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Classical Chaos in String Theoretic Black Holes

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Classical Black Holes

In classical physics, a black hole is a region of spacetime where the gravitational field is so strong that once a particle enters, it can never escape.

Black Hole Mass

$$\underbrace{ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2 d\Omega^2}_{\text{Schwarzschild metric}}$$

The surface r = 2M is known as the Event Horizon of the black hole. This is the point of no return.

Quantum Black Holes

In 1972, Jacob Bekenstein proposed that black holes should have entropy which should be proportional to the area of the event horizon.

In 1974, Stephen Hawking showed that black holes are thermal systems which readiate energy continuously. He was able to derive a precise formula for the entropy of a black hole.

Boltzmann's constant

Area of Event Horizon

Planck's constant

$$S_{BH} = \frac{k_B A}{4\hbar} \quad (1)$$

The statistical origin of the Bekenstein-Hawking entropy remained unclear

Black Hole Chaos

Thermal behaviour is closely related to chaos. Since black holes are thermal systems, it is natural to look for chaotic behaviour in black holes.

Recent work has placed an upper bound on the chaotic behvaiour of thermal systems.

Temperature

$$\lambda_L \leq \frac{2\pi k_B T}{\hbar} \tag{2}$$

Black holes saturate this bound and thus are the most chaotic systems found in nature.

Classical Chaos

Consider choosing a point x_0 in the phase space of some system and perturbing it slightly to obtain a new point δx_0.

Hamilton's equations describe the time evolution of each point. We can define a metric on phase space $|\delta X(t)| \equiv |x_0(t) - \delta x_0(t)|$ to define the distance between these two points at late time.

For chaotic systems, we expect the perturbation to grow exponentially

$$|\delta X(t)| \sim |\delta X(0)| e^{\lambda_L t} \tag{3}$$

System under analysis

- String theory: 9 space dimensions, 1 time dimension.
- Spatial position of one D0-brane requires 9 position coordinates.
- Let N be the number of D0-branes.
- X^i = N $\times$ N Hermitian traceless matrix, $i = 1, ..., 9$.

$$\mathsf{L} = \frac{1}{2g^2}\mathrm{Tr}\big(\sum_i [D_t X^i]^2\big) + \frac{1}{4g^2}\mathrm{Tr}\big(\sum_{j\neq i}[X^i, X^j]^2\big) + ...$$

$$D_t X^i = \partial_t X^i - [A_t, X^i].$$

$$A_t = 0.$$

$$[D_t X^i]^2 = [\dot{X}^i]^2$$

Classical Approximation

- Classical mechanics provides a good approximation in the weak coupling/high temperature limit:

$$\frac{g^2 N}{T^3} \sim small$$

- Lagrangian is simplified in the classical limit.

X^i Equations of motion:

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}^i}\right) - \frac{\partial L}{\partial X^i}.$$

$$\ddot{X}^i(t) = \sum_j [X^j, [X^i, X^j]].$$

A_t Equation of motion:

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{A}_t}\right) - \frac{\partial L}{\partial A_t}.$$

$$0 = \sum_i [X^i, \dot{X}^i].$$

Perturbation

- To perturb the system while preserving Gauss' law constraint:

$$\mathsf{L} = \frac{1}{2g^2}\mathrm{Tr}(\sum_i [D_t X^i]^2) + \frac{1}{4g^2}\mathrm{Tr}(\sum_{j\neq i}[X^i, X^j]^2) + \sum_{k=1}^{2} kc_k \mathrm{Tr}(\sum_i X^{i2})$$

$$c_k \sim N(0, 10^{-8})$$

- The perturbing equations of motion:

$$\ddot{X}^i = \sum_j [X^j, [X^i, X^j]] + \sum_{k=1}^{2} kc_k \{X^i, (\sum_j X_j^2)^{k-1}\}$$

Simulations

- Evolve initial system for 1 second with original equations of motion.
  State 1: X_1^i
- Perturb initial system for 1 second with perturbation equations of motion.
  State 2: X_2^i
- Evolve both states under original equations of motion for 100 seconds while measuring the distance between them.

$$|\delta X(t)| = \sqrt{\sum_{i=1}^{9} \mathrm{Tr}[(X_1^i(t) - X_2^i(t))^2]}$$

Lyapunov Exponent

- $|\delta X(t)| = |\delta X(0)| e^{\lambda_L t}$
- Expect $|\delta X(t)|$ against t to be linear on a log-scale plot.
- Solve for $\lambda_L t$:

$$\lambda_L t = \ln\left(\frac{|\delta X(t)|}{|\delta X(0)|}\right)$$

- Expect λ_L to be the slope of the plot of $\ln(\frac{|\delta X(t)|}{|\delta X(0)|})$ against t.

