathcal{W}^\dagger\right\} + \sum_{n=0}^{\Lambda-1}\mathrm{tr}\left\{\frac{1}{a}(X'^i_n)^2 - \frac{a}{4}[X'^i_n, X'^j_n]^2\right\}$$

Decomposing $\mathcal{W} = VDV^\dagger$ where $D = \mathrm{diag}\{e^{i\theta_1}, \ldots, e^{i\theta_N}\}$ (θ_N refer to angles associated with the rotation). However, it is possible to choose a gauge such that S_b becomes:

$$\boxed{S_b[X, D] = \mathrm{tr}\left\{-\frac{1}{a}\sum_{n=0}^{\Lambda-2} X^i_n X^i_{n+1} - \frac{1}{a}X^i_{\Lambda-1}DX^i_0D^\dagger + \sum_{n=0}^{\Lambda-1}\left[\frac{1}{a}(X^i_n)^2 - \frac{a}{4}[X^i_n, X^j_n]^2\right]\right\}}$$

Transporter Field Interactions

- It is necessary to account for the interaction between the transporter field and the gauge field (associated with the U(N) symmetry) as well as the dynamics of the transporter field itself, as gauge transformations are local. This can be achieved using the covariant derivative of $U_{n,n+1}$.

$$S_{\mathrm{FP}}[\theta] = -\sum_{l \neq m} \ln \left| \sin \frac{\theta_l - \theta_m}{2} \right| .$$

- This term is essentially a phase due to a change in coordinates. It will be added to the Hamiltonian of the system.

Bosonic BFSS Hamiltonian

- The Hamiltonian for this system is given by:

$$H = \frac{1}{2}\sum_{n=0}^{\Lambda-1} \mathrm{tr}\, P_n^i.P_n^i + \frac{1}{2}\sum_{l=0}^{N-1} {P_d^l}^2 + S_b[X, D(\theta)] + S_{\mathrm{FP}}[\theta]$$

- P_n^i and P_d^l (chosen randomly) are the canonical momenta corresponding to the hermitian matrices X_n^i and the angles θ_l, respectively.

$$\dot{q} = \partial H/\partial p \ . \qquad \dot{p} = -\partial H/\partial q = F(q).$$

- Now, all we need to do is find the forces using Hamilton's equations.

Forces

- After very careful differentiation, we have:

$$
\begin{aligned}
-\partial S_b/\partial X^i_{0,\,ml} &= \frac{1}{a}\left(X^i_1 - 2X^i_0 + D^\dagger X^i_{\Lambda-1} D\right)_{lm} + a\left[X^j_0, \left[X^i_0, X^j_0\right]\right]_{lm} \;, \\
-\partial S_b/\partial X^i_{n,\,ml} &= \frac{1}{a}\left(X^i_{n+1} - 2X^i_n + X^i_{n-1}\right)_{lm} + a\left[X^j_n, \left[X^i_n, X^j_n\right]\right]_{lm} \qquad n = 1, \ldots, \Lambda - 2 \;, \\
-\partial S_b/\partial X^i_{\Lambda-1,\,ml} &= \frac{1}{a}\left(D X^i_0 D^\dagger - 2X^i_{\Lambda-1} + X^i_{\Lambda-2}\right)_{lm} + a\left[X^j_{\Lambda-1}, \left[X^i_{\Lambda-1}, X^j_{\Lambda-1}\right]\right]_{lm} \;, \\
-\partial S_b/\partial \theta_l &= \frac{2}{a}\sum_{m=0}^{N-1} \mathrm{Re}\left(i X^i_{\Lambda-1\;ml} X^i_{0\;lm} e^{i(\theta_l-\theta_m)}\right) + \sum_{m\,,\,m\neq l} \cot\left(\frac{\theta_l-\theta_m}{2}\right) .
\end{aligned}
$$

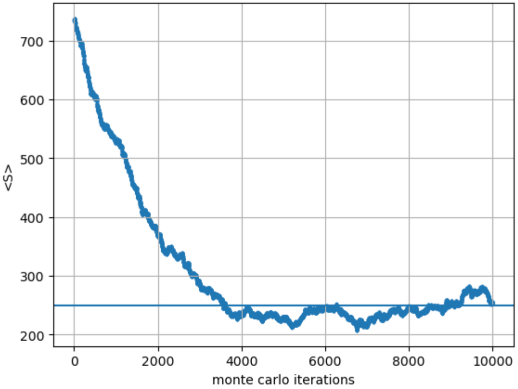
Expectation Value

- We used the partition function $Z = \int [dx] e^{-(S_2[x]+S_3[x]+...+S_k[x])}$, where $S_k[x]$ is the action (a polynomial of order k), to derive a check we could use for our code.
- We can define $Z_B = B^{-N^2\Lambda} Z$ where B is a dummy variable that will be set to 1 later. Introducing $\tilde{x} = x/B$ and taking dZ_B/dB, we eventually end up with:

$$N^2\Lambda = 2 < S_2[x] > +3 < S_3[x] > +... + k < S_k[x] > .$$

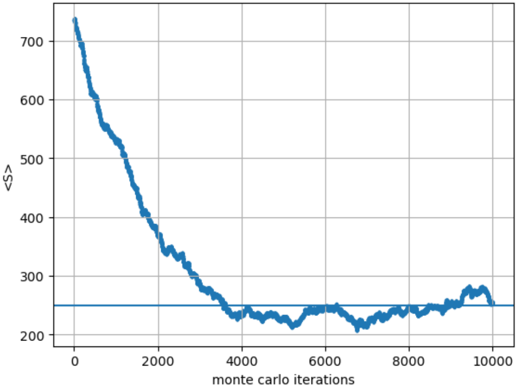
Simulation Results

- First, we checked if the expectation value of the action $< S >$ converged to the value we expected analytically.
- As can be seen in the plot, the action for a one-dimensional system where $N = 10, \Lambda = 5,\ \beta = 1$ converges and oscillates.
- The horizontal blue line represents what we would expect analytically.



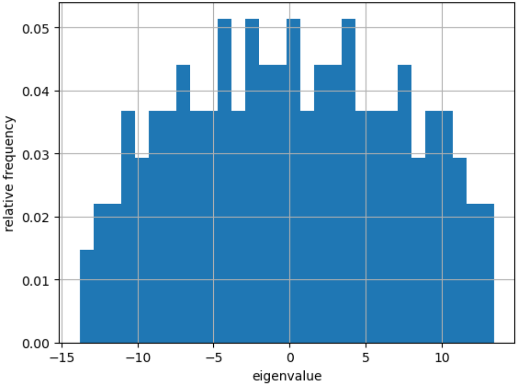
Simulation Results

- The action for a five-dimensional system where $N = 10, \Lambda = 4,\ \beta = 1$ also converges and oscillates.
- The horizontal blue line represents what we would expect analytically.



Simulation Results

- We can also use Wigner's semicircle law to check if our matrices behave as we would expect.
- **Wigner's semicircle law**: If we let X be a symmetric $N \times N$ matrix where the entries are independent and identically distributed random variables with bounded moments, then the eigenvalue distribution of X converges to a distribution in the shape of a semicircle as N goes to infinity.



Conclusion

- If we had more time, we could add fermionic interactions to our model. The infinite N limit of such a model is conjectured to be a formulation of a non-perturbative model of M-theory.
- With enough computational power, this algorithm could be used to study dynamics of M-theory. The low-energy limit ($\beta << 1$) BFSS model algorithm could be used to study supergravity, for example, or known results could be compared to this model to further investigate the proposal that the (low energy) BFSS model is equivalent to M-theory.