Results: Distance and Energy

- Simulate $N = 4$ case with varied total energy E.

Results: Distance and Energy

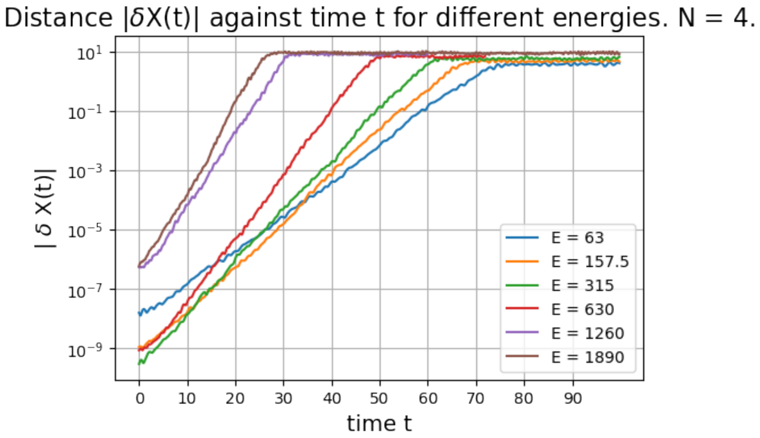

Figure: $|\delta X(t)|$ against t for multiple total energies for 4 D0-branes. Growth continues until the perturbation is of the same order as the system size.

Results: Distance and Energy

- Simulate N = 4 case with varied total energy E.
- Rescale the y-axis to be $\ln(\frac{|\delta X(t)|}{|\delta X(0)|})$

Results: Distance and Energy

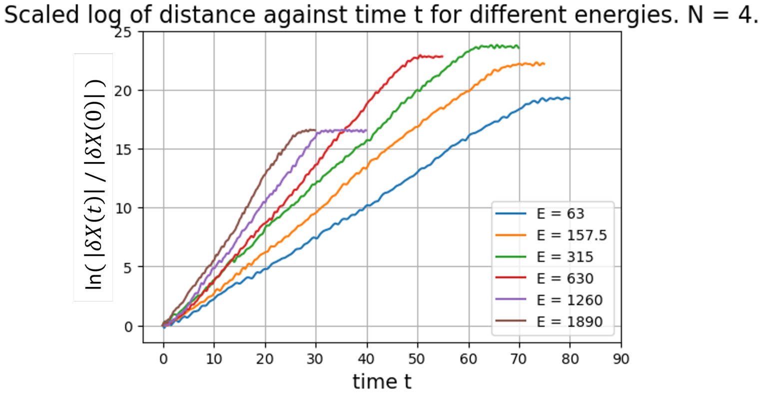

Figure: $\ln(\frac{|\delta X(t)|}{|\delta X(0)|})$ against t. Growth continues until the perturbation is of the same order as the system size. Increasing slope correlates with increasing energy of the system.

Results: Distance and Energy

- Simulate $N = 4$ case with varied total energy E.
- Rescale the y-axis to be $\ln(\frac{|\delta X(t)|}{|\delta X(0)|})$
- Linear fit the sloped sections of the data to find $\lambda_L(E)$
- Plot $\lambda_L(E)$ against E.

Results: Distance and Energy

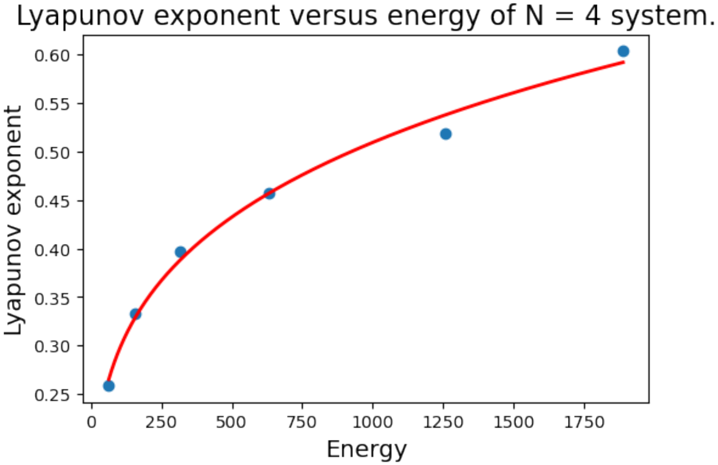

Figure: $\lambda_L(E)$ against total energy E. $\lambda_L(E) \propto E^{0.24}$.

Results: Distance and Energy

- Repeat for N = 7 case.

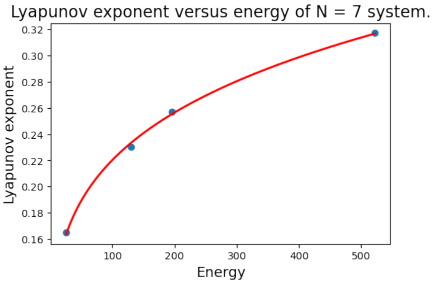

Figure: $\lambda_L(E)$ against total energy E. $\lambda_L(E) \propto E^{0.22}$.

Lyapunov exponent and Energy

- Experimental data:
- N $= 4$ case: $\lambda_L(E)= (0.100 \pm 0.0077)\ E^{0.24\pm0.011}$
- N $= 7$ case: $\lambda_L(E)= (0.080 \pm 0.0025) E^{0.220\pm0.0059}$
- An upper bound on the energy dependence of the Lyapunov exponent for any chaotic system has been conjectured*:

$$c \leq 1 \text{ for } \lambda_L(E) \propto E^c (E \to \infty)$$

*Hashimoto, K. Murata, K. Tanahashi, N. Watanabe, R. (2022). A bound on energy dependence of chaos. *Phys. Rev. D.* 106(12)

Lyapunov exponent comparison

- Compare λ_l with Shenker et. al.*

$$\lambda_L = (0.29252 - \frac{0.424}{N^2})(g^2 NT)^{\frac{1}{4}}$$

Case	*Equation*	*Experiment*
$N = 4$	$\lambda_L = 0.2655$	$\lambda_L = 0.270$
$N = 8$	$\lambda_L = 0.2853$	$\lambda_L = 0.2865$
$N = 12$	$\lambda_L = 0.2891$	$\lambda_L = 0.2893$
$N = 16$	$\lambda_L = 0.2903$	$\lambda_L = 0.2884$

* Gur-Ari, G. Hanada, M. Shenker, S. (2016). Chaos in Classical D0-Brane Mechanics. *Journal of High Energy Physics*. 91(2016).

Fast Scrambler

- A chaotic system is a 'fast scrambler' if the scrambling time t^* scales as $ln(N^2)$.
- For a fast scrambler: $t^* \sim \frac{k}{2} ln(N^2)$
- t^* has an inverse dependence on λ_L: $t^* \sim \frac{1}{\lambda_L}$
- $t^* \sim \frac{1}{\lambda_L} \frac{k}{2} ln(N^2)$
- Then: $e^{\lambda_L t^*} \sim N^k$
- $|\delta X(t^*)| \sim N^k$
- $\frac{|\delta X(t^*)|}{N^k} = \text{constant}$

* Gur-Ari, G. Hanada, M. Shenker, S. (2016). Chaos in Classical D0-Brane Mechanics. *Journal of High Energy Physics*. 91(2016).

Fast Scrambler

- $\frac{|\delta X(t^*)|}{N^k} = \text{constant}$
- Empirical answer: $k = \frac{1}{2}$
- $\frac{|\delta X(t^*)|}{\sqrt{N}} = \text{constant}$

Fast Scrambler

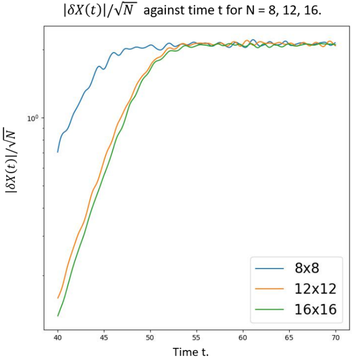

Figure: $|\delta X(t)|/\sqrt{N}$ against time for N = 8, 12, 16.

Conclusion

- Energy dependence of the Lyapunov exponent for four D0-branes and for seven D0-branes.

$$\lambda_L(E) = (0.100 \pm 0.0077)E^{0.24 \pm 0.011}$$
$$\lambda_L(E) = (0.080 \pm 0.0025)E^{0.220 \pm 0.0059}$$

- Good agreement with previous literature values for λ_L.
- System is a fast scrambler.

Numerical Integration

- Velocity Verlet algorithm:

$$X(t+\Delta t) = X(t) + \dot{X}(t)\Delta t + \frac{1}{2}\ddot{X}(t)\Delta t^2. \tag{4}$$

$$\dot{X}(t+\Delta t) = \dot{X}(t) + \frac{1}{2}(\ddot{X}(t) + \ddot{X}(t+\Delta t))\Delta t. \tag{5}$$

Algorithm:

- Set initial position X^i and velocity $\dot{X}^i$ matrices for the system.
- Use equations of motion to calculate the initial acceleration $\ddot{X}^i$ matrices.
- Use equation (4) to calculate $X^i(t+\Delta t)$.
- Use equations of motion to calculate $\ddot{X}(t+\Delta t)$.
- Use equation (5) to calculate $\dot{X}(t+\Delta t)$.
- Repeat the previous steps.